Existence of strong solutions to rate independent systems

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The model case

Let $u : [0, T] \times \Omega \to \mathbb{R}^N$, where Ω is a space domain, with C^1-boundary. We consider *purely dissipative* systems

$$\boxed{\begin{aligned} \alpha \frac{\dot{u}}{|\dot{u}|} - DW_0(u) + \mu \Delta u = f \qquad & \text{in } [0, T] \times \Omega, \\ u = 0 \qquad & \text{on } \partial\Omega, \\ u(0) = u_0 \qquad & \text{in } \Omega. \end{aligned}}$$

Here, DW_0 represents the impact of the (elastic) energy.
The term $\mu \Delta u$ is a regularizer, (damping term).
Important is the rate independent phenomenon. Scaling of the time variable, **does not** change the PDE.

One basic example.

Consider the 1-D example, $W_0(u) = \frac{(t-u)^2}{2}$ which represents a linear spring potential.
We assume α, to be the friction.
Then $u(t) = (t - \alpha)_+$ is a solution to

$$\alpha \frac{\dot{u}}{|\dot{u}|} - DW_0(u) \ni 0 \text{ and } u(0) = 0,$$

meaning, that $DW_0(u) = t - u = t \in [0, \alpha]$ for $t \in [0, \alpha]$
$DW_0(u) = \alpha$, for $t \geq \alpha$.
Consequently, solutions are expected to be at most Lipschitz in time direction.

Collected regularity, vector-valued.

Consider $u : \Omega \to \mathbb{R}^m$,

$$\alpha \frac{\dot{u}}{|\dot{u}|} - DW_0(u) + \mu \Delta u = f$$

1. **Expected regularity in time direction.** In the vector valued case only $b = 1$ possible. Therefore, $\|\nabla \dot{u}\|_{L^a(L^2)} \leq c\|\dot{f}\|_{L^a(L^2)}$.
2. **Expected space regularity:**
   Since $\frac{\dot{u}}{|\dot{u}|} \in L^\infty(L^\infty)$, we can use stationary theory:
   $\|\nabla^2 u(t)\|_{L^p} \leq c\|\mu\Delta(u(t)) + DW_0(u(t))\|_{L^p} \leq C(\|f(t)\|_{L^p}) + C$,
   in case DW_0 holds according assumptions.
3. **Combining time and space estimates.** If $\dot{f} \in L^a(L^2)$, we can expect, $u \in L^\infty(H^2)$ and $\dot{u} \in L^a(H^1_0)$.
   Moreover, $u \in C^\alpha([0, T] \times \Omega)$.

There is hope, that the non-linearity $DW_0(u)$, can be established, for a weakly converging sequence of discrete in time solutions.

The variational formulation.

We will introduce a **rate-independent dissipation pseudo potential.**
$\mathcal{R} : \mathbb{R}^m \to [0, \infty)$ 1-homogenous and convex.
Then we look for a solution to the following sub-differential inequality

$$\mathcal{R}(\dot{u}(t)) + \langle \mathcal{L}u(t) - \mathrm{D}\mathcal{W}_0(u(t)) + f(t), \xi(t) - \dot{u}(t)\rangle \leq \mathcal{R}(\xi(t))$$

for all $\xi \in L^2(H^1_0)$.
In other words, u satisfies

$$\begin{aligned} \partial\mathcal{R}(\dot{u}(t)) &\ni \mathcal{L}u(t) - \mathrm{D}\mathcal{W}_0(u(t)) + f \quad \text{in } [0, T] \times \Omega, \\ u(t)|_{\partial\Omega} &= 0 \quad \text{for } t \in [0, T], \\ u(0) &= u_0. \end{aligned}$$

$\mathcal{L}$ is an elliptic operator, with continuous coefficients.
On $D\mathcal{W}_0(u) = \int_\Omega DW_0(u)\,\mathrm{d}x$ we assume, $|\mathrm{D}W_0(v)| \leq C(1 + |v|^{q-1})$ and $-\nu|v - z|^2 \leq (\mathrm{D}W_0(v) - \mathrm{D}W_0(z)) \cdot (v - z)$, with ν small.

The existence theorem

Theorem (Rindler, Sch., Süli '17)

For $f \in \mathrm{W}^{1,a}(0,T;\mathrm{L}^p(\Omega;\mathbb{R}^m))$ for $a \in (1,\infty)$, $p \in [2,\infty)$ there exists a solution, such that

$$\mathcal{R}(\dot{u}(t)) + \langle \mathcal{L}u(t) - \mathrm{D}\mathcal{W}_0(u(t)) + f(t), \xi(t) - \dot{u}(t)\rangle \leq \mathcal{R}(\xi(t))$$

The solution has the following regularity.

(i) $\nabla^2 u \in \mathrm{L}^\infty(0,T;\mathrm{L}^p(\Omega;\mathbb{R}^m))$,

(ii) $\nabla \dot{u} \in \mathrm{L}^a(0,T;\mathrm{L}^2(\Omega;\mathbb{R}^m))$,

(iii) $u \in \mathrm{C}^{0,\gamma}([0,T] \times \overline{\Omega};\mathbb{R}^m)$ *for some* $\gamma \in (0,1)$,

(iv) *If* $p > d$, *then* $\nabla u \in \mathrm{C}^{0,\zeta}([0,T] \times \overline{\Omega};\mathbb{R}^m)$ *for some* $\zeta \in (0,1)$.

Important assumption:

$-\nu|v - z|^2 \leq (\mathrm{D}W_0(v) - \mathrm{D}W_0(z)) \cdot (v - z)$, with ν small.

