Gamma-convergence of a shearlet-based Ginzburg–Landau energy

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Abstract

We introduce two shearlet-based Ginzburg–Landau energies, based on the continuous and the discrete shearlet transform. The energies result from replacing the elastic energy term of a classical Ginzburg–Landau energy by the weighted L^2-norm of a shearlet transform. The asymptotic behaviour of sequences of these energies is analysed within the framework of Γ- convergence and the limit energy is identified. We show that the limit energy of a characteristic function is an anisotropic surface integral and we demonstrate that its anisotropy can be controlled by weighting the underlying shearlet transforms according to their directional parameter.

Keywords: Shearlets, Ginzburg–Landau energy, Γ-convergence

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1 Introduction

The Ginzburg–Landau energy [9], introduced initially to model phase-transitions in superconductors, has recently found many applications in image processing [1, 3, 8, 12]. In [7] a wavelet-based Ginzburg–Landau energy was introduced, which, in contrast with the original Ginzburg–Landau energy, is derivative-free and the solution of the associated minimisation problem does not require the solution of a partial differential equation. The wavelet-based Ginzburg–Landau energy is constructed by substituting the elastic energy part of a standard Ginzburg–Landau energy by a wavelet-based Besov seminorm. This seminorm is formed by a weighted L^2-norm of the semi-continuous wavelet transform. It was demonstrated in [7] that this generalised energy admits the following very attractive theoretical property: a sequence of wavelet-based Ginzburg–Landau energies Γ-converges to a limit that, for piecewise constant functions, can be described by an anisotropic surface integral over the interfaces. Thereby, the functional introduces a directional anisotropy. This is particularly convenient since modelling such anisotropies in the classical Ginzburg–Landau energy necessitates the solution of quasilinear PDEs for the numerical minimisation. The wavelet-based formulation, on the other hand, can be minimised by solving a linear system.

Unfortunately, the wavelet-based functional can only produce very simple anisotropies. It was conjectured in [8] that a similar approach using directional systems, such as curvelets [4] or shearlets [15, 13], would produce energies such that the anisotropy of the limiting energy could be more precisely controlled. Indeed, this approach was numerically analysed in [5], but no theoretical analysis of the Γ-convergence was included. One issue preventing direct translation of the theoretical arguments used for the convergence results of wavelet-based Ginzburg–Landau energies is that neither compactly supported shearlets, nor any known shearlet system on a bounded domain, not resulting from truncation, forms a tight frame. The proof of the convergence results for wavelet-based energies in [8] is mostly based on a projection procedure of an underlying function to a space spanned by wavelets up to a maximal scale. Such a projection operator must necessarily involve the dual frame. If the frame is not tight, then the dual frame is usually not known analytically. This severly complicates the analysis.

In this work, we present the first proof of the Γ-convergence of a sequence of shearlet-based Ginzburg–Landau energies in Theorem 2.9 and Theorem 3.1. The analysed energies are constructed similarly to the wavelet case, by replacing the elasticity term of a Ginzburg–Landau energy by a weighted L^2-norm of a shearlet transform. We present results on two types of shearlet transforms, the continuous shearlet transform and the discrete shearlet transform. We precisely identify the Γ-limit of the sequences of energies and demonstrate that the limit energy applied to characteristic functions of sets with finite perimeter is a surface integral over the boundaries which can be controlled by weighting the individual shearlet elements according to their orientation parameter. Our arguments are based on spatial and directional localisation which demonstrates that for certain functions a reweighted cone-adapted shearlet transform is asymptotically equivalent to a so-called homogeneous shearlet transform. Contrary to the cone-adapted shearlet transform, the homogeneous shearlet transform is an isometry. This shows that after spatial and directional localisation the shearlet transform is locally and asymptotically a tight frame.

1.1 Comparison to the wavelet-based Ginzburg–Landau energy

The arguments in [8] describe the Γ-convergence of a sequence of wavelet-based functionals to an anisotropic perimeter functional by studying the quotient of a wavelet-based Besov seminorm and the H^1 seminorm. It is shown that for a particular choice of sequences the quotient converges and the limit can be identified. To obtain convergence of the quotient for all sequences a lifting argument is used. This then establishes the asymptotic equivalence of the wavelet-based Ginzburg–Landau energy and the standard Ginzburg–Landau energy up to a multiplicative constant which is given by the limit of the quotient described above. Finally, the fact that the standard Ginzburg–Landau energy converges to the perimeter functional is used to establish the Γ-limit of the wavelet-based functional.

Our approach is considerably different. We employ a different technique to construct a recovery sequence in Subsection 2.7, which is not based on projections, but on spatial and directional filtering. The liminf condition in the Γ-convergence then results from our construction of the shearlet-based energy, without any lifting step.

In fact, we cannot apply the lifting technique of [8], since the argument there is incorrect. To elaborate on this, we provide a counterexample to the main theorem of [8] and identify a miscalculation in the lifting argument in Appendix E.

1.2 Notation

For $D \subset \mathbb{R}^2$ and $1 \le p \le \infty$, we denote by $L^p(D)$ the standard Lebesgue spaces with parameter p. We denote by $H^1(D)$ the Sobolev space of once weakly differentiable functions with $\|f\|_{H^1}^2 := \|f\|_{L^2}^2 + |f|_{H^1}^2 := \|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2$. We denote by BV the space of all functions of bounded variation on $(0,1)^2$ and by SBV the space of all special functions of bounded variation on $(0,1)^2$. The Fourier transform of a function $f \in L^1(\mathbb{R}^2)$ is defined by

$$\mathcal{F}(f)(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}^2} f(x)\, \mathrm{e}^{-2\pi i \langle x, \xi \rangle}\, \mathrm{d}x \quad \text{for all } \xi \in \mathbb{R}^2.$$

It is well known that $\mathcal{F} : L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)$ can be extended to an isomorphism on $L^2(\mathbb{R}^2)$, which yields a Fourier transform on $L^2(\mathbb{R}^2)$. We again denote this extended Fourier transform by $\mathcal{F}(f)$ or $\hat{f}$. For $f \in L^p(\mathbb{R}^2)$, $g \in L^1(\mathbb{R}^2)$, we write $f * g$ for the convolution of f and g. For $\phi_1, \phi_2 \in L^2(\mathbb{R})$ we define the tensor product by $(\phi_1 \otimes \phi_2)(x) := \phi_1(x_1)\, \phi_2(x_2)$, where $x = (x_1, x_2) \in \mathbb{R}^2$. We use the symbol $\mathbb{H}_1$ for the one-dimensional Hausdorff measure. For a set $D \subset \mathbb{R}^2$, we denote by χ_D the characteristic function of D.

For a vector $z \in \mathbb{R}^n$, we denote by z_i the entry at its i-th coordinate. We denote by $|\cdot|$ the Euclidean norm on $\mathbb{R}^n$, $n \in \mathbb{N}$, and $\|x\|_\infty := \sup_{i=1,\dots,n} |x_i|$ for $x \in \mathbb{R}^n$; $\mathbb{S}^1$ denotes the unit sphere in $\mathbb{R}^2$. If f, g are non-negative functions defined over X and there is a $c > 0$ such that $f(x) \le c g(x)$ for all $x \in X$, then we write $f \lesssim g$. If $f \lesssim g$ and $g \lesssim f$, then we write $f \sim g$.

2 Shearlet-based Ginzburg–Landau energy

2.1 Continuous shearlet systems

A shearlet system over $\mathbb{R}^2$ is a set of functions, generated from shifting, scaling, and shearing of one or more generator functions. For the scaling and shearing operations, we require the following definitions: for $a \in \mathbb{R}^+$, $s \in \mathbb{R}$, we define the *anisotropic scaling matrix* A_a and the *shear matrix* S_s by

$$A_a := \begin{pmatrix} a^{-1} & 0 \\ 0 & a^{-\frac{1}{2}} \end{pmatrix} \quad \text{and} \quad S_s := \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

We define two shearlet transforms, beginning with the homogeneous shearlet transform, [14, 6].

Definition 2.1. *Let $\psi \in L^2(\mathbb{R}^2)$; then we define*

$$\begin{aligned} \mathcal{SH} : L^2(\mathbb{R}^2) &\to L^\infty(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2), \\ f &\mapsto \big((a,s,t) \mapsto \mathcal{SH}(f)(a,s,t) := \langle f, \psi_{a,s,t} \rangle\big), \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of $L^2(\mathbb{R}^2)$ and for $x \in \mathbb{R}^2$:

$$\psi_{a,s,t}(x) := a^{-\frac{3}{4}} \psi\left(A_a^{-1} S_s (x - t)\right).$$

We call $\mathcal{SH}$ the homogeneous shearlet transform*. Moreover, we call $\mathcal{SH}(f)$ the* homogeneous shearlet transform of f*.*

It is clear from the Cauchy–Schwarz inequality that $\mathcal{SH}$ is well-defined. More importantly, under suitable assumptions on ψ, $\mathcal{SH}$ is also well-defined as a map to $L^2(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2, a^{-3}\,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t)$ and is, in fact, an isometry; we recall the relevant results below.

Definition 2.2 ([6])**.** *A function $\psi \in L^2(\mathbb{R})$ is called an* admissible shearlet *if*

$$\int_{\mathbb{R}^2} \frac{|\widehat{\psi}(\xi)|^2}{|\xi_1|^2}\,\mathrm{d}\xi =: C_\psi < \infty.$$

In the special case when $C_\psi = 1$ then we arrive at the previously announced norm equivalence, and the following result holds.

Theorem 2.3 ([6, 10])**.** *Let $\psi \in L^2(\mathbb{R}^2)$ be an admissible shearlet; then, for all $f \in L^2(\mathbb{R}^2)$,*

$$\|f\|^2_{L^2(\mathbb{R}^2)} = C_\psi^{-1} \int_{\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2} |\mathcal{SH}(f)(a,s,t)|^2 a^{-3}\,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t.$$

To prevent unequal treatment of directions or shearlet elements $\psi_{a,s,t}$ with very long supports if s becomes large, a cone-adapted shearlet system was introduced in [15]. We define

$$Q := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$\widetilde{A}_a := QA_a$, and $\widetilde{\psi}_{a,s,t} := a^{-\frac{3}{4}} \psi(Q\widetilde{A}_a^{-1} S_{-s}^T(\cdot - t))$ for $(a,s,t) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$. Cone-adapted systems are associated with a transform which, similarly to the homogeneous shearlet transform, admits a certain norm-equivalence with the $L^2(\mathbb{R}^2)$ norm.

