Limiting strain models in elasticity theory and variational integrals with linear growth

Miroslav Bulíček

Mathematical Institute of the Charles University
Sokolovská 83, 186 75 Prague 8, Czech Republic

Equadiff 2017

July 7, 2017

The talk is based on the following results

- M. Bulíček, J. Málek, K. R. Rajagopal and J. R. Walton: **Existence of solutions for the anti-plane stress for a new class of "strain-limiting" elastic bodies**, Calc. Var. Partial Differential Equations, 2015
- M. Bulíček, J. Málek and E. Süli: **Analysis and approximation of a strain-limiting nonlinear elastic model**, Mathematics and Mechanics of Solids, 2014
- M. Bulíček, J. Málek, K. R. Rajagopal and E. Süli: **On elastic solids with limiting small strain: modelling and analysis**, EMS Surveys in Mathematical Sciences, 2014.
- L. Beck, M. Bulíček, J. Málek and E. Süli: **Analysis and approximation of a strain-limiting nonlinear elastic model II**, ARMA 2017
- L. Beck, M. Bulíček, E. Maringová: **On regularity up to the boundary for variational problems with linear growth**, submitted

Linearized nonlinear elasticity

We consider the elastic deformation of the body $\Omega \subset \mathbb{R}^d$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\overline{\Gamma_D \cup \Gamma_N} = \partial\Omega$ described by

$$\begin{aligned} -\operatorname{div} \mathbf{T} = \boldsymbol{f} \qquad & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{u}_0 \qquad \text{on } \Gamma_D, \qquad \text{and} \qquad & \mathbf{T}\boldsymbol{n} = \boldsymbol{g} \quad \text{on } \Gamma_N. \end{aligned} \tag{El}$$

where $\boldsymbol{u}$ is displacement, $\mathbf{T}$ the Cauchy stress, $\boldsymbol{f}$ the external body forces, $\boldsymbol{g}$ the external surface forces and $\boldsymbol{\varepsilon}$ is the linearized strain tensor, i.e.,

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$$

Linearized nonlinear elasticity

We consider the elastic deformation of the body $\Omega \subset \mathbb{R}^d$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\overline{\Gamma_D \cup \Gamma_N} = \partial\Omega$ described by

$$\begin{aligned} -\operatorname{div} \mathbf{T} &= \boldsymbol{f} && \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{u}_0 && \text{on } \Gamma_D, \quad \text{and} \quad \mathbf{T}\boldsymbol{n} = \boldsymbol{g} \quad \text{on } \Gamma_N. \end{aligned} \tag{El}$$

where $\boldsymbol{u}$ is displacement, $\mathbf{T}$ the Cauchy stress, $\boldsymbol{f}$ the external body forces, $\boldsymbol{g}$ the external surface forces and $\boldsymbol{\varepsilon}$ is the linearized strain tensor, i.e.,

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$$

- The implicit relation between the Cauchy stress and the strain

$$\boxed{\mathbf{G}(\mathbf{T}, \boldsymbol{\varepsilon}) = \mathbf{0}}$$

Linearized nonlinear elasticity

We consider the elastic deformation of the body $\Omega \subset \mathbb{R}^d$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\overline{\Gamma_D \cup \Gamma_N} = \partial\Omega$ described by

$$\begin{aligned} -\operatorname{div} \mathbf{T} = \boldsymbol{f} \qquad & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{u}_0 \qquad \text{on } \Gamma_D, \qquad & \text{and} \qquad \mathbf{T}\boldsymbol{n} = \boldsymbol{g} \quad \text{on } \Gamma_N. \end{aligned} \tag{El}$$

where $\boldsymbol{u}$ is displacement, $\mathbf{T}$ the Cauchy stress, $\boldsymbol{f}$ the external body forces, $\boldsymbol{g}$ the external surface forces and $\boldsymbol{\varepsilon}$ is the linearized strain tensor, i.e.,

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$$

- The implicit relation between the Cauchy stress and the strain

$$\boxed{\mathbf{G}(\mathbf{T}, \boldsymbol{\varepsilon}) = \mathbf{0}}$$

- The key assumption in linearized elasticity

$$\boxed{|\boldsymbol{\varepsilon}| \ll 1}. \tag{A}$$

Motivation for symmetric p-Laplace like operator for $p = 1$ or $p = \infty$

- $p = 1$: the model of the plasticity, i.e.,

$$\mathbf{T} \sim \frac{\boldsymbol{\varepsilon}}{|\boldsymbol{\varepsilon}|} \qquad \text{for } |\boldsymbol{\varepsilon}| \gg 1$$

- $p = \infty$: the limiting strain model, i.e.,

$$\boldsymbol{\varepsilon} \sim \frac{\mathbf{T}}{|\mathbf{T}|} \qquad \text{for } |\mathbf{T}| \gg 1.$$

Limiting strain model

The standard linear models immediately may lead to the contradiction:

Limiting strain model

The standard linear models immediately may lead to the contradiction:

- Consider Ω a domain with non-convex corner at x_0, $\Gamma = \partial\Omega$, $\boldsymbol{u}_0 = \mathbf{0}$ and $\mathbf{G}$ of the form

$$\mathbf{T} = \varepsilon.$$

Limiting strain model

The standard linear models immediately may lead to the contradiction:

- Consider Ω a domain with non-convex corner at x_0, $\Gamma = \partial\Omega$, $\boldsymbol{u}_0 = \mathbf{0}$ and $\mathbf{G}$ of the form
$$\mathbf{T} = \varepsilon.$$
- There exists a smooth $\boldsymbol{f}$ such that the solution $(\mathbf{T}, \varepsilon)$ fulfils
$$|\mathbf{T}(x)| = |\varepsilon(x)| \overset{x \to x_0}{\to} \infty.$$

Limiting strain model

The standard linear models immediately may lead to the contradiction:

- Consider Ω a domain with non-convex corner at x_0, $\Gamma = \partial\Omega$, $\boldsymbol{u}_0 = \mathbf{0}$ and $\mathbf{G}$ of the form
$$\mathbf{T} = \varepsilon.$$
- There exists a smooth $\boldsymbol{f}$ such that the solution $(\mathbf{T}, \varepsilon)$ fulfils
$$|\mathbf{T}(x)| = |\varepsilon(x)| \stackrel{x \to x_0}{\to} \infty.$$

$\Longrightarrow$ contradicts the assumption of the model (A) $\Longrightarrow$ not valid model at least in the neighborhood of x_0.

