ADAPTIVE ERROR CONTROL FOR FINITE ELEMENT APPROXIMATIONS OF THE LIFT AND DRAG COEFFICIENTS IN VISCOUS FLOW

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ABSTRACT. We derive estimates for the error in a variational approximation of the lift and drag coefficients of a body immersed into a viscous flow governed by the Navier-Stokes equations. The variational approximation is based on computing a certain weighted average of a finite element approximation to the solution of the Navier-Stokes equations. Our main result is an *a posteriori* estimate that puts a bound on the error in the lift and drag coefficients in terms of the local mesh size, a local residual quantity, and a local weight describing the local stability properties of an associated linear dual problem. The weight may be approximated by solving the dual problem numerically. The error bound is thus computable and can be used for quantitative error estimation; we apply it to design an adaptive finite element algorithm specifically for the approximation of the lift and drag coefficients.

1. INTRODUCTION

Often, the purpose of numerical computations in mathematical modelling of physical phenomena is to approximate a functional of the analytical solution to a differential equation, represented as an integral average of the solution, rather than to obtain accurate pointwise values of the solution itself; in such instances solving the differential equation considered is only an intermediate stage in the process of computing the primary quantities of concern. For example, in fluid dynamics one may be concerned with calculating the lift and drag coefficients of a body immersed into a viscous incompressible fluid whose flow is governed by the Navier-Stokes equations. The lift and drag coefficients are defined as integrals, over the boundary of the body, of the stress tensor components normal and tangential to the flow, respectively. Similarly, in elasticity theory, the quantities of prime interest, such as the stress intensity factor, or the moments of a shell or a plate, are derived quantities.

The objective of this paper is to obtain *a posteriori* error bounds for finite element approximations of functionals that arise in fluid dynamics; specifically, we shall be concerned with the construction and the *a posteriori* error analysis of finite element approximations

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to the lift and drag coefficients in a viscous incompressible flow, and the implementation of these bounds into an adaptive finite element algorithm, with reliable and efficient quantitative control of the error.

It is frequently the case that the functional under consideration may be expressed in various forms which are mutually equivalent at the continuous level but result in very different approximations under discretisation. Thus it is important to select the appropriate representation of the functional before formulating its discretisation. While this basic idea has been widely exploited in structural mechanics, see [1, 2, 3] and [4], and heat conduction, see [17], in post-processing finite element approximations, it does not seem to have penetrated the field of computational fluid dynamics.

We approximate the solution $\hat{u} = (u, p)$ of the Navier-Stokes equations, where u represents the velocity of the fluid and p its pressure, by means of a finite element method using piecewise polynomials of degree k for u and degree $k-1$ for p. Let $N_\psi(\hat{u})$ denote the boundary integral representing the lift or drag coefficient, depending on the choice of the function ψ defined on the boundary of the computational domain; the precise definition of $N_\psi(\hat{u})$ will be given in Section 3. Using the differential equation we may express $N_\psi(\hat{u})$ in a variational form, involving test functions that are equal to ψ on the boundary of the domain; this form will be the basis for constructing an accurate approximation $N_\psi^h(\hat{u}_h)$ to $N_\psi(\hat{u})$. Provided the boundary is sufficiently smooth and there are no variational crimes, the order of convergence of $N_\psi^h(\hat{u}_h)$ to $N_\psi(\hat{u})$ is $2k$, while for the naive direct approximation $N_\psi(\hat{u}_h)$ the order of convergence is typically only k.

Our main result is a weighted *a posteriori* estimate for the error $N_\psi(\hat{u}) - N_\psi^h(\hat{u}_h)$ of the form

$$|N_\psi(\hat{u}) - N_\psi^h(\hat{u}_h)| \leq c \sum_{\tau \in \mathcal{T}_h} h_\tau^\alpha \mathcal{R}_\tau(\hat{u}_h) \omega_{\tau,\alpha},$$

where the sum is taken over the elements τ in a triangulation $\mathcal{T}_h$ of the computational domain, h_τ is the diameter of τ; given that α is a real number with $1 \leq \alpha \leq k+1$, $\mathcal{R}_\tau(\hat{u}_h)$ is an element residual quantity, $\omega_{\tau,\alpha}$ is a local weight, and $c = c(\psi)$ is a positive constant. The element residual quantity $\mathcal{R}_\tau(\hat{u}_h)$ is a computable bound on the actual residual obtained by inserting the approximate solution into the differential equation. The derivation of the estimate is based on a representation formula for the error $N_\psi(\hat{u}) - N_\psi^h(\hat{u}_h)$ in terms of the computed solution $\hat{u}_h$ and the solution to an associated linear dual equation with homogeneous right-hand side and non-homogeneous boundary data ψ. The weight $\omega_{\tau,\alpha}$ describes the local size of derivatives of order α of the solution to the dual problem. The data for the dual problem is known, and therefore the weight can be computed by solving the dual problem numerically. Therefore, the right-hand side of the estimate is computable and can be applied to design an adaptive finite element algorithm for approximating $N_\psi(\hat{u})$, and to ensure efficient quantitative control of the error $N_\psi(\hat{u}) - N_\psi^h(\hat{u}_h)$. The fact, stated above, that the order of convergence of $N_\psi^h(\hat{u}_h)$ is $2k$ follows from the error representation formula together with an *a priori* estimate of the global error $\hat{u} - \hat{u}_h$ in the finite element method. Indeed, for a linear elliptic model problem, it can be shown, using results obtained in [13], that the *a posteriori* estimate is sharp.

In [6] a weighted *a posteriori* estimate is given for the error in the lift and the drag; however, unlike our approach, it is based on a direct approximation using $N_\psi(\hat{u}_h)$. For *a posteriori* estimates of the error $\hat{u} - \hat{u}_h$ in finite element approximations of the solution to the Navier-Stokes equations in various norms, see [11], [12], [16], and references therein; a general introduction to *a posteriori* error analysis is given in [8]. Finally we mention that an *a priori* error analysis of the approximation method, employed here, for a linear convection-diffusion equation, is presented in [5].