Theorem 2.4 ([10])**.** *Let $\psi \in L^2(\mathbb{R}^2)$ be an admissible shearlet such that*

$$|\widehat{\psi}(\xi)| \leq \frac{\min\{|\xi_1|^2, 1\}}{(1+|\xi_1|^2)(1+|\xi_2|^2)}, \quad \textit{for all } \xi \in \mathbb{R}^2.$$

Then, there exist $\Gamma^* > 0$, $\Delta^* > 0$ such that for all $\Gamma \geq \Gamma^*$, $\Delta \geq \Delta^*$ and any function $K \in L^2(\mathbb{R}^2)$ with $\widehat{K}(\xi) \in [A, B]$ for all $\xi \in [-1,1]^2$, we have that, for all $f \in L^2(\mathbb{R}^2)$,

$$\|f\|^2_{L^2(\mathbb{R}^2)} \sim \int_{\mathbb{R}^2} |\langle f, K(\cdot - t)\rangle|^2 \,\mathrm{d}t + \int_{\mathbb{R}^2}\int_{-\Delta}^{\Delta}\int_0^{\Gamma} |\langle f, \psi_{a,s,t}\rangle|^2 a^{-3}\,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t + \int_{\mathbb{R}^2}\int_{-\Delta}^{\Delta}\int_0^{\Gamma} |\langle f, \widetilde{\psi}_{a,s,t}\rangle|^2 a^{-3}\,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t.$$

We have seen that the homogeneous and the cone-adapted shearlet transforms satisfy a certain norm equivalence with the L^2-norm. Additionally, it is also possible to characterise Sobolev norms by these transforms in the following way. Note that we have, for $f \in H^1(\mathbb{R}^2)$, by Parseval's identity, that

$$|f|^2_{H^1(\mathbb{R}^2)} = \left\|\frac{\partial f}{\partial x_1}\right\|^2_{L^2(\mathbb{R}^2)} + \left\|\frac{\partial f}{\partial x_2}\right\|^2_{L^2(\mathbb{R}^2)} = (2\pi)^2\left(\left\|\xi_1 \hat{f}\right\|^2_{L^2(\mathbb{R}^2)} + \left\|\xi_2 \hat{f}\right\|^2_{L^2(\mathbb{R}^2)}\right).$$

Let $\psi \in L^2(\mathbb{R}^2)$ be such that

$$\int_{\mathbb{R}^2} \frac{|\widehat{\psi}(\xi)|^2}{|\xi_1|^4}\,\mathrm{d}\xi = (2\pi)^2, \tag{2.1}$$

and set $\widehat{\mu}(\xi) := \widehat{\psi}(\xi)/\xi_1$, then $C_\mu = (2\pi)^2$. By Parseval's identity, we observe that

$$\begin{aligned}\int_{\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^2} a^{-2}|\langle f, \psi_{a,s,t}\rangle|^2 a^{-3}\,\mathrm{d}a\,\mathrm{d}s\ \mathrm{d}t &= \int_{\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^2} \left|\left\langle \frac{\widehat{\psi}_{a,s,t}}{a\xi_1}, \xi_1\hat{f}\right\rangle\right|^2 a^{-3}\,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t \\ &= \int_{\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^2} \left|\left\langle \widehat{\mu}_{a,s,t}, \xi_1\hat{f}\right\rangle\right|^2 a^{-3}\,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t = (2\pi)^2\left\|\xi_1\hat{f}\right\|^2_{L^2}.\end{aligned}$$

Additionally, it is not hard to see, and we will show it in Appendix A, that for any admissible shearlet $\psi \in L^2(\mathbb{R}^2)$ there exist $\Gamma^* > 0$, $\Delta^* > 0$ such that for all $\Gamma \geq \Gamma^*$, $\Delta \geq \Delta^*$ and any function $K \in L^2(\mathbb{R}^2)$ with $\widehat{K}(\xi)/|\xi| \in [A, B]$ on $[-1,1]^2$, there are $0 < c_1, c_2$ such that we have, for all $f \in H^1(\mathbb{R}^2)$,

$$\begin{aligned}c_1|f|^2_{H^1} \leq \int_{\mathbb{R}^2} |\langle f, K(\cdot - t)\rangle|^2 \,\mathrm{d}t &+ \int_{\mathbb{R}^2}\int_{-\Delta}^{\Delta}\int_0^{\Gamma} a^{-2}|\langle f, \psi_{a,s,t}\rangle|^2 a^{-3}\,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t \\ &+ \int_{\mathbb{R}^2}\int_{-\Delta}^{\Delta}\int_0^{\Gamma} a^{-2}\left|\left\langle f, \widetilde{\psi}_{a,s,t}\right\rangle\right|^2 a^{-3}\,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t \leq c_2|f|^2_{H^1}.\end{aligned} \tag{2.2}$$

Note that

$$\int_{\mathbb{R}^2} |\langle f, K(\cdot - t)\rangle|^2 \,\mathrm{d}t \leq B\|f\|^2_{L^2}. \tag{2.3}$$

The last step towards constructing a shearlet-based Besov seminorm is to introduce directional weights. Let $\Gamma > \Gamma^* > 0$, and $\Delta > \Delta^* > 0$, and let $\psi \in L^2(\mathbb{R}^2)$ be such that (2.2) holds. A *directional weight* $\omega : \{-1,1\} \times \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous map such that, for $\iota \in \{-1,1\}$,

$$\operatorname{supp}\omega(\iota,\cdot) \subset [-\Delta, \Delta] \qquad \text{and} \qquad \min_{\iota\in\{-1,1\}}\min_{s\in[-\Delta^*,\Delta^*]} \omega(\iota, s) > 0.$$

We define for $a \in \mathbb{R}^+$, $s \in \mathbb{R}$, $t \in \mathbb{R}^2$:

$$K_t := K(\cdot - t), \quad \psi^\omega_{a,s,t,1} := \omega(1,s)\psi_{a,s,t}, \quad \text{and} \quad \psi^\omega_{a,s,t,-1} := \omega(-1,s)\widetilde{\psi}_{a,s,t}.$$

If (2.2) holds for $\Delta > 0$ such that $\min_{\iota\in\{-1,1\}} \min_{s\in[-\Delta,\Delta]} \omega(\iota, s) > 0$, then we have that there exist $c_1, c_2 > 0$ such that

$$c_1|f|_{H^1}^2 \le \int_{\mathbb{R}^2} |\langle f, K_t\rangle|^2 \, \mathrm{d}t + \sum_{\iota=-1,1} \int_{\mathbb{R}^2} \int_{-\Delta}^{\Delta} \int_0^{\Gamma} a^{-2} |\langle f, \psi^{\omega}_{a,s,t,\iota}\rangle|^2 a^{-3} \, \mathrm{d}a \, \mathrm{d}s \, \mathrm{d}t \le c_2 |f|_{H^1}^2.$$

Finally, we introduce, for $f \in L^2(\mathbb{R}^2)$, the *shearlet-based Besov seminorm* $\|\cdot\|_B$ *on* $\mathbb{R}^2$ by

$$\|f\|_B^2 := \sum_{\iota=-1,1} \int_{\mathbb{R}^2} \int_{-\Delta}^{\Delta} \int_0^{\Gamma} a^{-2} |\langle f, \psi^{\omega}_{a,s,t,\iota}\rangle|^2 a^{-3} \, \mathrm{d}a \, \mathrm{d}s \, \mathrm{d}t \in [0, \infty].$$

2.2 Shearlet transform on a bounded domain and norm equivalence

To define a shearlet-based Besov seminorm on a bounded domain, we first need to adapt the shearlet system from $\mathbb{R}^2$ accordingly. A simple procedure to generate a shearlet system on $[0, 1]^2$ is to make the elements of the system periodic using the following approach:

$$\psi^{\omega,per}_{a,s,t,\iota} := \sum_{k\in\mathbb{Z}^2} \psi^{\omega}_{a,s,t,\iota}(\cdot - k), \qquad \text{and} \qquad K_t^{per} = \sum_{k\in\mathbb{Z}^2} K_t(\cdot - k). \tag{2.4}$$

In a different setting of discrete shearlet systems, there have been alternative constructions of shearlets on bounded domains, see [11, 17]. The associated constructions are a bit more involved than the approach via periodisation above. This is because in the discrete setting the approximation properties of the associated systems are analysed, which is not a concern in our approaches. Moreover, the approaches are based on discrete, boundary adapted wavelet systems which do not generalise to continuous parameter sets.

Based on the periodic construction of (2.4), we proceed to define the *shearlet-based Besov seminorm on* $[0, 1]^2$ by

$$|f|_{B,p}^2 := \sum_{\iota=-1,1} \int_{(0,1)^2} \int_{-\Delta}^{\Delta} \int_0^{\Gamma} a^{-2} \left|\langle f, \psi^{\omega,per}_{a,s,t,\iota}\rangle\right|^2 a^{-3} \, \mathrm{d}a \, \mathrm{d}s \, \mathrm{d}t \in [0, \infty], \quad \text{for } f \in L^2((0, 1)^2).$$

Sometimes it will be necessary to also study a *shearlet-based Besov-functional with nonperiodic elements* which, for $r > 0$, is defined as follows:

$$|f|_{B,[-r,r]}^2 := \sum_{\iota=-1,1} \int_{[-r,r]^2} \int_{-\Delta}^{\Delta} \int_0^{\Gamma} a^{-2} \left|\langle f, \psi^{\omega}_{a,s,t,\iota}\rangle\right|^2 a^{-3} \, \mathrm{d}a \, \mathrm{d}s \, \mathrm{d}t \in [0, \infty], \quad \text{for } f \in L^2(\mathbb{R}^2).$$

We shall be interested in the question as to whether $|f|_{B,p}$ is equivalent to the Sobolev seminorm $|f|_{H^1((0,1)^2)}$.