Limiting strain model

Limiting strain model

- Consider implicit models which a priori guarantees $|\varepsilon| \le K$:

$$\boxed{\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^*(\mathbf{T}) := \lambda_1(|\operatorname{tr}\mathbf{T}|)(\operatorname{tr}\mathbf{T})\mathbf{I} + \lambda_2(|\mathbf{T}|)\mathbf{T} + \lambda_3(|\mathbf{T}^d|)\mathbf{T}^d}, \qquad \text{(L-S)}$$

where

$$\mathbf{T}^d := \mathbf{T} - \frac{\operatorname{tr}\mathbf{T}}{d}, \qquad |\lambda_{1,2,3}(s)| \le \frac{K}{3(s+1)}.$$

Limiting strain model

- Consider implicit models which a priori guarantees $|\boldsymbol{\varepsilon}| \le K$:

$$\boxed{\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^*(\mathbf{T}) := \lambda_1(|\operatorname{tr}\mathbf{T}|)(\operatorname{tr}\mathbf{T})\mathbf{I} + \lambda_2(|\mathbf{T}|)\mathbf{T} + \lambda_3(|\mathbf{T}^d|)\mathbf{T}^d}, \tag{L-S}$$

where

$$\mathbf{T}^d := \mathbf{T} - \frac{\operatorname{tr}\mathbf{T}}{d}, \qquad |\lambda_{1,2,3}(s)| \le \frac{K}{3(s+1)}.$$

- A priori estimates: from (L-S)

$$|\boldsymbol{\varepsilon}| \le K.$$

From the equation, we may hope that

$$\int_\Omega \lambda_1(|\operatorname{tr}\mathbf{T}|)|\operatorname{tr}\mathbf{T}|^2 + \lambda_2(|\mathbf{T}|)|\mathbf{T}|^2 + \lambda_3(|\mathbf{T}^d|)|\mathbf{T}^d|^2 = \int_\Omega \mathbf{T}\cdot\boldsymbol{\varepsilon} \le C.$$

Limiting strain model

- Consider implicit models which a priori guarantees $|\varepsilon| \le K$:

$$\boxed{\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^*(\mathbf{T}) := \lambda_1(|\operatorname{tr}\mathbf{T}|)(\operatorname{tr}\mathbf{T})\mathbf{I} + \lambda_2(|\mathbf{T}|)\mathbf{T} + \lambda_3(|\mathbf{T}^d|)\mathbf{T}^d}, \tag{L-S}$$

  where

$$\mathbf{T}^d := \mathbf{T} - \frac{\operatorname{tr}\mathbf{T}}{d}, \qquad |\lambda_{1,2,3}(s)| \le \frac{K}{3(s+1)}.$$

- A priori estimates: from (L-S)

$$|\boldsymbol{\varepsilon}| \le K.$$

  From the equation, we may hope that

$$\int_\Omega \lambda_1(|\operatorname{tr}\mathbf{T}|)|\operatorname{tr}\mathbf{T}|^2 + \lambda_2(|\mathbf{T}|)|\mathbf{T}|^2 + \lambda_3(|\mathbf{T}^d|)|\mathbf{T}^d|^2 = \int_\Omega \mathbf{T}\cdot\boldsymbol{\varepsilon} \le C.$$

- The reasonable assumptions (∞-Laplacian-like problem):

$$\left.\lambda_{1,2,3}(s) \ge \frac{\alpha}{s+1}. \quad\right\} \implies \int_\Omega |\mathbf{T}| \le C.$$

Limiting strain model & monotonicity

- Apriori estimates for $\mathbf{T}$ in L^1
- For the convergence at least some monotonicity needed, the minimal assumption:

$$0 \le \frac{d}{ds}(\lambda_{1,2,3}(s)s). \tag{M}$$

- If we would have a sequence fulfilling

$$\int_{\Omega_0} |\mathbf{T}^n|^{1+\delta} \le C(\Omega_0) \qquad \text{for all } \Omega_0 \subset\subset \Omega,$$
$$\implies \mathbf{T}^n \rightharpoonup \mathbf{T} \qquad \text{weakly in } L^1_{loc}.$$

  then using (M) we can identify the limit.
- Assume kind of uniform monotonicity, i.e., for some $\alpha, a, K > 0$

$$\frac{\alpha}{(K+s)^{a+1}} \le \frac{d}{dt}(\lambda_i(s)s) \tag{UM}$$

  for example

$$\boxed{\lambda_i(s) := \frac{1}{(1+s^a)^{\frac{1}{a}}}} \quad \text{for simplicity} \quad \boxed{\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^*(\mathbf{T}) := \frac{\mathbf{T}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}}}.$$

Simplified setting - potential structure

We look for $(\boldsymbol{u}, \mathbf{T})$ such that $\boldsymbol{u} = \boldsymbol{u}_0$ on Γ_D and $\mathbf{T}\boldsymbol{n} = \boldsymbol{g}$ on Γ_N such that in Ω there holds

$$\left.\begin{aligned} -\operatorname{div} \mathbf{T} &= \boldsymbol{f}, \\ \varepsilon(\boldsymbol{u}) &= \varepsilon^*(\mathbf{T}). \end{aligned}\right\} \quad \Leftrightarrow \quad \left\{ -\operatorname{div} \mathbf{T}^*(\varepsilon(\boldsymbol{u})) = \boldsymbol{f}. \right.$$

with

$$\boxed{\varepsilon^*(\mathbf{T}) := \frac{\mathbf{T}}{(1 + |\mathbf{T}|^a)^{\frac{1}{a}}} \qquad \text{and} \qquad T^*(\mathbf{W}) := (\varepsilon^*)^{-1}(\mathbf{W}) := \frac{\mathbf{W}}{(1 - |\mathbf{W}|^a)^{\frac{1}{a}}}}$$

for all $\mathbf{T} \in \mathbb{R}^{d \times d}_{sym}$ and $\mathbf{W} \in \mathbb{R}^{d \times d}_{sym}$ such that $|\mathbf{W}| < 1$.