The remainder of this paper is organized as follows: in order to illustrate the key ideas, in Section 2 we outline the theory for a linear elliptic model problem. In Section 3 we consider the extension of this analysis to the Stokes system which models the flow of a viscous incompressible fluid at low velocities; we derive an *a posteriori* estimate of the error in a finite element approximation to the lift and drag coefficients. In Section 4 we further extend our results to the Navier-Stokes equations and present numerical computations for the drag coefficient of a cylinder in a channel; these illustrate the quality of the adaptive error control based on the *a posteriori* error bound. Finally, in Section 5, we summarise our results and draw some conclusions.

2. A model problem

2.1. **The model problem and the finite element method.** We begin by introducing some notation. Let Ω be a bounded domain in $\mathbf{R}^d$, $d = 2$ or 3, with Lipschitz-continuous boundary Γ. For an open set K in $\mathbf{R}^d$, let $L^2(K)$ signify the space of real-valued square-integrable functions on K, with norm $\|\cdot\|_K$; $H^s(K)$ will denote the Sobolev space of real index s, equipped with the norm $\|\cdot\|_{H^s(K)}$ and corresponding seminorm $|\cdot|_{H^s(K)}$; with a slight abuse of the notation, we shall frequently write $\|D^s u\|_K$ instead of $|u|_{H^s(K)}$, $s > 0$. Given that ψ is an element of $H^{1/2}(\Gamma)$, we let $H^1_\psi = H^1_\psi(\Omega)$ denote the space of all v in $H^1 = H^1(\Omega)$ which satisfy the Dirichlet boundary condition $v_{|\Gamma} = \psi$. Finally, we adopt the following notational convention: generic constants that are independent of the problem and the mesh size are denoted by c, while constants that depend on the problem but not on the mesh size are labelled C, with a subscript when necessary.

As a first model problem we consider the boundary-value problem

$$-\nabla \cdot \sigma(u) = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \Gamma, \tag{2.1}$$

where $f \in L^2 = L^2(\Omega)$ and $\sigma(u) = A\nabla u$, with A a uniformly positive definite $d \times d$ matrix, with continuous real-valued entries defined on $\overline{\Omega}$. This problem has a unique weak solution $u \in H^1_0$. In order to define the corresponding finite element approximation, we introduce a family of finite-dimensional spaces V^h contained in H^1, which consist of continuous piecewise polynomials of degree k defined on a triangulation $\mathcal{T}_h$ of Ω. We denote the diameter of a triangle $\tau \in \mathcal{T}_h$ by h_τ. It will always be assumed that the triangulation is shape-regular, i.e., there exists a positive constant c such that $\mathrm{vol}(\tau) \geq c\, h_\tau^d$, where $\mathrm{vol}(\tau)$ is the $d-$dimensional volume of τ. Further, for each function $\psi \in H^{1/2}(\Gamma)$ such that $\psi = v|_\Gamma$ for some $v \in V^h$, we let $V^h_\psi \subset V^h$ be the space of all $w \in V^h$ with $w|_\Gamma = \psi$. In particular, V^h_0 is the space of all $v \in V^h$ which vanish on the boundary Γ.

The finite element approximation of (2.1) reads: find $u_h \in V_0^h$ such that

$$a(u_h, v) = (f, v) \qquad \text{for all } v \in V_0^h. \tag{2.2}$$

Here and below $(\cdot,\cdot)$ denotes the scalar product in L^2 and $a(v,w) = (\sigma(v), \nabla w) = (\nabla v, A^T \nabla w)$, for $v, w \in H^1$, where A^T is the transpose of A.

2.2. **Approximation of the boundary flux.** In this section we shall construct a finite element approximation to the boundary flux

$$N(u) = \int_\Gamma n \cdot \sigma(u)\, ds,$$

where n denotes the unit outward normal vector to Γ. In order to do so, for $u \in H_0^1$ denoting the weak solution to problem (2.1) and $\psi \in H^{1/2}(\Gamma)$, we consider the weighted boundary flux, defined by

$$N_\psi(u) = \int_\Gamma n \cdot \sigma(u)\, \psi\, ds. \tag{2.3}$$

We note that since $\sigma(u) \in [L^2(\Omega)]^d$ and $\nabla \cdot \sigma(u) \in L^2(\Omega)$, according to the trace theorem (see Theorem 2.2 in [10]), $n \cdot \sigma(u)|_\Gamma$ is correctly defined as an element of $H^{-1/2}(\Gamma)$, and $N_\psi(u)$ is meaningful, provided that the integral over Γ is interpreted as a duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. Moreover, applying a generalisation of Green's Identity (see Theorem 2.2 in [10]), we deduce that, for any $v \in H_\psi^1$,

$$N_\psi(u) = (\sigma(u), \nabla v) - (f, v) = a(u, v) - (f, v). \tag{2.4}$$

Clearly, the value of the expression $a(u,v) - (f,v)$ on the right-hand side is independent of the choice of $v \in H_\psi^1$. From now on, for the sake of simplicity, we shall always assume that $\psi = v|_\Gamma$ for some $v \in V^h$; in other words, ψ will be supposed to be a continuous piecewice polynomial of degree k defined on Γ. This assumption will be satisfied in our applications. Motivated by the identity (2.4), we define the approximation $N_\psi^h(u_h)$ to $N_\psi(u)$ as follows:

$$N_\psi^h(u_h) = a(u_h, v) - (f, v), \quad v \in V_\psi^h. \tag{2.5}$$

We note that, because of (2.2), $N_\psi^h(u_h)$ is independent of the choice of $v \in V_\psi^h$. Furthermore, we observe that, in general,

$$N_\psi^h(u_h) \neq \int_\Gamma n \cdot \sigma(u_h)\, \psi\, ds = N_\psi(u_h),$$

in contrast with identity (2.4) satisfied by the analytical solution u. However, we shall show below that $N_\psi^h(u_h)$ is the appropriate approximation for $N_\psi(u)$, rather than $N_\psi(u_h)$.

