Limiting small strain problems with cracks

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Motivation

- In contrast to the linearized model, even when the strains are "small", e.g. metallic alloys response nonlinearly [Rajagopal (2014)]
- The boundedness, respectively smallness, of strains is required a-priori, within limiting strain model ensured by suitable nonlinear response function
- Since strains are constrained, they are complained by singular stresses within measure spaces [Beck, Buliček, Málek, Süli (2017), Buliček, Málek, Rajagopal, Süli (2014)]
- For cracks, contact conditions are reasoned physically and mathematically [Khludnev, Kovtunenko (2000)]
- While boundary tractions are problematic, contact conditions formulated in displacements are suitable for limiting strain within nonlinear elastic model [Itou, Kovtunenko, Rajagopal (2017a, 2017b)]

Outline

1. Nonlinear response function
   - General mathematical properties
   - Generic response functions
2. Governing equations and inequalities
   - Domain with crack
   - Non-penetration condition
   - Hierarchy of formulations
3. Well-posedness theorems
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   - Generalized variational formulation
   - Weak variational formulation
   - Anti-plane problem with non-penetration
4. Conclusion

Nonlinear response function

For an abstract response function given by a nonlinear map:

$$\mathcal{F}:\ \mathrm{Sym}(\mathbb{R}^{d\times d}) \mapsto \mathrm{Sym}(\mathbb{R}^{d\times d}), \quad \mathcal{F}(0)=0 \tag{1}$$

let constant $M_1, M_2, M_4 > 0$ and $M_3 \geq 0$ exist such that $\mathcal{F}$ is

uniform bounded:
$$\|\mathcal{F}(\sigma)\| \leq M_1 \tag{2}$$

monotone:
$$\big(\mathcal{F}(\sigma)-\mathcal{F}(\bar\sigma)\big):(\sigma-\bar\sigma)\geq 0 \tag{3}$$

Lipschitz-continuous:
$$(\mathcal{F}(\sigma)-\mathcal{F}(\bar\sigma)):(\sigma-\bar\sigma)\leq M_2\|\sigma-\bar\sigma\|^2 \tag{4}$$

semi-coercive:
$$-M_3+M_4\sum_{i,j=1}^{d}|\sigma_{ij}|\leq \mathcal{F}(\sigma):\sigma \tag{5}$$

Response function-1

Let $\Psi_1 : \mathbb{R} \mapsto \mathbb{R}$, $\Psi_1(0) = 0$ and $\Psi_2 : \mathbb{R}_+ \mapsto \mathbb{R}$ be continuous a.e. differentiable functions, constants $a_1, a_2, b_1, b_2, b_4 > 0$ and $a_3, b_3 \geq 0$ exist such that

$$|\Psi_1(y)| \leq a_1, \quad 0 \leq \Psi_1'(y) \leq a_2, \quad -a_3 \leq y\Psi_1(y) \text{ a.e. } y \in \mathbb{R} \tag{6}$$

$$y|\Psi_2(y)| \leq b_1, \quad y\big|\Psi_2'(y)\big| + \Psi_2(y) \leq b_2, \quad -b_3 + b_4 y \leq y^2 \Psi_2(y) \text{ a.e. } y \in \mathbb{R}_+ \tag{7}$$

and the two cases allowing switching hold in subintervals of $\mathbb{R}_+$:

$$\text{either } \Psi_2'(y) \geq 0,\ \Psi_2(y) \geq 0 \quad \text{or } \Psi_2'(y) < 0,\ (y\Psi_2(y))' \geq 0 \tag{8}$$