Simplified setting - potential structure

First, we introduce the space of functions having bounded the symmetric gradient

$$E := \{\boldsymbol{u} \in W^{1,1}(\Omega)^d;\ \boldsymbol{\varepsilon}(\boldsymbol{u}) \in L^\infty(\Omega)^{d\times d}\}.$$

and assume at least $\boldsymbol{u}_0 \in E$, $\boldsymbol{f} \in L^2(\Omega)^d$ and $\boldsymbol{g} \in L^1(\Gamma_N)^d$.

Simplified setting - potential structure

First, we introduce the space of functions having bounded the symmetric gradient

$$E := \{\boldsymbol{u} \in W^{1,1}(\Omega)^d;\ \boldsymbol{\varepsilon}(\boldsymbol{u}) \in L^\infty(\Omega)^{d\times d}\}.$$

and assume at least $\boldsymbol{u}_0 \in E$, $\boldsymbol{f} \in L^2(\Omega)^d$ and $\boldsymbol{g} \in L^1(\Gamma_N)^d$.

- the set of admissible displacement

$$\mathcal{V} := \{\boldsymbol{u} \in W^{1,1}(\Omega) :\ \boldsymbol{u} - \boldsymbol{u}_0 \in W^{1,1}_{\Gamma_D}(\Omega)^d,\ \boldsymbol{u} \in E\}$$

- the set of admissible stresses

$$\mathcal{S} := \left\{\mathbf{T} \in L^1(\Omega)^{d\times d}_{sym} :\ \forall \boldsymbol{v} \in E \cap W^{1,1}_{\Gamma_D}\ \int_\Omega \mathbf{T}\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} + \int_{\Gamma_N} \boldsymbol{g}\cdot\boldsymbol{v}\right\}$$

Weak solution: Find $(\boldsymbol{u}, \mathbf{T}) \in \mathcal{V} \times \mathcal{S}$ such that $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{\varepsilon}^*(\mathbf{T})$ a.e. in Ω.

Potential structure - primary formulation

Find potential $F : \mathbb{R}^{d\times d}_{sym} \to \mathbb{R}_+$ such that $F(0) = 0$ and

$$\frac{\partial F(\mathbf{W})}{\partial \mathbf{W}} = \mathbf{T}^*(\mathbf{W}) \qquad \text{if } |\mathbf{W}| < 1,$$
$$F(\mathbf{W}) = \infty \qquad \text{if } |\mathbf{W}| > 1.$$

Primary (variational) formulation: Find $\boldsymbol{u} \in \mathcal{V}$ such that for all $\boldsymbol{v} \in \mathcal{V}$

$$\boxed{\int_\Omega F(\varepsilon(\boldsymbol{u})) - f \cdot \boldsymbol{u} - \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{u} \le \int_\Omega F(\varepsilon(\boldsymbol{v})) - f \cdot \boldsymbol{v} - \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{v}}$$

Lemma

Let $\|\boldsymbol{\varepsilon}(\boldsymbol{u}_0)\|_\infty < 1$ (the safety strain condition). Then thee exists a unique $\boldsymbol{u}$ solving the primary formulation. Moreover there exists $\mathbf{T} \in L^1(\Omega)^{d\times d}$ such that $\varepsilon(\boldsymbol{u}) = \varepsilon^*(\mathbf{T})$ and for all $\boldsymbol{v} \in \mathcal{V}$ such that $\mathbf{T}^*(\boldsymbol{\varepsilon}(\boldsymbol{v})) \in L^1$ there holds

$$\int_\Omega \mathbf{T} \cdot \varepsilon(\boldsymbol{u} - \boldsymbol{v}) \le \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{u} - \boldsymbol{v}) + \int_{\Gamma_N} \boldsymbol{g} \cdot (\boldsymbol{u} - \boldsymbol{v})$$

In addition, if there is a weak solution then it also solves the primary formulation. Similarly, if $\boldsymbol{u}$ satisfies the safety strain condition, then $(\boldsymbol{u}, \mathbf{T})$ is a weak solution.

Potential structure - dual formulation

Find potential $F^* : \mathbb{R}^{d\times d}_{sym} \to \mathbb{R}_+$ such that $F(0) = 0$ and (note here that $F(\mathbf{W}) \sim |\mathbf{W}|$ at infinity

$$\frac{\partial F^*(\mathbf{W})}{\partial \mathbf{W}} = \boldsymbol{\varepsilon}^*(\mathbf{W}).$$

Dual (variational) formulation: Find $\mathbf{T} \in \mathcal{S}$ such that for all $\mathbf{W} \in \mathcal{S}$

$$\boxed{\int_\Omega F^*(\mathbf{T}) - \mathbf{T} \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_0) \le \int_\Omega F(\mathbf{W}) - \mathbf{W} \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_0)}$$

Lemma

The existence of weak solution is equivalent to the existence of the minimizer to the dual problem. Moreover, if $\|\varepsilon(\boldsymbol{u}_0)\|_\infty < 1$ (the safety strain condition) then there exists a finite infimum of the dual formulation which maybe attained by $\overline{\mathbf{T}} \in \mathcal{M}(\overline{\Omega})^{d\times d}_{sym}$.

Potential structure - relaxed dual formulation

- the relaxed set of admissible stresses

$$\mathcal{S}^m := \left\{ \mathbf{T} \in \mathcal{M}(\overline{\Omega})^{d\times d}_{sym} : \forall \boldsymbol{v} \in \mathcal{C}^1_{\Gamma_D}(\Omega)^d \int_\Omega \mathbf{T}\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} + \int_{\Gamma_N} \boldsymbol{g}\cdot\boldsymbol{v} \right\}$$

Dual (variational) relaxed formulation: For $\boldsymbol{u}_0 \in \mathcal{C}^1(\Omega)^d$, find $\mathbf{T} \in \mathcal{S}^m$ such that for all $\mathbf{W} \in \mathcal{S}^m$

$$\boxed{\int_\Omega F^*(\mathbf{T}^r) + (\mathbf{W}^r - \mathbf{T}^r)\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + |\mathbf{T}^s|(\overline{\Omega}) + \langle \mathbf{W}^s - \mathbf{T}^s, \boldsymbol{\varepsilon}(\boldsymbol{u}_0)\rangle \le \int_\Omega F^*(\mathbf{W}^r) + |\mathbf{W}^s|(\overline{\Omega})}$$

where $\mathbf{T} = \mathbf{T}^r + \mathbf{T}^s$ and $\mathbf{T}^r$ is a regular part (i.e., absolutely continuous w.r.t. Lebesgue measure) and $\mathbf{T}^s$ is a singular part (i.e., supported on the set of zero Lebesgue measure).