2.3. **Error representation using duality.** In order to derive a representation formula for the error $N_\psi(u) - N_\psi^h(u_h)$ in the boundary flux, we introduce the following dual problem in variational form: find $\phi \in H_\psi^1$ such that

$$a(v, \phi) = 0 \qquad \text{for all } v \in H_0^1. \tag{2.6}$$

Consider the global error $e = u - u_h$. Setting $v = e$ in (2.6) we obtain

$$0 = a(e, \phi) = a(e, \phi - \pi\phi) + a(e, \pi\phi),$$

where we made use of the fact that the error e is zero on the boundary Γ; here $\pi : H_\psi^1 \to V_\psi^h$ is a linear operator satisfying the approximation property (2.8) below. Since the definitions of $N_\psi(u)$ and $N_\psi^h(u_h)$ are independent of the choice of $v \in H_\psi^1$ and $v \in V_\psi^h$, respectively, we deduce that

$$a(e, \pi\phi) = \big(a(u, \pi\phi) - (f, \pi\phi)\big) - \big(a(u_h, \pi\phi) - (f, \pi\phi)\big) = N_\psi(u) - N_\psi^h(u_h).$$

Thus, we arrive at the error representation formula

$$N_\psi(u) - N_\psi^h(u_h) = a(e, \pi\phi - \phi) = a(u_h, \phi - \pi\phi) - (f, \phi - \pi\phi) \tag{2.7}$$

where, to obtain the last equality, we exploited the fact that u is the weak solution of (2.1) and that $\phi - \pi\phi$ belongs to H_0^1.

2.4. **The *a posteriori* and *a priori* error estimates.** In this section we introduce a residual quantity associated with the approximate solution u_h and derive a weighted *a posteriori* estimate for the error in the approximation to the boundary flux. We then investigate the order of convergence of the approximation through an *a priori* error analysis.

Two basic hypotheses will be required. First, we assume the following local approximation property: there is a linear operator $\pi : H_\psi^1 \to V_\psi^h$ such that

$$\|v - \pi v\|_\tau + h_\tau \|D(v - \pi v)\|_\tau \le c h_\tau^s \|D^s v\|_\tau, \quad 1 \le s \le k+1, \tag{2.8}$$

for each $\tau \in \mathcal{T}_h$, and c a positive constant, independent of τ, v and h. We note here that, without altering the basic error analysis that will follow, the right-hand side in this inequality can be replaced by $c h_{S(\tau)}^s \|D^s v\|_{S(\tau)}$, where $S(\tau)$ is the union of all elements in the partition whose closure has non-empty intersection with the closure of τ, and

$$h_{S(\tau)} = \max_{\sigma \in \mathcal{T}_h;\, \sigma \subset S(\tau)} h_\sigma;$$

an approximation property of this kind would arise if one took πv to be the quasi-interpolant of v (see, for example, [7]). For the sake of notational simplicity, we shall refrain from doing this and assume (2.8) instead. Second, we make an assumption concerning the regularity of the dual problem: there is a $t \ge 1$ such that for every s, $1 \le s \le t$, there is a constant C_s such that the solution ϕ to the dual problem (2.6) satisfies the following estimate:

$$\|D^s \phi\|_\Omega \le C_s \|D^{s-1/2} \psi\|_\Gamma, \tag{2.9}$$

whenever $\psi \in H^{s-1/2}(\Gamma)$. For instance, this bound holds when $\Gamma \in C^{s-1,1}$ and the entries of A belong to $C^{[s]}(\Omega)$, see [10]. For each triangle $\tau \in \mathcal{T}_h$ and u_h denoting the solution to (2.2) in V_0^h, we introduce the *residual quantity* $\mathcal{R}_\tau(u_h)$ by

$$\mathcal{R}_\tau(u_h) = \|\nabla \cdot \sigma(u_h) + f\|_\tau + h_\tau^{-1/2}\|[n \cdot \sigma(u_h)]/2\|_{\partial\tau\setminus\Gamma}, \tag{2.10}$$

where $[w]$ is the jump in w across the faces of elements in the partition, and n is the unit outward normal vector to $\partial\tau$.

Starting from the error representation (2.7), and estimating the right-hand side using the Cauchy-Schwarz inequality element-by-element together with the approximation property (2.8), we arrive the following theorem; its proof will be given in the next section.

Theorem 2.1. *Let (2.8) and (2.9) hold, and suppose that $\psi \in H^{\alpha-1/2}(\Gamma)$, $\alpha \geq 1$; then we have that*

$$|N_\psi(u) - N_\psi^h(u_h)| \leq c \sum_{\tau\in\mathcal{T}_h} h_\tau^\alpha \mathcal{R}_\tau(u_h)\omega_{\tau,\alpha}, \quad 1 \leq \alpha \leq \min(t, k+1),$$

where c is a constant, the local weight $\omega_{\tau,\alpha}$ is defined by $\omega_{\tau,\alpha} = \|D^\alpha\phi\|_\tau$, and ϕ is the weak solution of (2.6). Recall that the boundary flux $N_\psi(u)$ is defined in (2.3), the discrete approximation $N_\psi^h(u_h)$ in (2.5), and the residual quantity $\mathcal{R}_\tau(u_h)$ in (2.10).

Note, in particular, that the power α of $h = \max_\tau h_\tau$ is determined by the approximation properties of the finite element space, the parameter t in the regularity assumption (2.9), and the smoothness of the dual solution ϕ. To highlight the quality of the approximation $N_\psi^h(u_h)$ to $N_\psi(u)$, we state the following *a priori* error estimate; its proof is postponed until the next section.

Theorem 2.2. *Assume that (2.8) and (2.9) hold. Supposing that $\psi \in H^{\alpha-1/2}(\Gamma)$, $\alpha \geq 1$, and $u \in H^\beta$, $1 \leq \beta \leq k+1$, we have that*

$$|N_\psi(u) - N_\psi^h(u_h)| \leq ch^{\alpha+\beta-2}\|D^\beta u\|_\Omega \, \|D^\alpha\phi\|_\Omega, \quad 1 \leq \alpha \leq \min(t, k+1),$$

where ϕ is the weak solution of (2.6), c is a constant, and $h = \max_{\tau\in\mathcal{T}_h} h_\tau$.

This estimate shows that, for sufficiently smooth data, the order of convergence of $N_\psi^h(u_h)$ to $N_\psi(u)$ is $O(h^{2k})$. In general, this high order of convergence is not achieved for the naive approximation $\int_\Gamma n \cdot \sigma(u_h) ds$. At least on quasi-uniform meshes, the *a posteriori* bound stated in Theorem 2.1 can be proved to be sharp by applying the *a priori* residual estimate obtained in [13]; this shows that

$$\left(\sum_{\tau\in\mathcal{T}_h} \mathcal{R}_\tau^2(u_h)\right)^{1/2} = O(h^{k-1}),$$

and thus the right-hand side of the *a posteriori* estimate is $O(h^{2k})$, in agreement with Theorem 2.2. Furthermore, we observe that since the data ψ for the dual problem (2.6) is known, we may calculate the weight $\omega_{\tau,\alpha}$ by approximating the solution of the dual problem numerically. The right-hand side of the *a posteriori* error estimate is thus computable and can be used for direct quantitative error estimation.