For the nonlinear response function $\mathcal{F}$ defined by

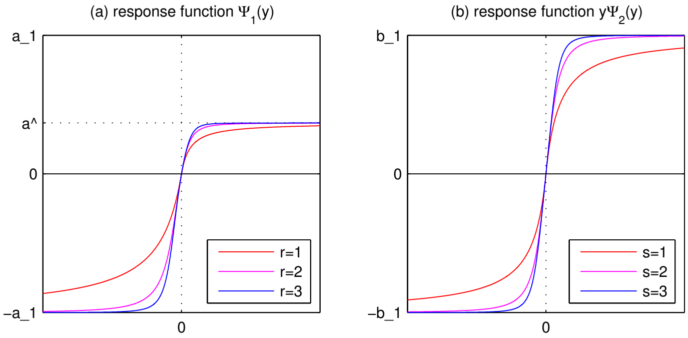
$$\mathcal{F}(\sigma) = \Psi_1(\mathrm{tr}(\sigma))I + \Psi_2(\|\sigma\|)\sigma, \tag{9}$$

the properties (2) are true with

$$M_1 = a_1\sqrt{d} + b_1, \quad M_2 = a_2 d + b_2, \quad M_3 = a_3 + b_3, \quad M_4 = \frac{b_4}{d} \tag{10}$$

Example response functions

$$\Psi_1(y) = \alpha\Big\{1 - \exp\Big(\frac{-\lambda y}{(1+\gamma|y|^r)^{1/r}}\Big)\Big\}, \quad \Psi_2(y) = \frac{\beta}{(1+\kappa y^s)^{1/s}} \qquad (11)$$



$a_1 = \alpha(e^{\frac{\lambda}{\gamma^{1/r}}} - 1)$, $a_2 = \alpha\lambda e^{\frac{\lambda}{\gamma^{1/r}}}$, $a_3 = 0$, $b_1 = \frac{\beta}{\kappa^{1/s}}$, $b_2 = 2\beta$, $b_3 = \frac{b_4}{\kappa^{1/s}}$, $b_4 = \frac{\beta}{c_s\kappa^{1/s}}$ where $c_s = 2^{1/s-1}$ for $s \in (0,1)$ and $c_s = 1$ for $s \geq 1$

Response function-2

Let $\widetilde{\Psi}_1$ and Ψ_2 satisfy (6)–(8), and constant $a_4 > 0$ exist such that

$$-a_3 + a_4|y| \leq y\widetilde{\Psi}_1(y) \quad \text{a.e. } y \in \mathbb{R} \tag{12}$$

For the deviatoric stress tensor $\sigma^{\text{dev}} := \sigma - \frac{\text{tr}(\sigma)}{d} I$ such that $\text{tr}(\sigma^{\text{dev}}) = 0$ and the nonlinear response function $\widetilde{\mathcal{F}}$ defined by

$$\widetilde{\mathcal{F}}(\sigma) = \widetilde{\Psi}_1(\text{tr}(\sigma))I + \Psi_2(\|\sigma^{\text{dev}}\|)\sigma^{\text{dev}} \tag{13}$$

- the properties (2) hold with M_1, M_2, M_3 given in (10) and $M_4 = \min\{a_4, \frac{b_4}{d}\}$

$$\text{For example,} \quad \widetilde{\Psi}_1(y) = \frac{\alpha y}{(1+\gamma|y|^r)^{1/r}}, \quad \alpha, \gamma, r > 0 \tag{14}$$

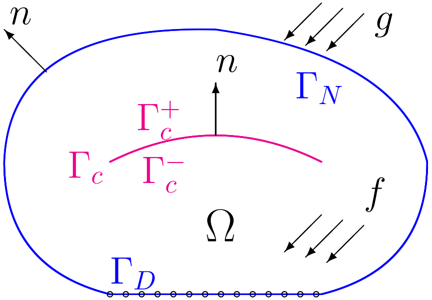
satisfies (6) and (12) with $a_1 = \frac{\alpha}{\gamma^{1/r}}$, $a_2 = \alpha$, $a_3 = \frac{a_4}{\gamma^{1/r}}$, $a_4 = \frac{\alpha}{c_r \gamma^{1/r}}$

- In the limit $\kappa \searrow 0^+$ in (11) and $\gamma \searrow 0^+$ in (14), from (13) we arrive at

$$\widetilde{\mathcal{F}}(\sigma) = \alpha \text{tr}(\sigma)I + \beta\sigma^* = (\alpha - \tfrac{\beta}{d})\text{tr}(\sigma)I + \beta\sigma,$$

which turns into the linearized elasticity in 3d when $\alpha = \frac{1-2\nu}{3E}$ and $\beta = \frac{1+\nu}{E}$