Lemma

Let $\|\boldsymbol{\varepsilon}(\boldsymbol{u}_0)\|_\infty < 1$. Then there exists a minimizer to relaxed dual formulation. Moreover, the regular part $\mathbf{T}^r$ is unique and satisfies $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{\varepsilon}^*(\mathbf{T}^r)$, where $\boldsymbol{u}$ is (unique) minimizer to primary formulation. In addition, if $\mathbf{T}^s_1$ and $\mathbf{T}^s_2$ are two singular parts then for all $\boldsymbol{v} \in \mathcal{C}^1_{\Gamma_D}(\Omega)^d$

$$|\mathbf{T}^s_1|(\overline{\Omega}) - \langle \mathbf{T}^s_1, \boldsymbol{\varepsilon}(\boldsymbol{u}_0)\rangle = |\mathbf{T}^s_2|(\overline{\Omega}) - \langle \mathbf{T}^s_2, \boldsymbol{\varepsilon}(\boldsymbol{u}_0)\rangle \ \textit{and}\ \langle \mathbf{T}^s_1 - \mathbf{T}^s_2, \nabla \boldsymbol{v}\rangle = 0$$

Conclusion

Conclusion

- We solved the problem completely. Natural setting is the relaxed dual formulation. The displacement is unique. The regular part of the Cauchy stress is unique. There is non-uniquely given singular part of the Cauchy stress.

Conclusion

- We solved the problem completely. Natural setting is the relaxed dual formulation. The displacement is unique. The regular part of the Cauchy stress is unique. There is non-uniquely given singular part of the Cauchy stress.
- Where is the singular measure supported? Is it really there? How do you explain that the regular part did not solve the balance equation? Is there some crack/damage possible region? Is there any influence of the shape Ω or the parameter a? etc. etc.

Limiting strain model - anti-plane stress

We consider the following special geometry

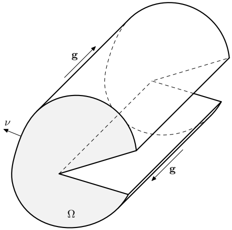

Figure: Anti-plane stress geometry.

and we look for the solution in the following from:

$$\boldsymbol{u} = \boldsymbol{u}(x_1, x_2) = (0, 0, u(x_1, x_2)), \quad \boldsymbol{g}(x) = (0, 0, g(x_1, x_2)),$$

and

$$\mathbf{T}(x) = \begin{pmatrix} 0 & 0 & T_{13}(x_1, x_2) \\ 0 & 0 & T_{23}(x_1, x_2) \\ T_{13}(x_1, x_2) & T_{23}(x_1, x_2) & 0 \end{pmatrix}. \tag{1}$$

Equivalent reformulation-simply connected domain

- Find $U : \Omega \to \mathbb{R}$ - the Airy stress function such that

$$T_{13} = \frac{1}{\sqrt{2}} U_{x_2} \quad \text{and} \quad T_{23} = -\frac{1}{\sqrt{2}} U_{x_1}.$$

$\Longrightarrow$ $\operatorname{div} \mathbf{T} = \mathbf{0}$ is fulfilled.

Equivalent reformulation-simply connected domain

- Find $U : \Omega \to \mathbb{R}$ - the Airy stress function such that

$$T_{13} = \frac{1}{\sqrt{2}} U_{x_2} \quad \text{and} \quad T_{23} = -\frac{1}{\sqrt{2}} U_{x_1}.$$

$\implies \operatorname{div} \mathbf{T} = \mathbf{0}$ is fulfilled.

- U must satisfy $(\varepsilon(\boldsymbol{u}) = \frac{\mathbf{T}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}})$

$$\operatorname{div}\left(\frac{\nabla U}{(1+|\nabla U|^a)^{\frac{1}{a}}}\right) = 0 \qquad \text{in } \Omega,$$

$$U_{x_2}\boldsymbol{n}_1 - U_{x_1}\boldsymbol{n}_2 = \sqrt{2}g \qquad \text{on } \partial\Omega.$$

Equivalent reformulation-simply connected domain

- Find $U : \Omega \to \mathbb{R}$ - the Airy stress function such that

$$T_{13} = \frac{1}{\sqrt{2}} U_{x_2} \quad \text{and} \quad T_{23} = -\frac{1}{\sqrt{2}} U_{x_1}.$$

$\implies \operatorname{div} \mathbf{T} = \mathbf{0}$ is fulfilled.

- U must satisfy $(\varepsilon(\boldsymbol{u}) = \frac{\mathbf{T}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}})$

$$\operatorname{div}\left(\frac{\nabla U}{(1+|\nabla U|^a)^{\frac{1}{a}}}\right) = 0 \qquad \text{in } \Omega,$$

$$U_{x_2} \boldsymbol{n}_1 - U_{x_1} \boldsymbol{n}_2 = \sqrt{2} g \qquad \text{on } \partial\Omega.$$

- Dirichlet problem, indeed assume that $\partial\Omega$ is parameterized by $\gamma(s) = (\gamma_1(s), \gamma_2(s))$. Then

$$U(\gamma(s_0)) = a_0 + \sqrt{2} \int_0^{s_0} g(\gamma(s)) \sqrt{(\gamma_1'(s))^2 + (\gamma_2'(s))^2} ds =: U_0(x).$$

Consequences for U

- We look for $U \in W^{1,1}(\Omega)$

$$\operatorname{div}\left(\frac{\nabla U}{(1+|\nabla U|^a)^{\frac{1}{a}}}\right) = 0 \quad \text{in } \Omega, \qquad U = U_0 \quad \text{on } \partial\Omega.$$

Consequences for U

- We look for $U \in W^{1,1}(\Omega)$
$$\operatorname{div}\left(\frac{\nabla U}{(1+|\nabla U|^a)^{\frac{1}{a}}}\right) = 0 \quad \text{in } \Omega, \qquad U = U_0 \quad \text{on } \partial\Omega.$$
- It is equivalent to find $U \in W^{1,1}(\Omega)$ such that $U = U_0$ on $\partial\Omega$ and
$$\int_\Omega F^*(\nabla U) \le \int_\Omega F^*(\nabla V).$$
- In general does not exists - relaxed formulation: fixed $\Omega \subset\subset \Omega_0$ and find $U \in BV(\Omega_0)$ such that $U = U_0$ in $\Omega_0 \setminus \overline{\Omega}$ and
$$\int_\Omega F^*((\nabla U)^r) + |\nabla U^s|(\overline{\Omega}) \le \int_\Omega F^*((\nabla V)^r) + |\nabla V^s|(\overline{\Omega}).$$