Since $\langle f, \psi^{\omega,per}_{a,s,t,\iota}\rangle = \langle f^{per}, \psi^{\omega}_{a,s,t,\iota}\rangle$, where f^{per} is a periodised version of f, we deduce from (2.2) that $|\cdot|_{B,p}^2$ is *not* equivalent to $|\cdot|_{H^1((0,1)^2)}$ if the situation $f^{per} \notin H^1(\mathbb{R}^2)$ is possible. In [7], a similar construction is presented for a wavelet-based Besov seminorm. It is claimed that the Besov seminorm on $[0, 1]^2$ defined by a sufficiently regular, periodic wavelet is equivalent to the $H^1((0, 1)^2)$ seminorm. As has just been demonstrated this assertion is invalid if $f^{per} \notin H^1(\mathbb{R}^2)$ is possible, i.e., if f has non-matching boundary conditions. However, the result does hold true for compactly supported wavelets over $H_0^1((0, 1)^2)$ or if another form of matching boundary condition at 0 and 1 is prescribed. A proof of such a result is not given in [7], but following the arguments of the proposition below and replacing shearlets by wavelets when appropriate one can deduce the asserted equivalence of seminorms in the regime of [7] as well.

Proposition 2.5. *Let* $0 < A, B$ *and* $\psi \in L^2(\mathbb{R}^2)$ *be such that* $\operatorname{supp} \psi \subset [0, 1]^2$ *and such that, for* $K \in L^2(\mathbb{R}^2)$ *with* $|\hat{K}(\xi)|/|\xi| \in [A, B]$ *for all* $\xi \in [-1, 1]^2$ *and for all* $f \in H^1(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} |\langle f, K_t\rangle|^2 \, \mathrm{d}t + \sum_{\iota=-1,1} \int_{\mathbb{R}^2} \int_{-\Delta}^{\Delta} \int_0^{\Gamma} a^{-2} |\langle f, \psi^{\omega}_{a,s,t,\iota}\rangle|^2 a^{-3} \, \mathrm{d}a \, \mathrm{d}s \, \mathrm{d}t \sim |f|_{H^1(\mathbb{R}^2)}^2. \tag{2.5}$$

Then, there exists a $B'>0$ such that

$$|f|^2_{H^1((0,1)^2)} \sim |f|^2_{B,p}, \quad \textit{for all } f \in H^1_0\left((0,1)^2\right) \textit{ with } \|f\|^2_{L^2((0,1)^2)} \le \frac{\|f\|^2_{H^1((0,1)^2)}}{2B'} \tag{2.6}$$

and

$$|f|^2_{B,p} \lesssim |f|^2_{H^1((0,1)^2)}, \quad \textit{for all } f \in H^1_0\left((0,1)^2\right). \tag{2.7}$$

Proof. By (2.3) and (2.5), there exist $c_1, c_2 > 0$ such that, for all $f \in H^1(\mathbb{R}^2)$,

$$c_1\left(|f|^2_{H^1(\mathbb{R}^2)} - \frac{B}{c_1}\|f\|^2_{L^2(\mathbb{R}^2)}\right) \le \sum_{\iota=-1,1}\int_{\mathbb{R}^2}\int_{-\Delta}^{\Delta}\int_0^{\Gamma} a^{-2}|\langle f, \psi^{\omega}_{a,s,t,\iota}\rangle|^2 a^{-3}\,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t \le c_2|f|^2_{H^1(\mathbb{R}^2)}. \tag{2.8}$$

For a function $f \in H^1_0((0,1)^2)$ and $N_1, N_2 \in \mathbb{N}$ with $N_1 < N_2$, we define $f_{[N_1,N_2]}(x) := f(x - \lfloor x \rfloor)$ for all $x \in [N_1, N_2]^2$ and 0 otherwise, where the floor function $\lfloor . \rfloor$ is applied component-wise. It can easily be verified that $f_{[N_1,N_2]} \in H^1_0((N_1,N_2)^2)$ and $|f_{[N_1,N_2]}|^2_{H^1} = (|N_2|^2 - |N_1|^2)|f|^2_{H^1}$. Moreover, it is not hard to see that for $N_1 \le M_1 \le M_2 \le N_2$ we have that

$$\left|f_{[N_1,N_2]} - f_{[M_1,M_2]}\right|^2_{H^1} \le ((N_2 - N_1)(M_1 - N_1) + (N_2 - N_1)(N_2 - M_2))|f|^2_{H^1}. \tag{2.9}$$

Without the assumption on the boundary condition for f, the conclusion $f_{[N_1,N_2]} \in H^1_0((N_1,N_2)^2)$ would not hold and the proof would fail.

It is not hard to see that there exists an $R := R(\Gamma, \Delta) \in \mathbb{N}$ such that $\operatorname{supp} \psi_{a,s,t,\iota} \subset B_R(t)$ for all $(a,s,t,\iota) \in (0,\Gamma] \times [-\Delta,\Delta] \times \mathbb{R}^2 \times \{-1,1\}$. For $f \in H^1_0((0,1)^2)$, we have

$$\begin{aligned} |f|^2_{B,p} &= \sum_{\iota=-1,1}\int_0^{\Gamma}\int_{-\Delta}^{\Delta} a^{-2}\int_{(0,1)^2}\left|\left\langle f, \psi^{\omega,per}_{a,s,t,\iota}\right\rangle_{L^2(\mathbb{R}^2)}\right|^2 a^{-3}\,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t \\ &= \sum_{\iota=-1,1}\int_0^{\Gamma}\int_{-\Delta}^{\Delta} a^{-2}\int_{(0,1)^2}\left|\left\langle f_{[-R,1+R]}, \psi^{\omega}_{a,s,t,\iota}\right\rangle_{L^2(\mathbb{R}^2)}\right|^2 a^{-3}\,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t =: \mathrm{I}. \end{aligned} \tag{2.10}$$

By the shift invariance of the shearlet system, we observe that

$$\begin{aligned} \mathrm{I} = &\frac{1}{N^2}\sum_{\iota=-1,1}\int_0^{\Gamma}\int_{-\Delta}^{\Delta} a^{-2}\int_{[0,N]^2}\left|\left\langle f_{[-R,N+R]}, \psi^{\omega}_{a,s,t,\iota}\right\rangle_{L^2(\mathbb{R}^2)}\right|^2 a^{-3}\ \mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a \\ = &\frac{1}{N^2}\sum_{\iota=-1,1}\int_0^{\Gamma}\int_{-\Delta}^{\Delta} a^{-2}\int_{\mathbb{R}^2}\left|\left\langle f_{[-R,N+R]}, \psi^{\omega}_{a,s,t,\iota}\right\rangle_{L^2(\mathbb{R}^2)}\right|^2 a^{-3}\ \mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a \\ &- \frac{1}{N^2}\sum_{\iota=-1,1}\int_0^{\Gamma}\int_{-\Delta}^{\Delta} a^{-2}\int_{\mathbb{R}^2\setminus[0,N]^2}\left|\left\langle f_{[-R,N+R]}, \psi^{\omega}_{a,s,t,\iota}\right\rangle_{L^2(\mathbb{R}^2)}\right|^2 a^{-3}\ \mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a =: \mathrm{II}. \end{aligned}$$

We proceed by using that $\langle f_{R,N-R}, \psi_{a,s,t,\iota}\rangle = 0$, for all $t \notin [0,N]^2$ since $\operatorname{supp} \psi_{a,s,t,\iota} \cap [R, N-R]^2 = \varnothing$. Moreover, by the seminorm equivalence of (2.8), $c_2|\cdot|^2_{H^1} \ge |\cdot|^2_{B,\mathbb{R}^2}$ and $|\cdot|^2_{B,\mathbb{R}^2} \ge c_1(|\cdot|^2_{H^1} - B/c_1\|\cdot\|^2_{L^2(\mathbb{R}^2)})$.

We compute that

$$
\begin{aligned}
\mathrm{II} &= \frac{1}{N^2}\sum_{\iota=-1,1}\int_0^\Gamma\int_{-\Delta}^{\Delta} a^{-2}\int_{\mathbb{R}^2}\left|\left\langle f_{[-R,N+R]},\psi^\omega_{a,s,t,\iota}\right\rangle_{L^2(\mathbb{R}^2)}\right|^2 a^{-3}\ \mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a\\
&\quad-\frac{1}{N^2}\sum_{\iota=-1,1}\int_0^\Gamma\int_{-\Delta}^{\Delta} a^{-2}\int_{\mathbb{R}^2}\left|\left\langle f_{[-R,N+R]}-f_{[R,N-R]},\psi^\omega_{a,s,t,\iota}\right\rangle_{L^2(\mathbb{R}^2)}\right|^2 a^{-3}\ \mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a\\
&\geq \frac{c_1}{N^2}\left(|f_{[-R,N+R]}|^2_{H^1(\mathbb{R}^2)}-\frac{B}{c_1}\|f_{[-R,N+R]}\|^2_{L^2(\mathbb{R}^2)}\right)-\frac{c_2}{N^2}|f_{[-R,N+R]}-f_{[R,N-R]}|^2_{H^1(\mathbb{R}^2)}\\
&= \frac{c_1((N+2R)^2)}{N^2}\left(|f_{[0,1]}|^2_{H^1((0,1)^2)}-\frac{B}{c_1}\|f_{[0,1]^2}\|^2_{L^2(\mathbb{R}^2)}\right)-\frac{c_2}{N^2}|f_{[-R,N+R]}-f_{[R,N-R]}|^2_{H^1(\mathbb{R}^2)}\\
&\geq \frac{c_1((N+2R)^2)}{2N^2}|f_{[0,1]}|^2_{H^1((0,1)^2)}-\frac{2(N+2R)2Rc_2}{2N^2}|f_{[0,1]}|^2_{H^1(\mathbb{R}^2)},
\end{aligned}
$$

where the last line holds if $\|f_{[0,1]}\|^2_{L^2(\mathbb{R}^2)}\leq c_1|f_{[0,1]}|^2_{H^1((0,1)^2)}/(2B)$ and by (2.9). Since N was arbitrary, this completes the proof of the lower bound of the Besov seminorm by the Sobolev seminorm. The upper bound follows trivially from the norm equivalence on $\mathbb{R}^2$, equation (2.10), and $|f_{[0,2]}|_{H^1(\mathbb{R}^2)}\leq 2|f|_{H^1((0,1)^2)}$. $\square$

2.3 The shearlet-based Ginzburg–Landau energy

We start by recalling the definition and some properties of the classical anisotropic Ginzburg–Landau energy. Let Ω be a norm on $\mathbb{R}^2$ and $\mathcal{W}(u)=u^2(1-u)^2$; then, for $u\in BV$,

$$
\mathrm{GL}^\Omega_\varepsilon(u):=\begin{cases}\varepsilon\int_{(0,1)^2}[\Omega(\nabla_x u)]^2\ \mathrm{d}x+\frac{1}{4\varepsilon}\int_{(0,1)^2}\mathcal{W}(u)(x)\ \mathrm{d}x, & \text{if } u\in H^1((0,1)^2),\\ \infty, & \text{if } u\in BV\setminus H^1((0,1)^2).\end{cases}
$$

The Ginzburg–Landau energies as defined above Γ-converge to an anisotropic perimeter functional for $\varepsilon\to 0$, see [2, Theorem 4.13]. Let us recall the definition of Γ-convergence before making the previous statement more precise.