2.5. **Proofs of Theorems 2.1 and 2.2.** In this section we present the proofs of Theorems 2.1 and 2.2.

Proof of Theorem 2.1. Starting from the error representation (2.7), and integrating triangle-by-triangle using Green's identity, we have

$$\begin{aligned} N_\psi(u) - N_\psi^h(u_h) &= (\sigma(u_h), \nabla(\phi - \pi\phi)) - (f, \phi - \pi\phi) \\ &= -\sum_{\tau\in\mathcal{T}_h} \int_\tau (\nabla\cdot\sigma(u_h) + f)(\phi - \pi\phi)dx \\ &\qquad + \sum_{\tau\in\mathcal{T}_h} \int_{\partial\tau\setminus\Gamma} ([n\cdot\sigma(u_h)]/2)(\phi - \pi\phi)ds \\ &= I + II, \end{aligned} \tag{2.11}$$

where we made use of the fact that ϕ, the weak solution of the dual problem (2.6) is in $C^{0,\alpha}(\Omega)$, for some α in $(0,1)$ (see Theorem 5.24 in [9]), so that $[\phi - \pi\phi] = 0$ across $\partial\tau \setminus \Gamma$ for each element τ in the partition. We now turn to estimating I and II. For I, we apply the Cauchy-Schwarz inequality and use the approximation property (2.8) to obtain

$$|I| \le c \sum_{\tau\in\mathcal{T}_h} h_\tau^s \|\nabla\cdot\sigma(u_h) + f\|_\tau \|D^s\phi\|_\tau.$$

Next we consider II. Applying the multiplicative trace inequality, see [7], followed by the approximation property (2.8) yields

$$\|\phi - \pi\phi\|_{\partial\tau} \le c h_\tau^{s-1/2} \|D^s\phi\|_\tau.$$

Thus,

$$|II| \le c \sum_{\tau\in\mathcal{T}_h} h_\tau^{s-1/2} \|[n\cdot\sigma(u_h)]\|_{\partial\tau\setminus\Gamma} \|D^s\phi\|_\tau.$$

Substituting the bounds on I and II into (2.11) and recalling the definition of the residual quantity (2.10), we deduce that

$$|N_\psi(u) - N_\psi^h(u_h)| \le c \sum_{\tau\in\mathcal{T}_h} h_\tau^s \mathcal{R}_\tau(u_h) \|D^s\phi\|_\tau,$$

which is the desired estimate. ☐

Proof of Theorem 2.2. It follows from the error representation formula (2.7) that

$$|N_\psi(u) - N_\psi^h(u^h)| \le c\|De\|_\Omega \|D(\phi - \pi\phi)\|_\Omega.$$

A standard energy norm error estimate gives

$$\|De\|_\Omega \le c h^{\beta-1} \|D^\beta u\|_\Omega, \quad 1 \le \beta \le k+1.$$

Further, using the approximation property (2.8), we obtain

$$|N_\psi(u) - N_\psi^h(u_h)| \le c h^{\alpha+\beta-2} \|D^\beta u\|_\Omega \|D^\alpha\phi\|_\Omega,$$

for $1 \le \alpha \le \min(k+1, t)$. ☐

3. The Stokes problem

We now turn to the approximation of the lift and drag coefficients for a flow governed by the stationary incompressible Navier-Stokes equations. First, we develop the theory for the Stokes problem, which is a linear elliptic system that approximates the Navier-Stokes equations at low velocities. In Section 4 we extend our results to the Navier-Stokes equations.

3.1. **The Stokes problem and its finite element approximation.** The Stokes problem, describing the stationary motion of a fluid in a bounded domain $\Omega \subset \mathbf{R}^d$, $d = 2, 3$, has the form: find $u : \Omega \to \mathbf{R}^d$, and $p : \Omega \to \mathbf{R}$ such that

$$\begin{aligned} -\nabla \cdot \sigma(u, p) &= f \quad \text{in} \quad \Omega, \\ \nabla \cdot u &= 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \Gamma, \end{aligned} \tag{3.1}$$

where the components of the stress tensor $\sigma(u, p)$ are defined by

$$\sigma_{ij}(u, p) = 2\mu\epsilon_{ij}(u) - p\delta_{ij}, \qquad i, j = 1, \ldots, d, \tag{3.2}$$

and $\epsilon(u)$ is the strain tensor, with entries,

$$\epsilon_{ij}(u) = (\partial_j u_i + \partial_i u_j)/2, \qquad i, j = 1, \ldots, d. \tag{3.3}$$

Here and below $\partial_j = \partial/\partial x_j$. In these equations u denotes the velocity vector of the fluid, $\mu > 0$ is the (constant) viscosity coefficient, p is the pressure, and f denotes the body forces. Introducing the space $\widehat{W}_0 = W_0 \times M$, where $W_0 = [H_0^1]^d$ and $M = L^2/\mathbf{R}$, and the bilinear forms

$$a(v, w) = 2\mu \sum_{i,j=1}^{d} (\epsilon_{ij}(v), \epsilon_{ij}(w)), \quad b(v, q) = -(\nabla \cdot v, q), \tag{3.4}$$

the variational form of the Stokes problem reads: find $\hat{u} = (u, p) \in \widehat{W}_0$ such that

$$A(\hat{u}, \hat{v}) \equiv a(u, v) + b(v, p) + b(u, q) = (f, v) \quad \text{for all} \quad \hat{v} \in \widehat{W}_0. \tag{3.5}$$

Here we made use of the fact that $(\epsilon_{ij}(v), \partial_j w_i) = (\epsilon_{ij}(v), \epsilon_{ij}(w))$, which follows from the symmetry of the strain tensor. Clearly, $A(\cdot, \cdot)$ is a bilinear form on $\widehat{W}_0 \times \widehat{W}_0$.