Consequences for U

- We look for $U \in W^{1,1}(\Omega)$

$$\operatorname{div}\left(\frac{\nabla U}{(1+|\nabla U|^a)^{\frac{1}{a}}}\right) = 0 \quad \text{in } \Omega, \qquad U = U_0 \quad \text{on } \partial\Omega.$$

- It is equivalent to find $U \in W^{1,1}(\Omega)$ such that $U = U_0$ on $\partial\Omega$ and

$$\int_\Omega F^*(\nabla U) \le \int_\Omega F^*(\nabla V).$$

- In general does not exists - relaxed formulation: fixed $\Omega \subset\subset \Omega_0$ and find $U \in BV(\Omega_0)$ such that $U = U_0$ in $\Omega_0 \setminus \overline{\Omega}$ and

$$\int_\Omega F^*((\nabla U)^r) + |\nabla U^s|(\overline{\Omega}) \le \int_\Omega F^*((\nabla V)^r) + |\nabla V^s|(\overline{\Omega}).$$

- We have the same result as before:(But consider $a = 2$ then we know that $(\nabla U)^s$ is supported only on $\partial\Omega$ and we have "half"-relaxed formulation: Find $u \in W^{1,1}(\Omega)$ such that

$$\int_\Omega \sqrt{1+|\nabla U|^2} + \int_{\partial\Omega} |U - U_0| \le \int_\Omega \sqrt{1+|\nabla V|^2} + \int_{\partial\Omega} |V - U_0|.$$

Consequences for U II

- $a = 2$ - the minimal surface equation, you know everything that means you know nothing in general:

Consequences for U II

- $a = 2$ - the minimal surface equation, you know everything that means you know nothing in general: for convex domains and smooth data the classical solution exists, for non-convex domains the weak solution does not exist in general, the proper function space is BV, the trace is not attained

Consequences for U II

- $a = 2$ - the minimal surface equation, you know everything that means you know nothing in general: for convex domains and smooth data the classical solution exists, for non-convex domains the weak solution does not exist in general, the proper function space is BV, the trace is not attained
- $a = 2$ what does it say for "physics"?

Consequences for U II

- $a = 2$ - the minimal surface equation, you know everything that means you know nothing in general: for convex domains and smooth data the classical solution exists, for non-convex domains the weak solution does not exist in general, the proper function space is BV, the trace is not attained
- $a = 2$ what does it say for "physics"? the solution $\mathbf{T}$ must be of the prescribed form due to the uniqueness, g cannot be prescribed arbitrarily to get the weak solution, if g attains some critical value something very "bad" happens - either the model is not valid (there is not deformation for large g) or the body is no more continuum

Consequences for U II

- $a = 2$ - the minimal surface equation, you know everything that means you know nothing in general: for convex domains and smooth data the classical solution exists, for non-convex domains the weak solution does not exist in general, the proper function space is BV, the trace is not attained
- $a = 2$ what does it say for "physics"? the solution $\mathbf{T}$ must be of the prescribed form due to the uniqueness, g cannot be prescribed arbitrarily to get the weak solution, if g attains some critical value something very "bad" happens - either the model is not valid (there is not deformation for large g) or the body is no more continuum
- $a \neq 2$ we cannot use all the geometrical machinery, but on convex domains we can prove $|\nabla U| \leq C$

Consequences for U II

- $a = 2$ - the minimal surface equation, you know everything that means you know nothing in general: for convex domains and smooth data the classical solution exists, for non-convex domains the weak solution does not exist in general, the proper function space is BV, the trace is not attained
- $a = 2$ what does it say for “physics”? the solution $\mathbf{T}$ must be of the prescribed form due to the uniqueness, g cannot be prescribed arbitrarily to get the weak solution, if g attains some critical value something very “bad” happens - either the model is not valid (there is not deformation for large g) or the body is no more continuum
- $a \neq 2$ we cannot use all the geometrical machinery, but on convex domains we can prove $|\nabla U| \leq C$
- $a < 2$ we can localize and prove $\nabla U \in L^\infty_{loc}$

Consequences for U II

- $a = 2$ - the minimal surface equation, you know everything that means you know nothing in general: for convex domains and smooth data the classical solution exists, for non-convex domains the weak solution does not exist in general, the proper function space is BV, the trace is not attained
- $a = 2$ what does it say for "physics"? the solution $\mathbf{T}$ must be of the prescribed form due to the uniqueness, g cannot be prescribed arbitrarily to get the weak solution, if g attains some critical value something very "bad" happens - either the model is not valid (there is not deformation for large g) or the body is no more continuum
- $a \neq 2$ we cannot use all the geometrical machinery, but on convex domains we can prove $|\nabla U| \leq C$
- $a < 2$ we can localize and prove $\nabla U \in L^{\infty}_{loc}$
- $a \in (1, 2)$ the weak solution may not exists eg. for $\Omega = B_2 \setminus B_1$
- on the flat part of the boundary, one can extend the solution outside

Consequences for solution in general case/geometry

- Bildhauer & Fuchs (2001–): General theory for $a \in (0,2]$ there exists $u \in W^{1,1}(\Omega)$

$$\int_\Omega F^*(\nabla U) + \int_{\partial\Omega} |U - U_0| \le \int_\Omega F^*(\nabla V) + \int_{\partial\Omega} |V - U_0|.$$

i.e., smoothness locally in Ω, the trace may not be attained; for convex domains everything is nice up to the boundary

Consequences for solution in general case/geometry

- Bildhauer & Fuchs (2001–): General theory for $a \in (0, 2]$ there exists $u \in W^{1,1}(\Omega)$

$$\int_\Omega F^*(\nabla U) + \int_{\partial\Omega} |U - U_0| \leq \int_\Omega F^*(\nabla V) + \int_{\partial\Omega} |V - U_0|.$$

  i.e., smoothness locally in Ω, the trace may not be attained; for convex domains everything is nice up to the boundary

- We cannot solve the problem in general for the Neumann data - counterexamples
- Maybe we can avoid to be **T** measure in the interior of Ω at last for some a's