The uniqueness result

Theorem

The strong solution is unique among all strong solutions.
Moreover, let u be the strong solution constructed above and another solution

$$v \in \mathrm{L}^1(0,T;(\mathrm{W}^{1,2}_0 \cap \mathrm{L}^q)(\Omega;\mathbb{R}^m)) \qquad \textit{with} \qquad \dot{v} \in \mathrm{L}^1(0,T;\mathrm{W}^{1,1}_0(\Omega;\mathbb{R}^m)),$$

satisfies $v(0) = u_0$.
Then, $v = u$.

- Strong solution exists **uniquely** as long as the functional is convex.
- Weak solutions are not unique **before** leaving the convex regime.

The Rothe method

For simplicity let $\mathcal{L} \equiv \mu\Delta$ and $\mathcal{R}(q) = \int_\Omega |q| \,\mathrm{d}x$.
Implicit time-stepping.

$$0 = t_0^N < t_1^N < \cdots < t_N^N = T, \qquad \text{where} \qquad t_k^N - t_{k-1}^N = \frac{T}{N}, \quad N \in \mathbb{N},$$

and look for corresponding discrete-time approximations $(u_k^N)_{k=0,\ldots,N} \subset (\mathrm{W}_0^{1,2} \cap \mathrm{L}^q)(\Omega;\mathbb{R}^m)$, that minimize

$$\mathcal{F}_k^N(v) := \int_\Omega |v - u_{k-1}^N| \,\mathrm{d}x + \int_\Omega \frac{\mu}{2}|\nabla v|^2 + W_0(v) - f_k^N \cdot v \,\mathrm{d}x,$$

then for any $\xi \in (\mathrm{W}_0^{1,2} \cap \mathrm{L}^q)(\Omega;\mathbb{R}^m)$ we have

$$0 \le \frac{\mathcal{F}_k^N(u_k^N + h(u_{k-1}^N - u_k^N - \xi)) - \mathcal{F}_k^N(u_k^N)}{h}.$$

The Rothe Method: Establishing the sub differential.

Hence

$$\int_\Omega |u_k^N + h(u_{k-1}^N - u_k^N - \xi) - u_{k-1}^N| \,\mathrm{d}x - \int_\Omega |u_k^N - u_{k-1}^N| \,\mathrm{d}x$$
$$= \int_\Omega |-h\xi + (1-h)(u_k^N - u_{k-1}^N)| - |u_k^N - u_{k-1}^N| \,\mathrm{d}x$$
$$\leq h \int_\Omega |\xi| - |u_k^N - u_{k-1}^N| \,\mathrm{d}x.$$

This implies the sub-differential inequality

$$\int_\Omega |u_k^N - u_{k-1}^N| \,\mathrm{d}x + \langle \Delta u_k^N - DW_0(u_k^N) + f, \xi - (u_k^N - u_{k-1}^N)\rangle \leq \int_\Omega |\xi| \,\mathrm{d}x$$

and by devidng by h and defining $\delta_h^N = \frac{u_k^N - u_{k-1}^N}{h}$ we gain

$$\int_\Omega |\delta_k^N| \,\mathrm{d}x + \langle \nabla u_k^N, \nabla(\delta_k^N - \xi)\rangle + \langle DW_0(u_k^N) - f, \delta_k^N - \xi\rangle \leq \int_\Omega |\xi| \,\mathrm{d}x$$

Numeric result

Theorem

There is a sequence of a fully discrete numerical solutions $(u_h^\tau)_{h>0,\tau>0}$*, on a quasiuniform family of triangulations of* Ω *parametrized by the spatial discretization parameter* h*, and with time step* τ*. And*

- *it converges to the unique strong solution* u*, with a rate*

$$\int_0^T \|u_\tau^h - u\|_{\mathrm{W}^{1,2}}^2 \,\mathrm{d}t \leq C\left(h + \tau^{\min\{1,a-1\}}\right),$$

  where $a \in (1,\infty]$ *is the assumed integrability exponent of* $\dot{f}$ *in time.*
- *it satisfies for some* $\tilde{\beta} \in (0,1)$*, satisfies, uniformly in* h, τ*,*

$$[u_\tau^h]_{\mathrm{C}^{\tilde{\beta}}([0,T]\times\Omega)} + \|\nabla u_\tau^h\|_{\mathrm{L}^\infty(\mathrm{L}^6(\Omega))} + \|\nabla \dot{u}_\tau^h\|_{\mathrm{L}^a(\mathrm{L}^2(\Omega))} \leq C.$$

Outlook

1. Behavior in non-convex regimes:
   - Jump behavior
   - Uniqueness
   - Regularity of the set of non-convexity
2. Hyperbolic approach: $\lambda^2 \ddot{u} + \alpha \frac{\dot{u}}{|\dot{u}|} + DW_0(u) - \mu \Delta u = f$. Existence and Regularity (for non-convex potentials) and the quasi static limit $\lambda \to 0$ (in preparation).
3. Different rates. $\alpha |\dot{u}|^{a-1} \dot{u} + DW_0(u) - \mu \Delta u = f$, for $a \in [0, 1]$. Helps with rough f and non-convexity. Stability of the limit of $a \to 0$.
4. Limit problems of the purely elastic case. The limit $\mu \to 0$ of $\alpha |\dot{u}|^{a-1} \dot{u} + DW_0(u) - \mu \Delta u = f$.
5. Combining with other PDE in the context of damage, crack propagation, elasto-plasticity, avalanches.
6. Development of numerical schemes refining around jump-regions.