The finite element approximation of the Stokes problem has the form: find $\hat{u}_h = (u_h, p_h) \in \widehat{W}_0^h = W_0^h \times M^h$ such that

$$A(\hat{u}_h, \hat{v}) = (f, v) \quad \text{for all} \quad \hat{v} \in \widehat{W}_0^h, \tag{3.6}$$

where $W_0^h = [V_0^h]^d$ and $M^h = V^h$. In order to ensure that this problem has a unique solution, it suffices to suppose that the spaces W_0^h and M^h satisfy the inf-sup condition [10]; we note, however, that the inf-sup condition does not enter explicitly into the *a posteriori* error analysis that will be described below.

3.2. **Approximation of the lift and drag coefficients.** The traction vector on the boundary Γ has components $\sum_{j=1}^{d} \sigma_{ij} n_j$, where n is the unit outward normal to Γ, and thus, given that $\psi \in [H^{1/2}(\Gamma)]^d$, the force in the direction ψ on Γ is given by

$$N_\psi(\hat{u}) = \sum_{i,j=1}^{d} \int_\Gamma \sigma_{ij}(\hat{u}) n_j \psi_i ds.$$

If ψ is a unit vector parallel with the direction of the flow $N_\psi(\hat{u})$ is called the drag on Γ, and if ψ is a unit vector perpendicular to the direction of the flow $N_\psi(\hat{u})$ is referred to as the lift on Γ. If only a part Γ_0 of the boundary Γ is of concern, then ψ can be taken to have its support in Γ_0. Indeed, in most problems of practical interest, Γ_0 is a closed surface, representing the boundary of an object immersed into the fluid, and $\Gamma \setminus \Gamma_0$ will then denote the boundary of the container. In addition, in the rest of the paper we shall assume that

$$\sum_{i=1}^{d} \int_\Gamma n_i \, \psi_i \, ds = 0,$$

in order to ensure that ψ represents the trace on Γ of a divergence-free function that is in $[H^1(\Omega)]^d$.

In complete analogy with the derivation in Section 2.2, we note that upon multiplying the first equation in (3.1) by $v \in W_\psi = H^1_{\psi_1} \times \cdots \times H^1_{\psi_d}$, integrating over Ω, and using Green's identity, we obtain

$$N_\psi(\hat{u}) = (\sigma(\hat{u}), \nabla v) - (f, v) = A(\hat{u}, \hat{v}) - (f, v),$$

where in the last equality we made use of the fact that $b(u, q) = 0$. Further, we note that the right-hand side is independent of the choice of $\hat{v} \in \widehat{W}_\psi$. Motivated by this formula and our analysis in Section 2, we define the following approximation to $N_\psi(\hat{u})$:

$$N^h_\psi(\hat{u}_h) = (\sigma(\hat{u}_h), \nabla v) - (f, v) = A(\hat{u}_h, \hat{v}) - (f, v),$$

where $\hat{v} = (v, q) \in \widehat{W}^h_\psi = V^h_{\psi_1} \times \cdots \times V^h_{\psi_d} \times M^h$. Again the right-hand side is independent of the choice of $\hat{v} \in \widehat{W}^h_\psi$, and thus the definition is correct.

3.3. **Error representation using duality.** For the representation of the error $N_\psi(\hat{u}) - N^h_\psi(\hat{u}_h)$ we introduce the following dual problem in variational form: find $\hat{\phi} = (\phi, \chi) \in \widehat{W}_\psi$ such that

$$A(\hat{v}, \hat{\phi}) = 0 \quad \text{for all} \quad \hat{v} \in \widehat{W}_0. \tag{3.7}$$

This problem has a unique weak solution, provided that $\psi \in [H^{1/2}(\Gamma)]^d$. Setting $\hat{v} = \hat{u} - \hat{u}_h$ in (3.7), we obtain

$$0 = A(\hat{u} - \hat{u}_h, \hat{\phi}) = A(\hat{u} - \hat{u}_h, \hat{\phi} - \hat{\pi}\hat{\phi}) + A(\hat{u} - \hat{u}_h, \hat{\pi}\hat{\phi}),$$

where we added and subtracted $\hat{\pi}\hat{\phi} = (\pi_1\phi, \pi_2\chi) \in \widehat{W}_\psi^h$, with π_1 and π_2 denoting two bounded linear operators, to be defined below. Next we observe that

$$A(\hat{u} - \hat{u}_h, \hat{\pi}\hat{\phi}) = N_\psi(\hat{u}) - N_\psi^h(\hat{u}_h).$$

Thus we obtain the following error representation formula:

$$N_\psi(\hat{u}) - N_\psi^h(\hat{u}_h) = A(\hat{u} - \hat{u}_h, \hat{\pi}\hat{\phi} - \hat{\phi}) = A(\hat{u}_h, \hat{\phi} - \hat{\pi}\hat{\phi}) - (f, \phi - \pi_1\phi), \tag{3.8}$$

where, in the last equality, we used the fact that $\hat{u}$ is the weak solution of (3.5) and that $\hat{\phi} - \hat{\pi}\hat{\phi} \in [H_0^1(\Omega)]^d$.

3.4. The *a posteriori* estimate.

In order to derive an *a posteriori* error estimate, we shall make two assumptions. First, we adopt the following approximation property: there exists a linear operator $\hat{\pi} : \widehat{W}_\psi \to \widehat{W}_\psi^h$, with $\hat{\pi}\hat{v} = (\pi_1 v, \pi_2 q)$, such that, for each $\tau \in \mathcal{T}_h$, we have

$$\|v - \pi_1 v\|_\tau + h_\tau \|D(v - \pi_1 v)\|_\tau \leq c h_\tau^s \|D^s v\|_\tau, \quad 1 \leq s \leq k+1, \tag{3.9}$$

$$\|q - \pi_2 q\|_\tau \leq c h_\tau^s \|D^s q\|_\tau, \quad 1 \leq s \leq k. \tag{3.10}$$

Next we make the following assumption concerning the regularity of the dual problem (3.7): there exists $t \geq 1$ such that for every s, $1 \leq s \leq t$, there is a constant C_s such that the solution ϕ of the dual problem (3.7) satisfies the following bound:

$$\|D^s \phi\|_\Omega + \|D^{s-1}\chi\|_\Omega \leq C_s \|D^{s-1/2}\psi\|_\Gamma, \tag{3.11}$$

whenever $\psi \in H^{s-1/2}(\Gamma)$. For instance, this estimate is valid when $\Gamma \in C^{s-1,1}$, see [10], [15]. Starting from the error representation formula (3.8) and estimating the right-hand side using the same technique as in Theorem 2.1, we arrive at the following result.