Consequences for solution in general case/geometry

- Bildhauer & Fuchs (2001–): General theory for $a \in (0,2]$ there exists $u \in W^{1,1}(\Omega)$
$$\int_\Omega F^*(\nabla U) + \int_{\partial\Omega} |U - U_0| \le \int_\Omega F^*(\nabla V) + \int_{\partial\Omega} |V - U_0|.$$
i.e., smoothness locally in Ω, the trace may not be attained; for convex domains everything is nice up to the boundary
- We cannot solve the problem in general for the Neumann data - counterexamples
- Maybe we can avoid to be **T** measure in the interior of Ω at last for some a's
- Maybe for $a \in (0,1)$ the theory can be built up to the boundary

Consequences for solution in general case/geometry

- Bildhauer & Fuchs (2001–): General theory for $a \in (0,2]$ there exists $u \in W^{1,1}(\Omega)$

$$\int_\Omega F^*(\nabla U) + \int_{\partial\Omega} |U - U_0| \le \int_\Omega F^*(\nabla V) + \int_{\partial\Omega} |V - U_0|.$$

  i.e., smoothness locally in Ω, the trace may not be attained; for convex domains everything is nice up to the boundary

- We cannot solve the problem in general for the Neumann data - counterexamples
- Maybe we can avoid to be **T** measure in the interior of Ω at last for some a's
- Maybe for $a \in (0,1)$ the theory can be built up to the boundary
- Maybe the Dirichlet problem is easier to handle - we do not need the estimates up to the boundary

Consequences for solution in general case/geometry

- Bildhauer & Fuchs (2001–): General theory for $a \in (0,2]$ there exists $u \in W^{1,1}(\Omega)$
$$\int_\Omega F^*(\nabla U) + \int_{\partial\Omega} |U - U_0| \le \int_\Omega F^*(\nabla V) + \int_{\partial\Omega} |V - U_0|.$$
i.e., smoothness locally in Ω, the trace may not be attained; for convex domains everything is nice up to the boundary
- We cannot solve the problem in general for the Neumann data - counterexamples
- Maybe we can avoid to be **T** measure in the interior of Ω at last for some a's
- Maybe for $a \in (0,1)$ the theory can be built up to the boundary
- Maybe the Dirichlet problem is easier to handle - we do not need the estimates up to the boundary
- But in all cases we need to face the problem with symmetric gradient contrary to the full gradient as in Bildhauer & Fuchs
- Is really the assumption $a \le 2$ essential? Counterexamples only for non-smooth data

Limiting strain - anti-plane stress geometry

Theorem (anti-plane stress)

Let U_0 be arbitrary. Then there exists unique weak solution U provided that one of the following holds.

- Ω *is uniformly convex,* $a > 0$ *is arbitrary and* U_0 *smooth.*
- $a \in (0,2)$ *and* $\partial\Omega = \bigcup_{i=1}^{N} \Gamma_i$ *such that either* Γ_i *is uniformly convex and* U_0 *is smooth on* Γ_i *or* Γ_i *is flat and* U_0 *is constant there.*
- $a \in (0,1]$, Ω *arbitrary piece-wise* $\mathcal{C}^{1,1}$ *and* U_0 *piece-wise in* $\mathcal{C}^{1,1}$. *Moreover, if* U_0 *and* Ω *smooth then* U *is* $\mathcal{C}^{1,\alpha}(\overline{\Omega})$.

Limiting strain - anti-plane stress geometry

Theorem (anti-plane stress)

Let U_0 be arbitrary. Then there exists unique weak solution U provided that one of the following holds.

- *Ω is uniformly convex, $a > 0$ is arbitrary and U_0 smooth.*
- *$a \in (0,2)$ and $\partial\Omega = \bigcup_{i=1}^{N} \Gamma_i$ such that either Γ_i is uniformly convex and U_0 is smooth on Γ_i or Γ_i is flat and U_0 is constant there.*
- *$a \in (0,1]$, Ω arbitrary piece-wise $\mathcal{C}^{1,1}$ and U_0 piece-wise in $\mathcal{C}^{1,1}$. Moreover, if U_0 and Ω smooth then U is $\mathcal{C}^{1,\alpha}(\overline{\Omega})$.*

Theorem (anti-plane stress II)

Let $a \in (0,2]$, U_0 and $\Omega \subset \mathbb{R}^d$ be arbitrary. Then there exists unique weak solution $U \in W^{1,1}(\Omega)$ in the following sense

$$\int_\Omega F(\nabla U) + \int_{\partial\Omega} |U - U_0| \le \int_\Omega F(\nabla V) + \int_{\partial\Omega} |V - U_0| \qquad \forall V \in W^{1,1}(\Omega).$$

Limiting strain - anti-plane stress geometry

Theorem (anti-plane stress)

Let U_0 be arbitrary. Then there exists unique weak solution U provided that one of the following holds.

- *Ω is uniformly convex, $a > 0$ is arbitrary and U_0 smooth.*
- *$a \in (0,2)$ and $\partial\Omega = \bigcup_{i=1}^N \Gamma_i$ such that either Γ_i is uniformly convex and U_0 is smooth on Γ_i or Γ_i is flat and U_0 is constant there.*
- *$a \in (0,1]$, Ω arbitrary piece-wise $\mathcal{C}^{1,1}$ and U_0 piece-wise in $\mathcal{C}^{1,1}$. Moreover, if U_0 and Ω smooth then U is $\mathcal{C}^{1,\alpha}(\overline{\Omega})$.*

Theorem (anti-plane stress II)

Let $a \in (0,2]$, U_0 and $\Omega \subset \mathbb{R}^d$ be arbitrary. Then there exists unique weak solution $U \in W^{1,1}(\Omega)$ in the following sense

$$\int_\Omega F(\nabla U) + \int_{\partial\Omega} |U - U_0| \le \int_\Omega F(\nabla V) + \int_{\partial\Omega} |V - U_0| \qquad \forall V \in W^{1,1}(\Omega).$$

Defining $\mathbf{T}_{13} := U_{x_2}$ and $\mathbf{T}_{23} := -U_{x_1}$ we have $\operatorname{div} \mathbf{T} = 0$ but $\mathbf{T}\boldsymbol{n} = \boldsymbol{g}$ is not attained but we have "best approximation".