Governing equations

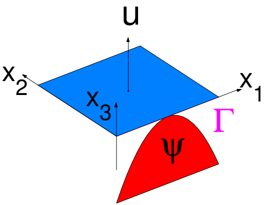


A reference domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$, consisted of the Dirichlet Γ_D and the Neumann Γ_N parts with the normal vector n, which contains crack $\Gamma_c \subset \Omega$ with opposite crack faces $\Gamma_c^{\pm}$

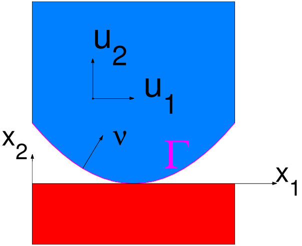
The domain with crack $\Omega_c := \Omega \setminus \overline{\Gamma}_c$ with boundary $\partial\Omega \cup \Gamma_c^+ \cup \Gamma_c^-$

Non-penetration condition

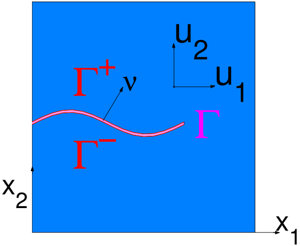
contacting obstacle: | Signorini's contact: | contacting crack faces:



$u \geq \psi$



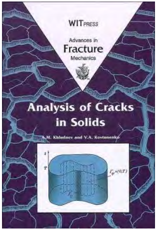
$u \cdot \nu \geq \psi$



Non-penetration condition for the jump:

$$[\![u \cdot \nu]\!] := u|_{\Gamma^+} \cdot \nu - u|_{\Gamma^-} \cdot \nu \geq 0$$

[Khludnev, Kovtunenko (2000)]



Boundary-value problem

Find:

vector of the displacement $u(t,x) = (u_1, \dots, u_d)$

tensor of the Cauchy–Green strain $\varepsilon(t,x) \in \mathrm{Sym}(\mathbb{R}^{d \times d})$

tensor of the Cauchy stress $\sigma(t,x) \in \mathrm{Sym}(\mathbb{R}^{d \times d})$

satisfying the boundary-value problem in Ω_c:

equilibrium equation:
$$-\mathrm{div}\sigma = f \tag{15}$$

constitutive equation:
$$\varepsilon(u) = \mathcal{F}(\sigma) \tag{16}$$

linearized strain:
$$\varepsilon_{ij}(u) := \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) \tag{17}$$

for $i, j = 1, \dots, d$

Boundary and non-penetration conditions

under boundary and non-penetration conditions:

$$\text{Dirichlet boundary condition on } \Gamma_D: \quad u = 0 \tag{18}$$

$$\text{Neumann boundary condition on } \Gamma_N: \quad \sigma \cdot n = g \tag{19}$$

non-penetration conditions on the crack Γ_c:

$$\begin{aligned} &\sigma \cdot n - \big((\sigma \cdot n) \cdot n\big)n = 0, \quad [\![(\sigma \cdot n) \cdot n]\!] = 0 \\ &[\![u \cdot n]\!] \geq 0, \quad (\sigma \cdot n) \cdot n \leq 0, \quad \big((\sigma \cdot n) \cdot n\big)[\![u \cdot n]\!] = 0 \end{aligned} \tag{20a}$$

for the given body force $f(x) = (f_1, \ldots, f_d)$
and the boundary traction $g(x) = (g_1, \ldots, g_d)$