Theorem 3.1. *Let $\hat{u}$ and $\hat{u}_h$ be the solutions of (3.1) and (3.6) respectively, and suppose that $\psi \in [H^{\alpha-1/2}(\Gamma)]^d$, $\alpha \geq 1$, and that (3.9), (3.10) and (3.11) hold with $t = \alpha$; then,*

$$|N_\psi(\hat{u}) - N_\psi^h(\hat{u}_h)| \leq c \sum_{\tau \in \mathcal{T}_h} h^\alpha \mathcal{R}_\tau(\hat{u}_h)\omega_{\tau,\alpha}, \quad \alpha \leq \min(t, k+1),$$

where, for each triangle $\tau \in \mathcal{T}_h$, the residual quantity $\mathcal{R}_\tau(\hat{u}_h)$ is defined by

$$\mathcal{R}_\tau(\hat{u}_h) = \|\nabla \cdot \sigma(\hat{u}_h) + f\|_\tau + h_\tau^{-1}\|\nabla \cdot u_h\|_\tau + h_\tau^{-1/2}\|[n \cdot \sigma(\hat{u}_h)]/2\|_{\partial\tau}, \tag{3.12}$$

and the local weight $\omega_{\tau,\alpha}$ is given by

$$\omega_{\tau,\alpha} = \|D^\alpha \phi\|_\tau + \|D^{\alpha-1}\chi\|_\tau,$$

where $\hat{\phi}$ is the solution to (3.7).

Proof. By considering the error representation formula (3.8), and integrating triangle-by-triangle using Green's identity, we have

$$
\begin{aligned}
N_\psi(u) - N_\psi^h(u_h) &= -\sum_{\tau\in\mathcal{T}_h}\int_\tau (\nabla\cdot\sigma(\hat{u}_h)+f)(\phi-\pi_1\phi)dx \\
&\quad + \sum_{\tau\in\mathcal{T}_h}\int_\tau (\nabla\cdot u_h)(\chi-\pi_2\chi)dx \\
&\quad + \sum_{\tau\in\mathcal{T}_h}\int_{\partial\tau\setminus\Gamma} ([n\cdot\sigma(\hat{u}_h)]/2)(\phi-\pi_1\phi)ds \\
&= I + II + III, \qquad (3.13)
\end{aligned}
$$

where we made use of the fact that ϕ, the weak velocity-solution of the dual problem (3.7) is in $C^{0,\alpha}(\Omega)$, for some α in $(0,1)$, so that $[\phi-\pi_1\phi]=0$ across $\partial\tau\setminus\Gamma$ for each element τ in the partition. Now let us estimate I, II and III. For I, we apply the Cauchy-Schwarz inequality and use the approximation property (3.9) to obtain

$$|I| \le c\sum_{\tau\in\mathcal{T}_h} h_\tau^s \|\nabla\cdot\sigma(\hat{u}_h)+f\|_\tau \|D^s\phi\|_\tau.$$

Similarly,

$$|II| \le c\sum_{\tau\in\mathcal{T}_h} h_\tau^{s-1}\|\nabla\cdot u_h\|_\tau \|D^{s-1}\chi\|_\tau.$$

To deal with III, we argue in the same manner as for Term II in the proof of Theorem 2.1 to arrive at the following bound:

$$|III| \le c\sum_{\tau\in\mathcal{T}_h} h_\tau^{s-1/2}\|[n\cdot\sigma(\hat{u}_h)]\|_{\partial\tau\setminus\Gamma}\|D^s\phi\|_\tau.$$

Substituting the bounds on I, II and III into (3.13) and recalling the definition of the residual quantity (3.12), we deduce the desired *a posteriori* error bound. ☐

3.5. **The *a priori* estimate.** Proceeding in the same manner as in the case of the scalar elliptic equation discussed in the previous section, it is also possible to derive an *a priori* estimate for the error in the lift and drag coefficients. Indeed, it follows from the error representation formula (3.8) that

$$N_\psi(\hat{u}) - N_\psi^h(\hat{u}_h) = a(u-u_h,\pi_1\phi-\phi) + b(\pi_1\phi-\phi,p-p_h) + b(u-u_h,\pi_2\chi-\chi),$$

where (ϕ,χ) is the dual velocity-pressure solution pair. Thus, by the Cauchy-Schwarz inequality,

$$
\begin{aligned}
|N_\psi(\hat{u}) - N_\psi^h(\hat{u}_h)| \;&\le\; 2\mu\|D(u-u_h)\|_\Omega\,\|D(\phi-\pi_1\phi)\|_\Omega \\
&\quad + \|D(\phi-\pi_1\phi)\|_\Omega\,\|p-p_h\|_\Omega + \|D(u-u_h)\|_\Omega\,\|\chi-\pi_2\chi\|_\Omega.
\end{aligned}
$$

Assuming that the pair of finite element spaces (W_0^h, M^h) satisfies the inf-sup condition (see, [10]) and that the solution to the Stokes problem and its dual are in $[H^{k+1}(\Omega)]^d \times H^k(\Omega)$, it follows from this inequality and the approximation properties (3.9), (3.10) that the error between $N_\psi(u)$ and $N_\psi^h(u_h)$ is of size $O(h^{2k})$.