For a space X, we say that a sequence of functionals $F_\varepsilon : X\to[-\infty,\infty]$ Γ-converges to a functional $F : X\to[-\infty,\infty]$ for $\varepsilon\to 0$, if for all $u\in X$ and all sequences $(u_\varepsilon)_{\varepsilon>0}\subset X$ such that $u_\varepsilon\to u$ we have that

$$
\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon)\geq F(u) \tag{2.11}
$$

and there exists a *recovery sequence* $(\widetilde{u}_\varepsilon)_{\varepsilon>0}\subset X$ with $\widetilde{u}_\varepsilon\to u$ such that

$$
\limsup_{\varepsilon\to 0} F_\varepsilon(\widetilde{u}_\varepsilon)\leq F(u).
$$

For a norm Ω on $\mathbb{R}^2$, we define, the *perimeter functional*

$$
P_\Omega(u):=\begin{cases} c\int_{S(u)}\Omega(\vec{n}_x)\,\mathrm{d}\mathbb{H}_1, & \text{if } u\in SBV\left((0,1)^2\right) \text{ and } u\in\{0,1\} \text{ a.e.,}\\ \infty, & \text{otherwise,}\end{cases}
$$

where $c:=\int_0^1\sqrt{\mathcal{W}(s)}ds$, $S(u)$ denotes the jump-set of u and $\vec{n}_x$ is the measure theoretic outer normal of $S(u)$ at x. As already previously announced, we have that $\mathrm{GL}^\Omega_\varepsilon$ Γ-converges to P_Ω.

Next, we aim at constructing a corresponding shearlet-based Ginzburg–Landau energy. We begin by placing some assumptions on the underlying shearlet systems and the weight ω.

Assumption 2.6. *We require the generator function* $\psi\in L^2(\mathbb{R}^2)$ *to satisfy* (2.1) *and* $\operatorname{supp}\psi\subset[0,1]^2$. *Additionally, let* $\Delta^*>0$ *and* $\Gamma^*>0$ *be such that the shearlet transform with* $\Delta>\Delta^*$ *and* $\Gamma>\Gamma^*$ *satisfies* (2.5).

For any norm Ω in $\mathbb{R}^2$, we say that ω is an *associated directional weight to* Ω if

$$\max\{|\vec{n}_1|, |\vec{n}_2|\}\left(\vec{n}_1\omega\left(1, \frac{\vec{n}_2}{\vec{n}_1}\right)^2 + \vec{n}_2\omega\left(-1, \frac{\vec{n}_1}{\vec{n}_2}\right)^2\right) := \Omega(\vec{n})^2, \quad \text{for all } \vec{n} \in \mathbb{S}^1.$$

where we define $\omega(\pm 1, \pm\infty) := 0$. We define the following space of *tame functions*.

Definition 2.7. *Let* Ω *be a norm on* $\mathbb{R}^2$ *and let* $|\cdot|_{B,p}$ *be a shearlet-based Besov seminorm from a weighted shearlet system with weight associated to* Ω*. We define, for* $U(x) := x/\log_2(2+x)$*, the set*

$$\mathcal{B}_\Omega := \left\{ u \in H^1(\Omega) : |u|_{B,p}^2 \geq \int_{(0,1)^2} \Omega(\nabla u(x))^2 \, \mathrm{d}x - U(|u|_{H^1}^2) \right\}.$$

This is the set of functions that we will use to define the shearlet-based Ginzburg–Landau energy. Admittedly, the definition of $\mathcal{B}_\Omega$ is very specific to the shearlet system. Nonetheless, we will observe that $\mathcal{B}_\Omega$ contains all functions that admit a smooth phase-transition along a polygonal curve as well as smooth pertubations of such functions. For a more detailed analysis of this set, we refer to Subsection 2.8.

Definition 2.8. *Let* Ω *be a norm on* $\mathbb{R}^2$ *and let* $|\cdot|_{B,p}$ *be a shearlet-based Besov seminorm from a weighted shearlet system with weight associated to* Ω*. We define, for* $u \in BV$*,*

$$\mathrm{SGL}_\varepsilon^\omega(u) := \begin{cases} \varepsilon|u|_{B,p}^2 + \frac{1}{4\varepsilon}\int_{(0,1)^2} \mathcal{W}(u)(x) \, \mathrm{d}x, & \text{if } u \in \mathcal{B}_\Omega, \\ \infty, & \text{if } u \in BV \setminus \mathcal{B}_\Omega. \end{cases} \tag{2.12}$$

It turns out that the sequence of energies $\mathrm{SGL}_\varepsilon^\omega$ described above Γ-converges to a perimeter functional with norm Ω, where Ω is such that ω is the associated directional weight.

Theorem 2.9. *Let* Ω *be a norm on* $\mathbb{R}^2$ *and* $|\cdot|_{B,p}$ *a shearlet-based Besov seminorm with a shearlet system satisfying Assumption 2.6 and*

$$|\widehat{\phi^1}(\xi)| \lesssim (1+|\xi|)^{-4} \quad \text{and} \quad |\widehat{\psi^1}(\xi)| \lesssim \frac{\min\{|\xi|, 1\}^4}{(1+|\xi|)^4}, \quad \text{for all } \xi \in \mathbb{R}.$$

Moreover, let the directional weight of $|\cdot|_{B,p}$ *be associated to* Ω*. Then, for all* $u \in SBV$*,* $u(x) \in \{0,1\}$ *for almost every* $x \in (0,1)^2$*, and* $(u_\varepsilon)_{\varepsilon>0} \subset H^1((0,1)^2)$ *such that* $u_\varepsilon \to u$ *in* L^1*, we have*

$$\liminf_{\varepsilon\to 0} \mathrm{SGL}_\varepsilon^\omega(u_\varepsilon) \geq P_\Omega(u).$$

Additionally, for every set of finite perimeter $D \subset (0,1)^2$ *we have that there exists a sequence* $(\widetilde{u}_\varepsilon)_{\varepsilon>0} \subset H_0^1((0,1)^2)$ *such that* $\widetilde{u}_\varepsilon \to \chi_D$ *in* L^1 *and*

$$\lim_{\varepsilon\to 0} \mathrm{SGL}_\varepsilon^\omega(\widetilde{u}_\varepsilon) = P_\Omega(\chi_D).$$

Proof. We shall develop the proof in the remainder of the paper. The liminf condition will be verified in Proposition 2.10 and the existence of a recovery sequence will be demonstrated in Proposition 2.15. ☐

2.4 The liminf condition

The definitions of $\mathcal{B}_\Omega$ and $\mathrm{SGL}_\varepsilon^\omega$ are quite convenient as they immediately allow us to obtain the liminf inequality, equation (2.11), for the shearlet-based Ginzburg–Landau energy.

Proposition 2.10. *Let* $(u_\varepsilon)_{\varepsilon>0} \subset H^1((0,1)^2)$*,* $u \in SBV$*,* $u(x) \in \{0,1\}$ *for almost every* $x \in [0,1]^2$*, and let* $u_\varepsilon \to u$ *in* $L^1((0,1)^2)$ *as* $\varepsilon \to 0$*; then,*

$$\liminf_{\varepsilon>0} \mathrm{SGL}_\varepsilon^\omega(u_\varepsilon) \geq P_\Omega(u). \tag{2.13}$$

Proof. If $u \in H^1((0,1)^2)$, then the jump-set $S(u)$ satisfies $S(u) = \varnothing$. Hence $P_\Omega(u) = 0$ and therefore (2.13) holds. If $u \notin H^1((0,1)^2)$, then, since $u_\varepsilon \to u$ in $L^1((0,1)^2)$, we have that $|u_\varepsilon|_{H^1} \to \infty$ as $\varepsilon \to 0$. If $\liminf_{\varepsilon>0} \mathrm{SGL}^\omega_\varepsilon(u_\varepsilon) = \infty$, then (2.13) holds. We assume that $\liminf \mathrm{SGL}^\omega_\varepsilon(u_\varepsilon) < \infty$. For any subsequence $(u_{\varepsilon'})_{\varepsilon'>0}$ such that $\sup_{\varepsilon'>0} \mathrm{SGL}^\omega_{\varepsilon'}(u_{\varepsilon'}) < \infty$ we have by the definition of $\mathrm{SGL}^\omega_{\varepsilon'}$ that

$$\varepsilon'|u_{\varepsilon'}|^2_{B,p} + \frac{1}{4\varepsilon'}\int_{(0,1)^2} \mathcal{W}(u_{\varepsilon'})(x)\,\mathrm{d}x \geq \mathrm{GL}_{\varepsilon'}(u_{\varepsilon'}) - \varepsilon' U(|u_{\varepsilon'}|^2_{H^1}). \tag{2.14}$$

Let $(u_{\varepsilon'})_{\varepsilon'>0}$ be any subsequence of $(u_\varepsilon)_{\varepsilon>0}$ such that $\varepsilon' U(|u_{\varepsilon'}|^2_{H^1}) \to 0$ for $\varepsilon' \to 0$. Then, by (2.14),

$$\liminf_{\varepsilon'>0} \varepsilon'|u_{\varepsilon'}|^2_{B,p} + \frac{1}{4\varepsilon'}\int_{(0,1)^2} \mathcal{W}(u_{\varepsilon'})(x)\,\mathrm{d}x \geq \liminf_{\varepsilon'>0} \mathrm{GL}(u_{\varepsilon'}) \geq P_\Omega(u).$$

If $(u_{\varepsilon'})_{\varepsilon'>0}$ is a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ such that $\varepsilon' U(|u_{\varepsilon'}|^2_{H^1}) \not\to 0$ as $\varepsilon' \to \infty$, then we have by the definition of U that $U(x) = o(x)$ for $x \to \infty$ and hence $\liminf_{\varepsilon'>0} \varepsilon'|u_{\varepsilon'}|^2_{H^1} \to \infty$. Let $(u_{\varepsilon''})_{\varepsilon''>0}$ be a subsequence of $(u_{\varepsilon'})_{\varepsilon'>0}$ such that $\lim_{\varepsilon''\to 0} \mathrm{SGL}^\omega_{\varepsilon''}(u_{\varepsilon''}) = \liminf_{\varepsilon'>0} \mathrm{SGL}^\omega_{\varepsilon'}(u_{\varepsilon'})$.