Hierarchy of formulations

Weak variational formulation

$$[\![u\cdot n]\!]\ge 0,\quad \int_{\Omega_c}\sigma:\varepsilon(\overline{u}-u)\,dx \ge \int_{\Omega_c} f\cdot(\overline{u}-u)\,dx+\int_{\Gamma_N} g\cdot(\overline{u}-u)\,dS_x$$

$$\mathcal{F}(\sigma)=\varepsilon(u)$$

Generalized variational formulation

$$[\![u\cdot n]\!]\ge 0,\quad \int_{\Omega_c}\sigma:\varepsilon(\overline{u})\,dx \ge \int_{\Omega_c} f\cdot\overline{u}\,dx+\int_{\Gamma_N} g\cdot\overline{u}\,dS_x$$

$$\int_{\Omega}(\sigma-\overline{\sigma}):\mathcal{F}(\overline{\sigma})\,dx+\int_{\Omega_c}\varepsilon(u):\overline{\sigma}\,dx\le\int_{\Omega_c} f\cdot u\,dx+\int_{\Gamma_N} g\cdot u\,dS_x$$

Elliptic regularization and penalization

$$\int_{\Omega_c}\big(\delta\varepsilon(u^\delta)+\sigma^\delta\big):\varepsilon(\overline{u})\,dx+\frac{1}{\delta}\int_{\Gamma_c}[\![u^\delta\cdot n]\!]^-[\![\overline{u}\cdot n]\!]\,dS_x=\int_{\Omega_c} f\cdot\overline{u}\,dx+\int_{\Gamma_N} g\cdot\overline{u}\,dS_x$$

$$\delta\sigma^\delta+\mathcal{F}(\sigma^\delta)=\varepsilon(u^\delta)$$

Well-posedness theorems

In order to express the right-hand side $f \in L^2(\Omega_c;\mathbb{R}^d)$ and $g \in L^2(\Gamma_N;\mathbb{R}^d)$, an auxiliary linearized elasticity problem: Find

the displacement $u^E \in H^1(\Omega_c;\mathbb{R}^d)$ such that $u^E = 0$ on Γ_D,
the strain $\varepsilon(u^E) \in L^2(\Omega_c;\mathrm{Sym}(\mathbb{R}^{d\times d}))$,
and the stress $\sigma^E \in L^2(\Omega_c;\mathrm{Sym}(\mathbb{R}^{d\times d}))$ such that

$$\int_{\Omega_c} \sigma^E : \varepsilon(\overline{u})\, dx = \int_{\Omega_c} f\cdot\overline{u}\, dx + \int_{\Gamma_N} g\cdot\overline{u}\, dS_x \tag{21a}$$

$$\sigma^E = \varepsilon(u^E) \tag{21b}$$

for all test functions $\overline{u} \in H^1(\Omega_c;\mathbb{R}^d)$ such that $\overline{u} = 0$ on Γ_D

Elliptic regularization and penalization

For a small parameter $\delta > 0$, the regularized and penalized problem: Find

the displacement $u^\delta \in H^1(\Omega_c; \mathbb{R}^d)$ such that $u^\delta = 0$ on Γ_D,
the strain $\varepsilon(u^\delta) \in L^2(\Omega_c; \mathrm{Sym}(\mathbb{R}^{d\times d}))$,
and the stress $\sigma^\delta \in L^2(\Omega_c; \mathrm{Sym}(\mathbb{R}^{d\times d}))$ such that

$$\int_{\Omega_c} \big(\delta\varepsilon(u^\delta) + \sigma^\delta - \sigma^E\big) : \varepsilon(\overline{u})\, dx + \frac{1}{\delta}\int_{\Gamma_c} [\![u^\delta \cdot n]\!]^- [\![\overline{u} \cdot n]\!]\, dS_x = 0 \tag{22a}$$

$$\delta\sigma^\delta + \mathcal{F}(\sigma^\delta) = \varepsilon(u^\delta) \tag{22b}$$

for all test functions $\overline{u} \in H^1(\Omega_c; \mathbb{R}^d)$ such that $\overline{u} = 0$ at Γ_D

Theorem (well-posedness of regularized and penalized problem)