4. The incompressible Navier Stokes Equation

4.1. **Analysis.** The stationary incompressible Navier Stokes equations have the form: find $u : \Omega \to \mathbf{R}^d, d = 2, 3$, and $p : \Omega \to \mathbf{R}$ such that

$$\begin{aligned} -\nabla \cdot \sigma(u,p) + \rho(u \cdot \nabla)u &= f \quad \text{in} \quad \Omega, \\ \nabla \cdot u &= 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \Gamma, \end{aligned} \tag{4.1}$$

where $((u \cdot \nabla)v)_i = \sum_{j=1}^d u_j \partial_j v_i$, and ρ is the density of the fluid. The corresponding weak form reads: find $\hat{u} \in \widehat{W}_0$ such that

$$A(u; \hat{u}, \hat{v}) = (f, v) \quad \text{for all} \quad \hat{v} \in \widehat{W}_0, \tag{4.2}$$

where

$$A(w, \hat{u}, \hat{v}) = a(u, v) + b(v, p) + c(w; u, v) + b(u, q).$$

Here the bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are defined in (3.4) and the trilinear form $c(\cdot; \cdot, \cdot)$ is given by

$$c(w; u, v) = \rho((w \cdot \nabla)u, v).$$

We now discretize this problem analogously to the Stokes problem. The convection term may be dealt with through the use of the streamline diffusion method, for example, [8]; indeed, in our numerical experiments we employ the streamline diffusion finite element method. However, for simplicity of presentation, we neglect the stabilizing terms in our analysis and consider the standard Galerkin finite element method, instead.

The approximation of the lift and drag coefficients follows in exactly the same way as for the Stokes problem. To be precise, the approximation of $N_\psi(\hat{u}) = A(u; \hat{u}, \hat{v}) - (f, v)$, $\hat{v} \in \widehat{W}_\psi$, is defined by

$$N_\psi^h(\hat{u}_h) = A(u_h; \hat{u}_h, \hat{v}) - (f, v),$$

with $\hat{v} \in \widehat{W}_\psi^h$, where ψ is as in Section 3.

For the sake of representing the error $N_\psi(\hat{u}) - N_\psi^h(\hat{u}_h)$, we introduce the following linearized dual problem in variational form: find $\hat{\phi} = (\phi, \chi) \in \widehat{W}_\psi$ such that

$$L(u, u_h; \hat{v}, \hat{\phi}) = 0 \quad \text{for all} \quad \hat{v} = (v, q) \in \widehat{W}_0, \tag{4.3}$$

where

$$L(u, u_h; \hat{v}, \hat{\phi}) = a(v, \phi) + b(\phi, q) + \overline{c}(u, u_h; v, \phi) + b(v, \chi),$$

and

$$\overline{c}(u,u_h;v,\phi) = \rho\left((-u\cdot\nabla)\phi + (\phi\cdot\nabla u_h), v\right),$$

with $(\phi\cdot\nabla u_h)_i = \sum_{j=1}^{d}\phi_j\partial_i u_j$. This definition of the linearized dual problem is motivated by the following identity

$$L(u,u_h;\hat{u}-\hat{u}_h,\hat{\phi}) = A(u;\hat{u},\hat{\phi}) - A(u_h;\hat{u}_h,\hat{\phi}).$$

Choosing $\hat{v} = \hat{u}-\hat{u}_h$ in (4.3) we thus obtain

$$\begin{aligned} 0 &= L(u,u_h;\hat{u}-\hat{u}_h,\hat{\phi}) \\ &= A(u;\hat{u},\hat{\phi}) - A(u_h;\hat{u}_h,\hat{\phi}) \\ &= A(u;\hat{u},\hat{\phi}-\hat{\pi}\hat{\phi}) - A(u_h;\hat{u}_h,\hat{\phi}-\hat{\pi}\hat{\phi}) \\ &\qquad + A(u;\hat{u},\hat{\pi}\hat{\phi}) - A(u_h;\hat{u}_h,\hat{\pi}\hat{\phi}), \end{aligned}$$

where, as above, we added and subtracted $\hat{\pi}\hat{\phi}\in\widehat{W}_\psi$. Next, observing that

$$A(u;\hat{u},\hat{\pi}\hat{\phi}) - A(u_h;\hat{u}_h,\hat{\pi}\hat{\phi}) = N_\psi(\hat{u}) - N_\psi^h(\hat{u}_h),$$

we finally arrive at the error representation formula

$$\begin{aligned} N_\psi(\hat{u}) - N_\psi^h(\hat{u}_h) &= A(u_h;\hat{u}_h,\hat{\phi}-\hat{\pi}\hat{\phi}) - A(u;\hat{u},\hat{\phi}-\hat{\pi}\hat{\phi}) \\ &= A(u_h,\hat{u}_h,\hat{\phi}-\hat{\pi}\hat{\phi}) - (f,\phi-\pi_1\phi) \end{aligned} \tag{4.4}$$

where, in the last transition, we made use of the fact that $\hat{\phi}-\hat{\pi}\hat{\phi}\in\widehat{W}_0$ and that $\hat{u}$ is the weak solution of (4.2).

Estimating the right-hand side in a similar fashion as in Theorem 2.1, we obtain the following *a posteriori* estimate.

Theorem 4.1. *Let $\hat{u}$ and u_h be the solutions of (4.1) and (4.2), and assume that (3.9), (3.10) and (3.11) hold for the solution $\hat{\phi}$ of (4.3). Supposing that $\psi\in[H^{\alpha-1/2}(\Gamma)]^d$, $\alpha\geq 1$, we have that*

$$|N_\psi(\hat{u}) - N_\psi^h(\hat{u}_h)| \leq c\sum_{\tau\in\mathcal{T}_h} h^\alpha \mathcal{R}_\tau(\hat{u}_h)\omega_{\tau,\alpha}, \quad \alpha\leq\min(t,k+1),$$

where, for each triangle $\tau\in\mathcal{T}_h$, the residual quantity $\mathcal{R}_\tau(\hat{u}_h)$ is defined by

$$\begin{aligned} \mathcal{R}_\tau(\hat{u}_h) &= \|\nabla\cdot\sigma(\hat{u}_h) - \rho(u_h\cdot\nabla)u_h + f\|_\tau \\ &\qquad + h_\tau^{-1}\|\nabla\cdot u_h\|_\tau + h_\tau^{-1/2}\|[n\cdot\sigma(\hat{u}_h)]/2\|_{\partial\tau}, \end{aligned}$$

and the weight $\omega_{\tau,\alpha}$ is defined by

$$\omega_{\tau,\alpha} = \|D^s\phi\|_\tau + \|D^{s-1}\chi\|_\tau.$$

4.2. **A numerical example.** In this section we present a numerical example illustrating the practical use of our estimates. We consider the computation of the drag coefficient for a cylinder immersed into a two-dimensional viscous incompressible fluid in a channel, whose flow is governed by the incompressible Navier-Stokes equations (4.1) with prescribed inflow velocity, no-slip conditions on the walls of the channel and the cylinder, and free flow conditions at the outflow. This problem is one of the benchmark problems presented in [14], and the value, 5.57, of the drag coefficient is determined experimentally.