If $\limsup_{\varepsilon''>0} \|u_{\varepsilon''}\|_{L^2} < \infty$, then we conclude by the seminorm equivalence of (2.6), that $\lim_{\varepsilon''\to 0} \varepsilon''|u_{\varepsilon''}|^2_{B,p} \to \infty$ and hence $\lim_{\varepsilon''\to 0} \mathrm{SGL}^\omega_{\varepsilon''}(u_{\varepsilon''}) = \infty$.

If, on the other hand $\limsup_{\varepsilon''>0} \|u_{\varepsilon''}\|_{L^2} = \infty$, then we define $R_{\varepsilon''} \coloneqq \{x \in (0,1)^2 : |u_{\varepsilon''}(x)| < 2\}$ and obtain that

$$\infty = \limsup_{\varepsilon''>0} \|u_{\varepsilon''}\|_{L^2} \leq \limsup_{\varepsilon''>0} \|u_{\varepsilon''}\chi_{R_{\varepsilon''}}\|_{L^2} + \|u_{\varepsilon''}\chi_{(0,1)^2\setminus R_{\varepsilon''}}\|_{L^2}. \tag{2.15}$$

Since $(u_{\varepsilon''})_{\varepsilon''>0}$ converges in $L^1((0,1)^2)$, we have that

$$\limsup_{\varepsilon''>0} \|u_{\varepsilon''}\chi_{R_{\varepsilon''}}\|^2_{L^2} \leq 2\limsup_{\varepsilon''>0} \|u_{\varepsilon''}\|^2_{L^1} < \infty.$$

As a consequence of (2.15), we conclude

$$\limsup_{\varepsilon''>0} \|u_{\varepsilon''}\chi_{(0,1)^2\setminus R_{\varepsilon''}}\|_{L^2} = \infty$$

and hence

$$\begin{aligned}\int_{(0,1)^2} \mathcal{W}(u_{\varepsilon''})(x)\ \mathrm{d}x &\geq \int_{(0,1)^2\setminus R_{\varepsilon''}} u_{\varepsilon''}(x)^2(1-u_{\varepsilon''}(x))^2\ \mathrm{d}x \\ &\geq \int_{(0,1)^2\setminus R_{\varepsilon''}} |u_{\varepsilon''}(x)|^2\ \mathrm{d}x \to \infty \text{ for } \varepsilon'' \to \infty,\end{aligned}$$

at least up to a subsequence. This implies that $\lim_{\varepsilon''\to 0} \mathrm{SGL}^\omega_{\varepsilon''}(u_{\varepsilon''}) = \infty$. Hence, we conclude that $\liminf_{\varepsilon'>0} \mathrm{SGL}^\omega_{\varepsilon'}(u_{\varepsilon'}) = \infty$ contradicting the assumption $\liminf_{\varepsilon>0} \mathrm{SGL}^\omega_\varepsilon(u_\varepsilon) < \infty$. ☐

We see that the definition of $\mathcal{B}_\Omega$ together with the convergence of the classical Ginzburg–Landau energy, directly yields the liminf condition of the Γ-convergence for $(SGL^\omega_\varepsilon)_{\varepsilon>0}$. The existence of a recovery sequence on the other hand, necessitates a more refined analysis of the behavior of $\mathrm{SGL}^\omega_\varepsilon$ and in particular $|\cdot|_{B,p}$. We will start by analysing $|\cdot|_{B,p}$ for characteristic functions of sets with linear boundaries, then extend the estimate to characteristic sets of polygons and finally use a density argument to conclude the behavior for characteristic functions of sets of finite perimeter.

2.5 Estimating the shearlet-based Ginzburg–Landau energy for Heaviside functions

Let $H \coloneqq \chi_{\mathbb{R}^-\times\mathbb{R}}$ be the *vertical Heaviside function* and $\widetilde{H} \coloneqq \chi_{\mathbb{R}\times\mathbb{R}^-}$ be the *horizontal Heaviside function*. We define for $\vec{n} \in \mathbb{S}^1$ the *Heaviside function with normal* $\vec{n}$ by

$$H^{\vec{n}}(x) \coloneqq \begin{cases} H(S_{\vec{n}_2/\vec{n}_1}x), & \text{if } \left|\frac{\vec{n}_2}{\vec{n}_1}\right| \leq 1, \\ \widetilde{H}(\widetilde{S}_{\vec{n}_1/\vec{n}_2}x), & \text{if } \left|\frac{\vec{n}_1}{\vec{n}_2}\right| < 1, \end{cases} \quad \text{for } x \in \mathbb{R}^2.$$

One readily verifies that $H^{\vec{n}}$ is well-defined and has a linear jump with normal $\vec{n}$. Note, that if $\vec{n}_2/\vec{n}_1 > 0$, then $H(S_{\vec{n}_2/\vec{n}_1}x) = \chi_{x_1+\vec{n}_2/\vec{n}_1 x_2 \le 0} = \widetilde{H}(\widetilde{S}_{\vec{n}_1/\vec{n}_2}x)$. If $\vec{n}_2/\vec{n}_1 < 0$, then we instead have that $H(S_{\vec{n}_2/\vec{n}_1}x) = \chi_{x_1+\vec{n}_2/\vec{n}_1 x_2 \le 0} = 1 - \widetilde{H}(\widetilde{S}_{\vec{n}_1/\vec{n}_2}x)$ almost everywhere. In the sequel, we always analyse inner products of $H^{\vec{n}}(x)$ with functions ψ that have vanishing moments and by the previous considerations we will always have $\langle H(S_{\vec{n}_2/\vec{n}_1}), \psi\rangle = \langle \widetilde{H}(\widetilde{S}_{\vec{n}_1/\vec{n}_2}\cdot), \psi\rangle$ whenever $\vec{n}_1, \vec{n}_2 \neq 0$. Additionally, we will be interested in functions that admit a smooth transition along a linear jump. A function $\Xi : [-1/2, 1/2] \to [-1/2, 1/2]$ is called a *transition profile* if

$$\Xi \in C^1\left(\left[\frac{1}{2}, \frac{1}{2}\right]\right), \quad \Xi\left(-\frac{1}{2}\right) = -\frac{1}{2}, \quad \text{and} \quad \Xi\left(\frac{1}{2}\right) = \frac{1}{2}.$$

Prototypes of functions with smooth transitions along linear jumps can now be defined as follows: for $\varepsilon > 0$, $w > 0$, and $x \in \mathbb{R}^2$ let

$$H_{\varepsilon,\Xi,w}(x) := \begin{cases} 1, & \text{if } x_1 < -\frac{w\varepsilon}{2}, \\ \frac{1}{2} - \Xi\left(\frac{x_1}{w\varepsilon}\right), & \text{if } -\frac{w\varepsilon}{2} \le x_1 \le \frac{w\varepsilon}{2}, \\ 0, & \text{if } \frac{w\varepsilon}{2} < x_1, \end{cases}$$

$$\widetilde{H}_{\varepsilon,\Xi,w}(x) := \begin{cases} 1, & \text{if } x_2 < -\frac{w\varepsilon}{2}, \\ \frac{1}{2} - \Xi\left(\frac{x_2}{w\varepsilon}\right), & \text{if } -\frac{w\varepsilon}{2} \le x_2 \le \frac{w\varepsilon}{2}, \\ 0, & \text{if } \frac{w\varepsilon}{2} < x_2. \end{cases}$$

In the sequel, we also need to analyse functions with smooth transitions along linear jumps that are not parallel to the horizontal or vertical axes. To model this, we rotate the functions $H_{\varepsilon,\Xi,w}$ and $\widetilde{H}_{\varepsilon,\Xi,w}$ described above, but to make the analysis in the sequel simpler, we present a construction based on shearing instead of rotation. We define, for $x \in \mathbb{R}^2$,

$$H^{\vec{n},1}_{\varepsilon,\Xi,w}(x) := H_{\varepsilon,\Xi,w/|\vec{n}_1|}(S_{\vec{n}_2/\vec{n}_1}x), \text{ if } \vec{n}_1 \neq 0, \quad \text{and} \quad H^{\vec{n},-1}_{\varepsilon,\Xi,w}(x) := \widetilde{H}_{\varepsilon,\Xi,w/|\vec{n}_2|}\left(\widetilde{S}_{\vec{n}_1/\vec{n}_2}x\right) \text{ if } \vec{n}_2 \neq 0.$$

Clearly, $H^{\vec{n},1}_{\varepsilon,\Xi,w}$ and $H^{\vec{n},-1}_{\varepsilon,\Xi,w}$ are functions that are constant along directions perpendicular to $\vec{n}$ and behave like $\Xi(\cdot/(w\varepsilon))$ or $\Xi(-\cdot/(w\varepsilon))$ along $\vec{n}$, depending on the sign of $\vec{n}$. This implies that for all functions ψ with a vanishing moment we have

$$\left\langle H^{\vec{n},1}_{\varepsilon,\Xi,w}, \psi\right\rangle = \left\langle 1 - H^{\vec{n},-1}_{\varepsilon,\widetilde{\widetilde{\Xi}},w}, \psi\right\rangle = \left\langle H^{\vec{n},-1}_{\varepsilon,\widetilde{\widetilde{\Xi}},w}, \psi\right\rangle,$$

as long as $\vec{n}_1, \vec{n}_2 \neq 0$, where $\widetilde{\widetilde{\Xi}} = \Xi$ or $\widetilde{\widetilde{\Xi}} = 1 - \Xi(-\cdot)$ depening on $\vec{n}$. The exact form of $\widetilde{\widetilde{\Xi}}$ will never appear in the sequel. The only relation needed is that

$$\left|H^{\vec{n},-1}_{\varepsilon,\widetilde{\widetilde{\Xi}},w}\right|_{H^1} = \left|H^{\vec{n},-1}_{\varepsilon,\Xi,w}\right|_{H^1}.$$

Finally, we define, for $x \in \mathbb{R}^2$,

$$H^{\vec{n}}_{\varepsilon,\Xi,w}(x) := \begin{cases} H^{\vec{n},1}_{\varepsilon,\Xi,w}(x), & \text{if } \left|\frac{\vec{n}_2}{\vec{n}_1}\right| \le 1, \\ H^{\vec{n},-1}_{\varepsilon,\Xi,w}(x), & \text{if } \left|\frac{\vec{n}_1}{\vec{n}_2}\right| < 1. \end{cases} \tag{2.16}$$