General result

Theorem (Beck, Bulíček, Maringová)

Let $F \in \mathcal{C}^2(0,\infty)$ *be increasing strictly convex fulfilling*

$$\lim_{s\to\infty} \frac{F(s)}{s} = \lim_{s\to\infty} F'(s) = K > 0.$$

Then the following is equivalent

- *For any* $\Omega \in \mathcal{C}^{1,1}$ *and any* $u_0 \in \mathcal{C}^{1,1}(\overline{\Omega})$ *there exists unique* $u \in W^{1,\infty}(\Omega)$ *fulfilling*

$$\boxed{\int_\Omega F(|\nabla u|) \le \int_\Omega F(|\nabla u_0 + \nabla\varphi|) \qquad \text{for all } \varphi \in W_0^{1,1}(\Omega).}$$

General result

Theorem (Beck, Bulíček, Maringová)

Let $F \in \mathcal{C}^2(0,\infty)$ be increasing strictly convex fulfilling

$$\lim_{s\to\infty} \frac{F(s)}{s} = \lim_{s\to\infty} F'(s) = K > 0.$$

Then the following is equivalent

- *For any $\Omega \in \mathcal{C}^{1,1}$ and any $u_0 \in \mathcal{C}^{1,1}(\overline{\Omega})$ there exists unique $u \in W^{1,\infty}(\Omega)$ fulfilling*

$$\boxed{\int_\Omega F(|\nabla u|) \le \int_\Omega F(|\nabla u_0 + \nabla \varphi|) \qquad \textit{for all } \varphi \in W_0^{1,1}(\Omega).}$$

- $$\int_1^\infty sF''(s) = \infty.$$

General result

Theorem (Beck, Bulíček, Maringová)

Let $F \in \mathcal{C}^2(0,\infty)$ be increasing strictly convex fulfilling

$$\lim_{s\to\infty} \frac{F(s)}{s} = \lim_{s\to\infty} F'(s) = K > 0.$$

Then the following is equivalent

- *For any $\Omega \in \mathcal{C}^{1,1}$ and any $u_0 \in \mathcal{C}^{1,1}(\overline{\Omega})$ there exists unique $u \in W^{1,\infty}(\Omega)$ fulfilling*

$$\boxed{\int_\Omega F(|\nabla u|) \le \int_\Omega F(|\nabla u_0 + \nabla\varphi|) \qquad \textit{for all } \varphi \in W_0^{1,1}(\Omega).}$$

- $$\int_1^\infty sF''(s) = \infty.$$

The second condition is equivalent to the fact that

$$\lim_{s\to K_-} F^*(s) = \infty.$$

Result for particular model and general geometry

Consider $\boldsymbol{\varepsilon}^*(\mathbf{T}) = \mathbf{T}/(1+|\mathbf{T}|^a)^{\frac{1}{a}}$:

Theorem (General result for $a > 0$)

Let $a > 0$ and $\boldsymbol{u}_0$ satisfy the safety strain condition. Then there exists a unique triple $(\boldsymbol{u}, \mathbf{T}, \tilde{\boldsymbol{g}}) \in \mathcal{V} \times L^1(\Omega)^{d\times d}_{sym} \times (\mathcal{C}^1_0(\Gamma_N))^*$ such that for all $\boldsymbol{v} \in \mathcal{C}^1_{\Gamma_D}(\overline{\Omega})$

$$\varepsilon(\boldsymbol{u}) = \varepsilon^*(\mathbf{T})$$

$$\int_\Omega \mathbf{T} \cdot \varepsilon(\boldsymbol{u}-\boldsymbol{w}) \le \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{u}-\boldsymbol{w}) + \int_{\Gamma_N} \boldsymbol{g} \cdot (\boldsymbol{u}-\boldsymbol{w})$$

$$\boldsymbol{u} = \boldsymbol{u}_0 \ on \ \Gamma_D,$$

where $\boldsymbol{w} \in \mathcal{V}$ is arbitrary such that there exists $\tilde{\mathbf{T}} \in L^1$ fulfilling $\varepsilon(\boldsymbol{w}) = \varepsilon^*(\tilde{\mathbf{T}})$.

Result for particular model and general geometry

Consider $\boldsymbol{\varepsilon}^*(\mathbf{T}) = \mathbf{T}/(1+|\mathbf{T}|^a)^{\frac{1}{a}}$:

Theorem (General result for $a > 0$)

Let $a > 0$ and $\boldsymbol{u}_0$ satisfy the safety strain condition. Then there exists a unique triple $(\boldsymbol{u}, \mathbf{T}, \tilde{\boldsymbol{g}}) \in \mathcal{V} \times L^1(\Omega)^{d\times d}_{sym} \times (\mathcal{C}^1_0(\Gamma_N))^*$ such that for all $\boldsymbol{v} \in \mathcal{C}^1_{\Gamma_D}(\overline{\Omega})$

$$\varepsilon(\boldsymbol{u}) = \varepsilon^*(\mathbf{T})$$

$$\int_\Omega \mathbf{T} \cdot \varepsilon(\boldsymbol{u}-\boldsymbol{w}) \le \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{u}-\boldsymbol{w}) + \int_{\Gamma_N} \boldsymbol{g} \cdot (\boldsymbol{u}-\boldsymbol{w})$$

$$\boldsymbol{u} = \boldsymbol{u}_0 \ on\ \Gamma_D,$$

where $\boldsymbol{w} \in \mathcal{V}$ is arbitrary such that there exists $\tilde{\mathbf{T}} \in L^1$ fulfilling $\varepsilon(\boldsymbol{w}) = \varepsilon^*(\tilde{\mathbf{T}})$.Moreover,

$$\boxed{\int_\Omega \mathbf{T} \cdot \nabla \boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} + \langle \boldsymbol{g} - \tilde{\boldsymbol{g}}, \boldsymbol{v} \rangle_{\Gamma_N}}$$

Assumptions for general model

Assumptions on ε^*: Denote $\mathbf{A}(\mathbf{T}) := \frac{\partial \boldsymbol{\varepsilon}^*(\mathbf{T})}{\partial \mathbf{T}}$.