We approximate the exact flow in the channel by means of the streamline diffusion finite element method using stabilised piecewise linear approximation for both the velocity u and the pressure p, see for instance [8]. The discrete equations are solved using a multigrid method, and the adaptive algorithm is designed so that approximately 40% of the triangles are refined in each step, depending on the size of $h^\alpha \mathcal{R}(u_j)\omega_{\tau,\alpha}$. In order to compute the quantities $h^\alpha \mathcal{R}(u_j)\omega_{\tau,\alpha}$, for each triangle $\tau \in \mathcal{T}_h$, we solve the dual problem numerically and approximate the weight $\omega_{\tau,\alpha}$ using difference quotients. In this case, ignoring variational crimes, we expect $\alpha = 2$, since the boundary data for the dual problem is smooth. In Figure 1 we present the computed error bound given in Theorem 4.1 and the error in the drag coefficient as functions of the number of degrees of freedom. The constant c in Theorem 4.1 is chosen equal to 1/10. The wiggles in the curves arise from the refinement of the triangles and cancellation phenomena in the computation of $N_\psi^h(\hat{u}_h)$.

In Figure 2 we show the final grid. Note that the refinement is located close to the cylinder, which is what we expect. Furthermore, in Figures 3 and 4 we present the level curves of the velocity parallel to the channel of the flow and the dual flow, respectively. Note that the dual solution is large close to the cylinder indicating that it is important that the residual is small in this area. For simplicity, we have neglected the influence of approximating the curved boundary and boundary conditions in the computations.

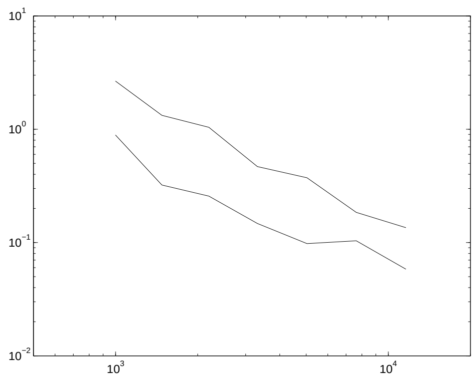

FIGURE 1. The error in the drag coefficient and the a posteriori bound in Theorem 4.1 as functions of the number of degrees of freedom.

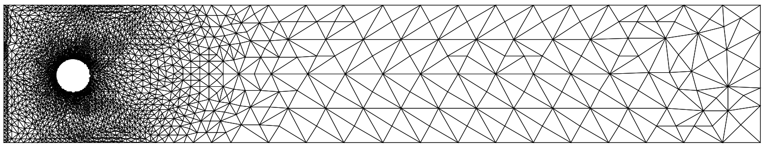

FIGURE 2. The final mesh.

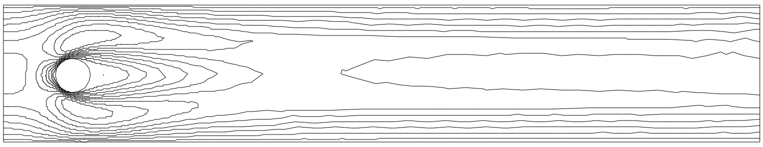

FIGURE 3. The first component of velocity of the flow.

FIGURE 4. The first component of the velocity of the dual flow.

REFERENCES

[1] I. Babuška and A. Miller. The post processing approach in the finite element method, part 1: calculation of displacements, stresses and other higher derivatives of the displacements. *Internat. J. Numer. Methods. Engrg*, 34:1085–1109, 1984.

[2] I. Babuška and A. Miller. The post processing approach in the finite element method, part 2: the calculation of stress intensity factors. *Internat. J. Numer. Methods. Engrg*, 34:1111–1129, 1984.

[3] I. Babuška and A. Miller. The post processing approach in the finite element method, part 3: a posteriori estimates and adaptive mesh selection. *Internat. J. Numer. Methods. Engrg*, 34:1131–1151, 1984.

[4] I. Babuška and M. Suri. The p and h-p versions of the finite element method, basic principles and properties. *SIAM. Review*, 36(4):578–632, 1994.

[5] J. W. Barrett and C. M. Elliott. Total flux estimates for a finite element approximation of elliptic equations. *I.M.A. J. Numer. Anal.*, 7:129–148, 1987.

[6] R. Becker and R. Rannacher. Weighted a posteriori error control in FE methods. preprint 1, Interdisziplinäres Zentrum für Wissenschaftliches Rechnen, Universität Heidelberg, 1996.

[7] S. C. Brenner and L. R. Scott. *The Mathematical Theory of Finite Element Methods.* Springer-Verlag, Berlin, 1994.

[8] K. Eriksson, D. Estep, P. Hansbo, and C. Johnson. Introduction to numerical methods for differential equations. *Acta Numerica*, pages 105–158, 1995.

[9] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Springer Verlag, 2nd edition, 1983.

[10] V. Girault and P.-A. Raviart. *Finite Element Methods for Navier-Stokes equations.* Springer Verlag, 1986.

[11] C. Johnson and R. Rannacher. On error control in CFD. Procedings from Conference on Navier-Stokes equations 93, 1994.

[12] C. Johnson, R. Rannacher, and M. Boman. Numerics and hydrodynamic stability theory: towards error control in CFD. *SIAM J. Numer. Anal.*, 32:1058–1079, 1995.

[13] M. G. Larson. A new error analysis for finite element approximations of indefinite linear elliptic problems. Ph.d. thesis, Dept. of Mathematics, Chalmers University of Technology, Sweden, 1997.

[14] M. Schäfer and S. Turek. The benchmark problem "flow around a cylinder". In E. H. Hirschel, editor, *Flow Simulation with High-Performance Computers*. Vieweg, 1996.

[15] R. Temam. *Theory and Numerical Analysis of the Navier-Stokes Equations*. North-Holland, 1977.

[16] R. Verfürth. *A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques*. Wiley and Teubner, 1996.

[17] J. A. Wheeler. Simulation of heat transfer from a warm pipeline buried in permafrost. Proceedings of the 74th National meeting of the American Institute of Chemical Engineering, 1973.

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