It is not hard to see, considering the lengh of the jump curve and the rotation invariance of the H^1 seminorm, that

$$\left|H^{\vec{n}}_{\varepsilon,\Xi,w}\right|^2_{H^1((-r,r)^2)} = \sqrt{\left(1 + \min\left\{\left|\frac{\vec{n}_2}{\vec{n}_1}\right|^2, \left|\frac{\vec{n}_1}{\vec{n}_2}\right|^2\right\}\right)}\left|H^{(1,0)}_{\varepsilon,\Xi,w}\right|^2_{H^1((-r,r)^2)} + \mathcal{O}(\varepsilon). \tag{2.17}$$

Additionally, we observe that, for $n \neq 0$ and $\varepsilon > 0$ sufficiently small

$$\left|H^{(1,0)}_{\varepsilon,\Xi,w/|n|}\right|^2_{H^1((-r,r)^2)} = |n|\left|H^{(1,0)}_{\varepsilon,\Xi,w}\right|^2_{H^1((-r,r)^2)}. \tag{2.18}$$

Understanding the asymptotic behaviour of the shearlet-based Besov seminorm of $H^{\vec{n}}_{\varepsilon,\Xi,w}$ requires a bit more technical work. We will observe below that the shearlet-based Besov seminorm of $H^{\vec{n}}_{\varepsilon,\Xi,w}$ is asymptotically equivalent to the H^1 seminorm. We study the nonperiodic Besov functional here because we will later only apply the following proposition to analyse functions supported away from the boundary of our domain.

Proposition 2.11. *Let $\psi \in L^2(\mathbb{R}^2)$ satisfy Assumption 2.6 and*

$$|\widehat{\phi^1}(\xi)| \lesssim (1+|\xi|)^{-4} \quad \text{and} \quad |\widehat{\psi^1}(\xi)| \lesssim \min\{|\xi|,1\}^4/(1+|\xi|)^4, \quad \text{for all } \xi \in \mathbb{R}^2. \tag{2.19}$$

Let ω be a directional weight and let $r \in (0,1/2)$. We then have that for all $\varepsilon > 0$ with $r/((\Delta+2)w) > \varepsilon$:

$$\begin{aligned}|H^{\vec{n}}_{\varepsilon,\Xi,w}|^2_{B,[-r,r]^2} = \Omega(\vec{n})^2 \left|H^{\vec{n}}_{\varepsilon,\Xi,w}\right|^2_{H^1((-r,r)^2)} \\ + r\, o\left(\frac{1}{\varepsilon}\log_2\left(\frac{1}{\varepsilon}\right)^{-1}\right) + o\left(\frac{1}{\sqrt{\varepsilon}}\right) + \mathcal{O}\left(\frac{1}{(r-w\varepsilon(\Delta+2))^4}\right).\end{aligned} \tag{2.20}$$

where the implied constant in the first asymptotic term depends quadratically on $1+\|\Xi'\|_\infty/w$, and the implied constants in the second and third asymptotic term depend quadratically on $1+\|\Xi'\|_\infty$.

Proof. The proof proceeds in several steps.

Step 1: (Splitting into cones)

First, we split the shearlet-based Besov-functional into two parts, each associated with one of the cones of the shearlet system:

$$\begin{aligned}\mathrm{I}_1^{\varepsilon,\omega,\vec{n}} &:= \int_0^\Gamma \int_{-\Delta}^\Delta \int_{[-r,r]^2} a^{-2} \left|\langle H^{\vec{n}}_{\varepsilon,\Xi,w}, \psi^\omega_{a,s,t,1}\rangle\right|^2 a^{-3}\, \mathrm{d}t\, \mathrm{d}s\, \mathrm{d}a,\\ \mathrm{I}_{-1}^{\varepsilon,\omega,\vec{n}} &:= \int_0^\Gamma \int_{-\Delta}^\Delta \int_{[-r,r]^2} a^{-2} \left|\langle H^{\vec{n}}_{\varepsilon,\Xi,w}, \psi^\omega_{a,s,t,-1}\rangle\right|^2 a^{-3}\, \mathrm{d}t\, \mathrm{d}s\, \mathrm{d}a.\end{aligned}$$

We have that $|H^{\vec{n}}_{\varepsilon,\Xi,w}|^2_{B,[-r,r]^2} = \mathrm{I}_1^{\varepsilon,\omega,\vec{n}} + \mathrm{I}_{-1}^{\varepsilon,\omega,\vec{n}}$. If $\vec{n}_1 \neq 0$, then, by the considerations after the definition of $H^{\vec{n}}_{\varepsilon,\Xi,w}$ in (2.16), we have that

$$\mathrm{I}_1^{\varepsilon,\omega,\vec{n}} = \int_0^\Gamma \int_{-\Delta}^\Delta \int_{[-r,r]^2} a^{-2} \left|\left\langle H_{\varepsilon,\widetilde{\Xi},w/|\vec{n}_1|}(S_{\vec{n}_2/\vec{n}_1}x), \psi^\omega_{a,s,t,1}\right\rangle\right|^2 a^{-3}\, \mathrm{d}t\, \mathrm{d}s\, \mathrm{d}a,$$

where $\widetilde{\Xi} = \Xi$ or $\widetilde{\Xi} = 1-\Xi(-\cdot)$. We assume that $\widetilde{\Xi} = \Xi$. The other case follows similarly. On the other hand, if $\vec{n}_1 = 0$, then $H^{\vec{n}}_\varepsilon$ is constant along the x_1-direction. As all $\psi_{a,s,t,1}$ have a vanishing moment in the x_1-direction, this then implies that $\mathrm{I}_1^{\varepsilon,\omega,\vec{n}} = 0$. Similarly we get that if $\vec{n}_2 \neq 0$ then

$$\mathrm{I}_{-1}^{\varepsilon,\omega,\vec{n}} = \int_0^\Gamma \int_{-\Delta}^\Delta \int_{[-r,r]^2} a^{-2} \left|\left\langle \widetilde{H}_{\varepsilon,\widetilde{\Xi},w/|\vec{n}_2|}\left(\widetilde{S}_{\vec{n}_1/\vec{n}_2}x\right), \psi^\omega_{a,s,t,-1}\right\rangle\right|^2 a^{-3}\, \mathrm{d}t\, \mathrm{d}s\, \mathrm{d}a,$$

where $\widetilde{\Xi} = \Xi$ or $\widetilde{\Xi} = 1-\Xi(-\cdot)$. We assume that $\widetilde{\Xi} = \Xi$, the other case follows similarly. If $\vec{n}_2 = 0$, then $\mathrm{I}_{-1}^{\varepsilon,\omega,\vec{n}} = 0$.

Step 2: (Partial integration)

We define

$$h^1_{\varepsilon,\Xi,w/|\vec{n}_1|} := \frac{\partial}{\partial x_1} H_{\varepsilon,\Xi,w/|\vec{n}_1|} \quad \text{and} \quad h^{-1}_{\varepsilon,\Xi,w/|\vec{n}_2|} := \frac{\partial}{\partial x_2} \widetilde{H}_{\varepsilon,\Xi,w/|\vec{n}_2|},$$

and we denote from now on $\theta := -\vec{n}_2/\vec{n}_1$ to shorten the notation. Setting $\mu := \mu_1 \otimes \phi_1$, with $\widehat{\mu_1}(\xi) := \frac{1}{2\pi i \xi}\widehat{\psi_1}(\xi)$, we have that

$$\left|\widehat{\phi_1}(\xi)\right| \le (1+|\xi|)^{-4} \quad \text{and} \quad |\widehat{\mu_1}(\xi)| \le \frac{\min\{|\xi|,1\}^3}{(1+|\xi|)^4}, \quad \text{for all } \xi \in \mathbb{R}.$$

Next, we compute

$$\begin{aligned}
\mathrm{I}_1^{\varepsilon,\omega,\vec{n}} &= \int_0^\Gamma \int_{-\Delta}^{\Delta} \int_{[-r,r]^2} \left| \left\langle h^1_{\varepsilon,\Xi,w/|\vec{n}_1|}(S_{-\theta}\cdot), \mu^\omega_{a,s,t,1} \right\rangle \right|^2 a^{-3} \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a \\
&= \int_0^\Gamma \int_{-\Delta}^{\Delta} \int_{[-r,r]^2} \left| \left\langle h^1_{\varepsilon,\Xi,w/|\vec{n}_1|}, \mu^{\omega_\theta}_{a,s+\theta,t,1} \right\rangle \right|^2 a^{-3} \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a \\
&= \int_0^\Gamma \int_{-\Delta+\theta}^{\Delta+\theta} \int_{[-r,r]^2} \left| \left\langle h^1_{\varepsilon,\Xi,w/|\vec{n}_1|}, \mu^{\omega_\theta}_{a,s,t,1} \right\rangle \right|^2 a^{-3} \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a,
\end{aligned}$$

where $\omega_\theta(\iota, s) = \omega(\iota, s-\theta)$. Similarly, we obtain that

$$\mathrm{I}_{-1}^{\varepsilon,\omega,\vec{n}} = \int_0^\Gamma \int_{-\Delta+\frac{1}{\theta}}^{\Delta+\frac{1}{\theta}} \int_{[-r,r]^2} \left| \left\langle h^1_{\varepsilon,\Xi,w/|\vec{n}_2|}, \mu^{\omega_{\frac{1}{\theta}}}_{a,s,t,-1} \right\rangle \right|^2 a^{-3} \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a.$$

Step 3: (Removing non-aligned parts)