For δ fixed, there exists the unique solution $(u^\delta, \varepsilon(u^\delta), \sigma^\delta)$ to the regularized and penalized problem (22) which satisfies a-priori estimates uniformly in $\delta \in (0, \delta_0)$:

$$\delta c_{\mathrm{KP}} \|u^\delta\|^2_{H^1(\Omega_c)} + \frac{\delta}{2}\|\sigma^\delta\|^2_{L^2(\Omega_c)} + M_4\|\sigma^\delta\|_{L^1(\Omega_c)} + \frac{1}{\delta}\|[\![u^\delta \cdot n]\!]^-\|^2_{L^2(\Gamma_c)} \leq M_3|\Omega_c| + \frac{\delta_0}{2}\|\sigma^E\|^2_{L^2(\Omega_c)} + M_1\|\sigma^E\|_{L^1(\Omega_c)} =: c_{RHS} \tag{23a}$$

$$c_{\mathrm{KP}}\|u^\delta\|^2_{H^1(\Omega_c)} \leq \|\varepsilon(u^\delta)\|^2_{L^2(\Omega_c)} \leq 2M_1^2|\Omega_c| + 4\delta_0 c_{RHS} \tag{23b}$$

where the linearized elastic stress tensor σ^E is from (21),
and constant $c_{\mathrm{KP}} > 0$ from the Korn–Poincaré inequality:

$$c_{\mathrm{KP}}\|u\|^2_{H^1(\Omega_c)} \leq \int_{\Omega_c} \|\varepsilon(u)\|^2\, dx \quad \text{for } u \in H^1(\Omega_c; \mathbb{R}^d) \text{ such that } u = 0 \text{ on } \Gamma_D$$

Generalized variational formulation

For the space of bounded measures $\mathcal{M}^1(\Omega_c)$ which is dual to the space $C_c(\Omega_c)$ of continuous functions with compact support in Ω_c,
the generalized variational formulation of the problem:

Find the displacement $u \in H^1(\Omega_c; \mathbb{R}^d)$ such that $u = 0$ on Γ_D,
the strain $\varepsilon(u) \in L^2(\Omega_c; \mathrm{Sym}(\mathbb{R}^{d\times d}))$,
and the stress $\sigma \in \mathcal{M}^1(\Omega_c; \mathrm{Sym}(\mathbb{R}^{d\times d}))$ such that

$$[\![u \cdot n]\!] \geq 0 \quad \text{on } \Gamma_c \tag{24a}$$

$$\langle \sigma : \varepsilon(\overline{u}) \rangle_{\Omega_c} \geq \int_{\Omega_c} \sigma^E : \varepsilon(\overline{u})\, dx \tag{24b}$$

$$\langle (\sigma - \overline{\sigma}) : \mathcal{F}(\overline{\sigma}) \rangle_{\Omega_c} + \int_{\Omega_c} \varepsilon(u) : \overline{\sigma}\, dx \leq \int_{\Omega_c} \sigma^E : \varepsilon(\overline{u})\, dx \tag{24c}$$

for all test functions $\overline{u} \in H^1(\Omega_c; \mathbb{R}^d)$ such that $[\![\overline{u} \cdot n]\!] \geq 0$ on Γ_c, $\overline{u} = 0$ at Γ_D,
and $\varepsilon(\overline{u}), \overline{\sigma} \in C_c(\Omega_c; \mathrm{Sym}(\mathbb{R}^{d\times d}))$

Well-posedness of generalized formulation

Theorem (well-posedness of generalized formulation)

(i) *As* $\delta \searrow 0^+$, *there exists an accumulation point* $(u, \varepsilon(u), \sigma)$ *of the solutions* $(u^\delta, \varepsilon(u^\delta), \sigma^\delta)$ *of the regularized and penalized problem* (22) *which solves the generalized variational problem* (24).

(ii) *If the stress is regular such that* $\sigma \in L^p(\Omega_c; \mathrm{Sym}(\mathbb{R}^{d\times d})$ *with* $p \in (1, \infty)$, *then*

- *the triple* $(u, \varepsilon(u), \sigma)$ *satisfies the weak variational formulation* (26)
- *the strain* $\varepsilon(u) \in L^\infty(\Omega_c; \mathrm{Sym}(\mathbb{R}^{d\times d}))$ *due to* (16) *and* (26c)
- *the a-priori estimate holds*

$$M_4 \|\sigma\|_{L^1(\Omega_c)} \leq M_3 |\Omega_c| + M_1 \|\sigma^E\|_{L^1(\Omega_c)} \quad (25)$$