- ε^* is coercive, i.e.,
$$\boldsymbol{\varepsilon}^*(\mathbf{T}) \cdot \mathbf{T} \geq C_1|\mathbf{T}| - C_2$$
- $\boldsymbol{\varepsilon}^*$ is h-elliptic, i.e., there exists nonincreasing function h such that for all $\mathbf{W} \neq 0$
$$0 < h(|\mathbf{T}|)|\mathbf{W}|^2 \leq (\mathbf{W}, \mathbf{W})_{\mathbf{A}(\mathbf{T})} \leq \frac{|\mathbf{W}|^2}{1 + |\mathbf{T}|},$$
where
$$(\mathbf{W}, \mathbf{W})_{\mathbf{A}(\mathbf{T})} := \sum \mathbf{A}^{\nu i}_{\mu j}(\mathbf{T})\mathbf{W}^{\nu i}\mathbf{W}^{\mu j}, \qquad \mathbf{A}^{\nu i}_{\mu j}(\mathbf{T}) := \frac{\partial (\boldsymbol{\varepsilon}^*)^{\nu i}(\mathbf{T})}{\partial \mathbf{T}^{\mu j}}.$$
- $\mathbf{A}$ is asymptotically symmetric, i.e.,
$$\frac{|\mathbf{A}^s(\mathbf{T}) - \mathbf{A}(\mathbf{T})|^2}{h(|\mathbf{T}|)} \leq \frac{C_2}{1 + |\mathbf{T}|}.$$
- either h does not decrease faster than $|\mathbf{T}|^{-1-2/d}$ or $\boldsymbol{\varepsilon}^*$ is asymptotically Uhlenbeck, i.e., there exists a function g such that $g(|\mathbf{T}|) \leq C(1 + |\mathbf{T}|)$ fulfilling
$$\frac{|g(|\mathbf{T}|)\varepsilon^*(\mathbf{T}) - \mathbf{T}|^2}{h(|\mathbf{T}|)} \leq C_2(1 + |\mathbf{T}|^3).$$

Assumptions for general models

Assumptions on data:

- $\boldsymbol{f} \in L^2$
- $\boldsymbol{g} \in L^1$
- $\boldsymbol{u}_0$ satisfies safety strain condition, i.e., there exists a compact set $K \subset \varepsilon^*(\mathbb{R}^{d\times d}_{sym})$ such that for almost all $x \in \Omega$

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_0(x)) \in K$$

.

Result for limiting strain models

Theorem (General result)

There exists a unique triple $(\boldsymbol{u}, \mathbf{T}, \tilde{\boldsymbol{g}}) \in W^{1,1}(\Omega)^d \times L^1(\Omega)^{d\times d}_{sym} \times (\mathcal{C}^1_0(\Gamma_d))^*$ *such that for all* $\boldsymbol{v} \in \mathcal{C}^1_{\Gamma_D}(\overline{\Omega})$ *and all*

$$\int_\Omega \mathbf{T}\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} + \langle \boldsymbol{g} - \tilde{\boldsymbol{g}}, \boldsymbol{v}\rangle_{\Gamma_N}$$

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \mathbf{D}(\mathbf{T}) \in L^\infty(\Omega; \mathbb{R}^{d\times d})$$

$$\int_\Omega \mathbf{T}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{w}) \le \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{u}-\boldsymbol{w}) + \int_{\Gamma_N} \boldsymbol{g}\cdot(\boldsymbol{u}-\boldsymbol{w})$$

$$\boldsymbol{u} = \boldsymbol{u}_0 \ \ on\ \Gamma_D,$$

where $\boldsymbol{w} \in W^{1,\infty}(\Omega)$ *is arbitrary function being equal to* $\boldsymbol{u}_0$ *on* Γ_D *such that there exists* $\tilde{\mathbf{T}} \in L^1(\Omega)^{d\times d}_{sym}$ *fulfilling* $\boldsymbol{\varepsilon}(\boldsymbol{w}) = \boldsymbol{\varepsilon}^*(\tilde{\mathbf{T}})$*.*

Conclusion II

- The first result for the symmetric gradient, where the Uhlenbeck setting plays the crucial role
- The same result obviously holds also for the full gradient case
- For any $\mathcal{C}^1$ strictly monotone operator being asymptotically symmetric and Uhlenbeck we avoided the presence of the singular part in the interior!
- At least in 2D and a simply connected domains, we can convert this setting to the minimal surface-like problems and get the same result. Improvement of the known results in a significant way!
- The method does not use the improved integrability result (which even may not be true)!
- The same theory for minimal surface-like problems and general geometries. Sharp identification of the cases when the theory can be built up to the boundary.

Scheme of the proof

We find a mollified problem for which we have a solution and then go to the limit. The approximation is of the form

$$\varepsilon_n^*(\mathbf{T}) := \varepsilon^*(\mathbf{T}) + n^{-1}\frac{\mathbf{T}}{(1+|\mathbf{T}|)^{1-\frac{1}{n}}}.$$

- The first a priori estimate

$$\int_\Omega |\mathbf{T}^n| \le C, \qquad \|\varepsilon(\boldsymbol{u}^n)\|_n \le C.$$

- $$\begin{aligned} \mathbf{T}^n &\rightharpoonup^* \overline{\mathbf{T}} && \text{in } \mathcal{M}(\overline{\Omega})^{d\times d}_{sym}, \\ \varepsilon(\boldsymbol{u}^n) &\rightharpoonup \varepsilon(\boldsymbol{u}) && \text{in } L^q(\Omega)^{d\times d}_{sym}, \text{ for all } q<\infty. \end{aligned}$$

  and $\overline{\mathbf{T}}$ solves the equation but we do not know that $\varepsilon(\boldsymbol{u}) = \varepsilon^*(\overline{\mathbf{T}})$

Scheme

- First we show that

$$\mathbf{T}^n \to \mathbf{T} \qquad \text{a.e. in } \Omega,$$

  where $\mathbf{T} \in L^1(\Omega)^{d\times d}_{sym}$ but we do not know that $\mathbf{T} = \overline{\mathbf{T}}$.
- Then due to the continuity of ε^* we have

$$\varepsilon(\boldsymbol{u}) = \varepsilon^*(\mathbf{T}) \text{ a.e. in } \Omega.$$

- Fatou lemma and monotonicity justifies the limit passage in

$$\int_\Omega \mathbf{T} \cdot \varepsilon(\boldsymbol{u} - \boldsymbol{w}) \le \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{u} - \boldsymbol{w}) + \int_{\Gamma_N} \boldsymbol{g} \cdot (\boldsymbol{u} - \boldsymbol{w})$$

- the final step is to show that

$$\boxed{-\operatorname{div} \mathbf{T} = \boldsymbol{f}}$$