We have that $h^1_{\varepsilon,\Xi,w/|\vec{n}_1|}(x) = \varepsilon^{-1}|\vec{n}_1|/wg(|\vec{n}_1|x_1/(w\varepsilon))$ for $g(x) = \Xi'(x)$ if $x \in [-1/2, 1/2]$ and $g(x) = 0$ otherwise. Moreover, $\|g\|_{L^1(\mathbb{R})} \leq \|\Xi'\|_\infty$ and thus $\|\hat{g}\|_\infty < \|\Xi'\|_\infty$. The Fourier transform of $h^1_{\varepsilon,\Xi,w/|\vec{n}_1|}$ is

$$\mathcal{F}\left(\frac{\partial}{\partial x_1} H_\varepsilon\right) = \hat{g}\left(\frac{w\varepsilon}{|\vec{n}_1|}(\cdot)_1\right) \otimes \delta_{\xi_2=0},$$

where $\delta_{\xi_2=0}$ denotes the δ-distribution in ξ_2. Parseval's identity shows that

$$\begin{aligned}
\left| \left\langle h^1_{\varepsilon,\Xi,w/|\vec{n}_1|}, \mu_{a,s,t,1} \right\rangle \right| &= a^{\frac{3}{4}} \left| \int_{\mathbb{R}} \hat{g}\left(\frac{w\varepsilon}{\vec{n}_1}\xi\right) \widehat{\mu_1}(a\xi)\widehat{\phi_1}(\sqrt{a}s\xi) \,\mathrm{d}\xi \right| \\
&\leq a^{\frac{3}{4}} \int_{\mathbb{R}} \left| \widehat{\mu}_1(a\xi)\widehat{\phi_1}(\sqrt{a}s\xi) \right| \,\mathrm{d}\xi \cdot \|\Xi'\|_\infty.
\end{aligned}$$

Since $|\widehat{\phi_1}(\xi)| \leq (1+|\xi|)^{-4}$ and $|\widehat{\mu}_1(\xi)| \leq \min\{|\xi|, 1\}^2/(1+|\xi|^2)$, we conclude for $s \neq 0$ that

$$\begin{aligned}
a^{\frac{3}{4}} \int_{\mathbb{R}} \left| \widehat{\mu}_1(a\xi)\widehat{\phi_1}(\sqrt{a}s\xi) \right| \,\mathrm{d}\xi &\leq a^{\frac{3}{4}} \int_{\mathbb{R}} \frac{\min\{|a\xi|, 1\}^3}{(1+|a\xi|)^2(1+|\sqrt{a}s\xi|)^4} \,\mathrm{d}\xi \\
&\leq a^{\frac{1}{4}} s^{-1} \int_{\mathbb{R}} \frac{|s^{-1}\sqrt{a}\xi|^3}{(1+|\xi|)^4} \,\mathrm{d}\xi \lesssim a^{\frac{7}{4}} s^{-3}. \qquad (2.21)
\end{aligned}$$

Next, we remove terms from $\mathrm{I}_1^{\varepsilon,\omega,\vec{n}}$ and $\mathrm{I}_{-1}^{\varepsilon,\omega,\vec{n}}$ that asymptotically decay very fast for $\varepsilon \to 0$. Let $\nu_s > 0$; then we define

$$\begin{aligned}
\mathrm{K}_1^{\varepsilon,\omega,\vec{n},\nu_s} := &\int_0^\Gamma \int_{(-\infty,-\nu_s]} \int_{[-r,r]^2} \left| \left\langle h^1_{\varepsilon,\Xi,w/|\vec{n}_1|}, \mu_{a,s,t,1} \right\rangle \right|^2 a^{-3} \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a \\
&+ \int_0^\Gamma \int_{[\nu_s,\infty)} \int_{[-r,r]^2} \left| \left\langle h^1_{\varepsilon,\Xi,w/|\vec{n}_1|}, \mu_{a,s,t,1} \right\rangle \right|^2 a^{-3} \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a.
\end{aligned}$$

Invoking (2.21), we get that

$$\mathrm{K}_1^{\varepsilon,\omega,\vec{n},\nu_s} \leq 2 \int_0^\Gamma \int_{(-\infty,-\nu_s]} \int_{[-r,r]^2} a^{\frac{1}{2}} s^{-6} \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a \lesssim \nu_s^{-5}.$$

Clearly, the same estimate can be performed for

$$\begin{aligned}
\mathrm{K}_2^{\varepsilon,\omega,\vec{n},\nu_s} := &\int_0^\Gamma \int_{(-\infty,-\nu_s]} \int_{[-r,r]^2} \left| \left\langle h^{-1}_{\varepsilon,\Xi,w/|\vec{n}_2|}, \mu_{a,s,t,-1} \right\rangle \right|^2 a^{-3} \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a \\
&+ \int_0^\Gamma \int_{[\nu_s,\infty)} \int_{[-r,r]^2} \left| \left\langle h^{-1}_{\varepsilon,\Xi,w/|\vec{n}_2|}, \mu_{a,s,t,-1} \right\rangle \right|^2 a^{-3} \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a.
\end{aligned}$$

We set from now on $v_s := \varepsilon^{\frac{1}{11}}$ and observe that $\mathrm{K}_1^{\varepsilon,\omega,\vec{n},\nu_s}, \mathrm{K}_2^{\varepsilon,\omega,\vec{n},\nu_s} = o(\varepsilon^{-1/2})$. Hence, we conclude that

$$\mathrm{I}_1^{\varepsilon,\omega,\vec{n}} = \int_0^\Gamma \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{[-r,r]^2} \left| \left\langle h^1_{\varepsilon,\Xi,w/|\vec{n}_1|}, \mu^{\omega_\theta}_{a,s,t,1} \right\rangle \right|^2 a^{-3} \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a + o\left(\varepsilon^{-\frac{1}{2}}\right).$$

Next, we observe that replacing the directional weight by a scalar only produces an asymptotically negligible error. In fact, by the Bessel property, see Appendix D, we can estimate using the Lipschitz continuity of ω and $\varepsilon^{-10/11} = o(\varepsilon^{-1}\log_2(\varepsilon^{-1})^{-1})$ that

$$\begin{aligned} &\int_0^\Gamma \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{[-r,r]^2} \left| \left\langle h^1_{\varepsilon,\Xi,w/|\vec{n}_1|}, \mu^{\omega_\theta}_{a,s,t,1} \right\rangle \right|^2 a^{-3} \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a \\ &\quad - \int_0^\Gamma \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{[-r,r]^2} \omega(1,-\theta)^2 \left| \left\langle h^1_{\varepsilon,\Xi,w/|\vec{n}_1|}, \mu_{a,s,t,1} \right\rangle \right|^2 a^{-3} \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a \\ &\le \sup_{s\in[-v_s,v_s]} |\omega(1,-\theta)^2 - \omega(1,s-\theta)^2| \cdot \left\| h^1_{\varepsilon,\Xi,w/|\vec{n}_1|} \right\|^2_{L^2((-r,r)^2)} \lesssim v_s\varepsilon^{-1} r = r\cdot o\left(\frac{1}{\varepsilon}\log_2\left(\frac{1}{\varepsilon}\right)^{-1}\right), \end{aligned}$$

where the implied constant depends quadratically on $\|\Xi'\|_\infty/w$. Consequentially, we have that

$$\begin{aligned} \mathrm{I}_1^{\varepsilon,\omega,\vec{n}} = \omega(1,-\theta)^2 \int_0^\Gamma \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{[-r,r]^2} \left| \left\langle h^1_{\varepsilon,\Xi,w/|\vec{n}_1|}, \mu_{a,s,t,1} \right\rangle \right|^2 a^{-3} \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a \\ + r\cdot o\left(\frac{1}{\varepsilon}\log_2\left(\frac{1}{\varepsilon}\right)^{-1}\right) + o\left(\varepsilon^{-\frac{1}{2}}\right). \end{aligned} \tag{2.22}$$

Similarly,

$$\begin{aligned} \mathrm{I}_{-1}^{\varepsilon,\omega,\vec{n}} = \omega(-1,-\theta)^2 \int_0^\Gamma \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{[-r,r]^2} \left| \left\langle h^{-1}_{\varepsilon,\Xi,w/|\vec{n}_2|}, \mu_{a,s,t,-1} \right\rangle \right|^2 a^{-3} \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a \\ + r\cdot o\left(\frac{1}{\varepsilon}\log_2\left(\frac{1}{\varepsilon}\right)^{-1}\right) + o\left(\varepsilon^{-\frac{1}{2}}\right). \end{aligned}$$

Step 4: (Rewrite as non-cone-adapted shearlet transform)

We want to show that $\mathrm{I}_1^{\varepsilon,\omega,\vec{n}}$ is asymptotically a rescaled version of the Sobolev seminorm of $H^{(1,0)}_{\varepsilon,\Xi,w}$. Towards this goal, we plan to invoke (2.1) and use the resulting equivalence of the homogeneous shearlet transform and the L^2-norm. The main obstacle here is to deal with the fact that the integral in (2.22) is taken only over a finite domain. Moreover, simply extending $h^1_{\varepsilon,\Xi,w/|\vec{n}_1|}$ to $\mathbb{R}^2$ does not yield an L^2 function. To overcome these issues, we use a blow-up technique, that was already applied in Proposition 2.5.

We define

$$\mathrm{B}_\varepsilon := \int_0^\Gamma \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{[-r,r]^2} \left| \left\langle h^1_{\varepsilon,\Xi,w/|\vec{n}_1|}, \mu_{a,s,t,1} \right\rangle \right|^2 a^{-3} \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a.$$

Since H_ε is constant in the x_2-direction we conclude that, for all $N \in \mathbb{N}$,

$$\mathrm{B}_\varepsilon = \frac{1}{N}\int_0^\Gamma \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{[-r,r]\times N[-r,r]} \left| \left\langle h^1_{\varepsilon,\Xi,w/|\vec{n}_1|}, \mu_{a,s,t,1} \right\rangle \right|^2 a^{-3} \,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t.$$

We replace $h^1_{\varepsilon,\Xi,w/|\vec{n}_1|}$ by a suitable $L^2(\mathbb{R}^2)$ function by using a mollifier $\gamma \in C^\infty(\mathbb{R})$ with $\operatorname{supp}\gamma \in [-1/2,1/2]$, $\gamma(x) > 0$ for $x \in (-1/2,1/2)$, and $\int_{-1/2}^{1/2}\gamma(x)\,\mathrm{d}x = 1$. Next, we set for $(x_1,x_2) \in \mathbb{R}^2$

$$f_N(x_1,x_2) := \frac{|\vec{n}_1|}{\varepsilon} g\left(\frac{|\vec{n}_1|}{w\varepsilon}x_1\right)\cdot\left(\gamma * \chi_{[-Nr-\Gamma,Nr+\Gamma]}\right)(x_2).$$

It is not hard to see that

$$\mathrm{B}_\varepsilon = \frac{1}{N}\int_0^\Gamma \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{[-r,r]\times N[-r,r]} |\langle f_N, \mu_{a,s,t,1}\rangle|^2 a^{-3} \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a.$$