- *If monotony* (3) *is strict, then the stress is unique*

The weak variational formulation of the problem:

Find the displacement $u \in H^1(\Omega_c; \mathbb{R}^d)$ such that $u = 0$ on Γ_D,
the strain $\varepsilon(u) \in L^\infty(\Omega_c; \mathrm{Sym}(\mathbb{R}^{d\times d}))$,
and the stress $\sigma \in L^p(\Omega_c; \mathrm{Sym}(\mathbb{R}^{d\times d}))$, $p > 1$, such that

$$[\![u \cdot n]\!] \geq 0 \quad \text{on } \Gamma_c \tag{26a}$$

$$\int_{\Omega_c} (\sigma - \sigma^E) : \varepsilon(\overline{u} - u)\, dx \geq 0 \tag{26b}$$

$$\mathcal{F}(\sigma) = \varepsilon(u) \tag{26c}$$

for all test functions $\overline{u} \in H^1(\Omega_c; \mathbb{R}^d)$ such that $[\![\overline{u} \cdot n]\!] \geq 0$ on Γ_c, $\overline{u} = 0$ at Γ_D,
and $\varepsilon(\overline{u}) \in L^\infty(\Omega_c; \mathrm{Sym}(\mathbb{R}^{d\times d}))$

Under the anti-plane strain assumption:

$$u(x_1, x_2) = (0, 0, u_3), \quad \varepsilon(u) = \begin{bmatrix} 0 & 0 & \frac{u_{3,1}}{2} \\ 0 & 0 & \frac{u_{3,2}}{2} \\ \frac{u_{3,1}}{2} & \frac{u_{3,2}}{2} & 0 \end{bmatrix} \quad \sigma = \begin{bmatrix} 0 & 0 & \sigma_{13} \\ 0 & 0 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & 0 \end{bmatrix}$$

for the inclined normal vector to crack Γ_c: $\quad n = \frac{1}{\sqrt{n_1^2+n_2^2+1}}(n_1, n_2, 1)$

Find the vertical displacement $u_3 \in H^1(\Omega_c; \mathbb{R})$ such that $u_3 = 0$ on Γ_D,
the gradient $\nabla u_3 \in L^\infty(\Omega_c; \mathbb{R}^2)$,
and the stress pair $(\sigma_{13}, \sigma_{23}) \in L^p(\Omega_c; \mathbb{R}^2)$, $p > 1$, such that

$$[\![u_3]\!] \geq 0 \quad \text{on } \Gamma_c \tag{27a}$$

$$\frac{1}{2}\int_{\Omega_c} (\sigma_{13}, \sigma_{23}) \cdot \nabla(\overline{u} - u_3)\, dx \geq \int_{\Omega_c} f_3(\overline{u} - u_3)\, dx + \int_{\Gamma_N} g_3(\overline{u} - u_3)\, dS_x \tag{27b}$$

$$\frac{\beta}{[1 + \kappa(2\sigma_{13}^2 + 2\sigma_{23}^2)^{s/2}]^{1/s}}(\sigma_{13}, \sigma_{23}) = \frac{1}{2}\nabla u_3 \quad (\mathcal{F}(\sigma) = \Psi_2(\|\sigma\|)\sigma) \tag{27c}$$

for $\overline{u} \in H^1(\Omega_c; \mathbb{R})$ such that $[\![\overline{u}]\!] \geq 0$ on Γ_c, $\overline{u} = 0$ at Γ_D, and $\nabla\overline{u} \in L^\infty(\Omega_c; \mathbb{R}^2)$

Conclusion

- Limiting strain models provide regularity, boundedness, smallness of strain
- Stress is defined by bounded measures
- Cracks are admissible within the modeling
- Contact conditions are suitable on the boundary/crack
- Elliptic regularization and penalization provide existence of generalized solution as an accumulation point
- If stress is smooth, then the generalized solution turns into weak solution
- We extend to nonlinear viscoelastic model for crack with stress-free faces [Itou, Kovtunenko, Rajagopal (2017c)]

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