Next, we need to extend the domain of integration to $\mathbb{R}^2$. We start by extending to $[-r,r]\times\mathbb{R}$. We have that

$$\begin{aligned}\mathrm{B}_\varepsilon = & \frac{1}{N}\int_0^\Gamma \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{\mathbb{R}^2} |\langle f_N, \mu_{a,s,t,1}\rangle|^2 a^{-3} \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a \\ & - \frac{1}{N}\int_0^\Gamma \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{\mathbb{R}^2\setminus((-r,r)\times N(-r,r))} |\langle f_N, \mu_{a,s,t,1}\rangle|^2 a^{-3} \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a. \end{aligned} \tag{2.23}$$

We have that

$$\mathbb{R}^2\setminus((-r,r)\times N(-r,r)) = \big((-r,r)\times(\mathbb{R}\setminus N(-r,r))\big)\cup\big((\mathbb{R}\setminus[-r,r])\times\mathbb{R}\big). \tag{2.24}$$

As $B_\varepsilon = 0$ if $|\theta|\geq\Delta+1$ we assume that $|\theta|\leq\Delta+1$ and hence

$$(\Delta+1)^2 \geq \frac{\vec{n}_2^2}{\vec{n}_1^2} = \frac{1}{\vec{n}_1^2} - 1$$

and thus $\vec{n}_1 \geq 1/(\Delta+2)$. Choose $R>0$ such that supp $\mu_{a,s,t,1}\subset B_{a^{1/2}R}(t)$ for all $(a,s,t)\in[0,\Gamma]\times[1,1]\times\mathbb{R}^2$. Hence we have that $\langle f_N, \mu_{a,s,t,1}\rangle = 0$ if $|t_2| > w\varepsilon/n_1 + R\max\{a^{1/2},a\}$. Thus, we set $\Gamma_{r,\varepsilon} = (r - w\varepsilon(\Delta+2))^2/R^2$ and compute

$$\begin{aligned} & \frac{1}{N}\left|\int_0^\Gamma \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{(\mathbb{R}\setminus[-r,r])\times\mathbb{R}} |\langle f_N, \mu_{a,s,t,1}\rangle|^2 a^{-3} \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a\right| \\ & = \frac{1}{N}\left|\int_{\Gamma_{r,\varepsilon}}^\Gamma \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{(\mathbb{R}\setminus[-r,r])\times\mathbb{R}} |\langle f_N, \mu_{a,s,t,1}\rangle|^2 a^{-3} \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a\right| \\ & \leq \frac{1}{N}\left|\int_{\Gamma_{r,\varepsilon}}^\Gamma \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{\mathbb{R}^2} |\langle f_N, \mu_{a,s,t,1}\rangle|^2 a^{-3} \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a\right|. \end{aligned}$$

We have by Plancherel's and Parseval's identities that

$$\begin{aligned} & \frac{1}{N}\left|\int_{\Gamma_{r,\varepsilon}}^\Gamma \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{\mathbb{R}^2} |\langle f_N, \mu_{a,s,t,1}\rangle|^2 a^{-3} \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a\right| \\ & = \frac{1}{N}\left|\int_{\Gamma_{r,\varepsilon}}^\Gamma \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{\mathbb{R}^2} \left|\left\langle \widehat{f_N}, \widehat{\mu}_{a,s,t,1}\right\rangle\right|^2 a^{-3} \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a\right| \\ & = \frac{1}{N}\int_{\Gamma_{r,\varepsilon}}^\Gamma \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{\mathbb{R}^2} \left|\int_{\mathbb{R}^2} \widehat{f_N}(\xi)\widehat{\mu}_{a,s,0,1}(\xi)\mathrm{e}^{-2\pi i\langle\xi,t\rangle}\,\mathrm{d}\xi\right|^2 a^{-3} \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a \\ & = \frac{1}{N}\int_{\Gamma_{r,\varepsilon}}^\Gamma \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{\mathbb{R}^2} \left|\widehat{f_N}(\xi)\right|^2 |\widehat{\mu}_{a,s,0,1}(\xi)|^2 a^{-3} \,\mathrm{d}\xi\,\mathrm{d}s\,\mathrm{d}a \\ & = \frac{1}{N}\int_{\mathbb{R}^2} \left|\widehat{f_N}(\xi)\right|^2 \int_{\Gamma_{r,\varepsilon}}^\Gamma \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} |\widehat{\mu}_{a,s,0,1}(\xi)|^2 a^{-3}\mathrm{d}s\,\mathrm{d}a\,\mathrm{d}\xi. \end{aligned}$$

A simple computation shows that, for all $\xi\in\mathbb{R}^2$,

$$\widehat{f_N}(\xi) = N\mathrm{sinc}((N+\Gamma)\xi_2)\widehat{\gamma}(\xi_2)\hat{g}\left(\frac{w\varepsilon}{|\vec{n}_1|}\xi_1\right). \tag{2.25}$$

Moreover, we show in Appendix C that, for all $\xi \in \mathbb{R}^2$,

$$\int_{\Gamma_{r,\varepsilon}}^{\Gamma} \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} |\widehat{\mu}_{a,s,0,1}(\xi)|^2 a^{-3} \mathrm{d}s\,\mathrm{d}a \lesssim \frac{1}{\Gamma_{r,\varepsilon}^2} \frac{1}{(1+|\xi_1|)^2}. \tag{2.26}$$

We conclude that

$$\frac{1}{N}\left|\int_{\Gamma_{r,\varepsilon}}^{\Gamma} \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{\mathbb{R}^2} |\langle f_N, \mu_{a,s,t,1}\rangle|^2 a^{-3}\ \mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a\right| = \mathcal{O}\left(\frac{1}{\Gamma_{r,\varepsilon}^2}\right),$$

where the implied constant depends quadratically on $\|\Xi'\|_\infty$. We proceed by estimating the integral over the second set given in (2.24):

$$\frac{1}{N}\left|\int_0^{\Gamma} \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{[-r,r]\times(\mathbb{R}\setminus N[-r,r])} |\langle f_N, \mu_{a,s,t,1}\rangle|^2 a^{-3}\ \mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a\right| =: \mathrm{II}.$$

Since $\operatorname{supp} \psi_{a,s,t,\iota} \subset B_{R\Gamma}(t)$ we conclude that

$$\begin{aligned}
\mathrm{II} &= \frac{1}{N}\left|\int_0^{\Gamma} \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{[-r,r]\times(\mathbb{R}\setminus N[-r,r])} \left|\left\langle f_N \chi_{\mathbb{R}\times[-Nr+R\Gamma,Nr-R\Gamma]^c}, \mu_{a,s,t,1}\right\rangle\right|^2 a^{-3}\ \mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a\right| \\
&\leq \frac{1}{N}\left|\int_0^{\Gamma} \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{\mathbb{R}^2} \left|\left\langle f_N \chi_{\mathbb{R}\times[-Nr+R\Gamma,Nr-R\Gamma]^c}, \mu_{a,s,t,1}\right\rangle\right|^2 a^{-3}\ \mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a\right| \\
&\lesssim \frac{1}{N}\|f_N \chi_{\mathbb{R}\times[-Nr+R\Gamma,Nr-R\Gamma]^c}\|_{L^2}^2,
\end{aligned} \tag{2.27}$$

where the last step is due to the Bessel inequality of the shearlet transform, see Appendix D. It is not hard to see, that $\|f_N \chi_{\mathbb{R}\times[-Nr+R\Gamma,Nr-R\Gamma]^c}\|_{L^2}^2 = \mathcal{O}(1/\varepsilon)$ with the implicit constant independent from N.

By Equations (2.23), (2.24) and (2.26), (2.27), we conclude that

$$B_\varepsilon = \frac{1}{N}\int_0^{\Gamma} \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{\mathbb{R}^2} |\langle f_N, \mu_{a,s,t,1}\rangle|^2 a^{-3}\ \mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a + \mathcal{O}\left(\frac{1}{\Gamma_{r,\varepsilon}^2}\right) + \frac{1}{N}\mathcal{O}(1/\varepsilon).$$

We have by the Plancherel and Parseval identities that

$$\begin{aligned}
&\frac{1}{N}\int_0^{\Gamma} \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{\mathbb{R}^2} |\langle f_N, \mu_{a,s,t,1}\rangle|^2 a^{-3}\ \mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a \\
&= \frac{1}{N}\int_0^{\Gamma} \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{\mathbb{R}^2} \left|\left\langle \widehat{f_N}, \widehat{\mu}_{a,s,t,1}\right\rangle\right|^2 a^{-3}\ \mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a \\
&= \frac{1}{N}\int_0^{\Gamma} \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{\mathbb{R}^2} \left|\int_{\mathbb{R}^2} \widehat{f_N}(\xi)\widehat{\mu}_{a,s,0,1}(\xi)\mathrm{e}^{-2\pi i\langle\xi,t\rangle}\,\mathrm{d}\xi\right|^2 a^{-3}\ \mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a \\
&= \frac{1}{N}\int_0^{\Gamma} \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{\mathbb{R}^2} \left|\widehat{f_N}(\xi)\right|^2 |\widehat{\mu}_{a,s,0,1}(\xi)|^2 a^{-3}\,\mathrm{d}\xi\,\mathrm{d}s\,\mathrm{d}a \\
&= \frac{1}{N}\int_{\mathbb{R}^2} \left|\widehat{f_N}(\xi)\right|^2 \int_0^{\Gamma} \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} |\widehat{\mu}_{a,s,0,1}(\xi)|^2 a^{-3}\mathrm{d}s\,\mathrm{d}a\,\mathrm{d}\xi.
\end{aligned}$$

Two simple but lengthy computations, that we postpone to Appendix C, show that

$$\int_{\Gamma}^{\infty} \int_{\mathbb{R}} |\widehat{\mu}_{a,s,0,1}(\xi)|^2 a^{-3}\mathrm{d}s\,\mathrm{d}a \lesssim (1+|\xi_1|)^{-2} \ \text{ for all } \xi \in \mathbb{R}^2, \tag{2.28}$$

and if $|\nu_s \xi_1|/2 > |\xi_2|$, then

$$\int_0^\Gamma \int_{-\infty}^{-\nu_s} |\widehat{\mu}_{a,s,0,1}(\xi)|^2 a^{-3} \mathrm{d}s\,\mathrm{d}a \lesssim \frac{|\xi_1|^2}{(1+|\nu_s\xi_1|)^4} \quad \text{and} \quad \int_0^\Gamma \int_{\nu_s}^{\infty} |\widehat{\mu}_{a,s,0,1}(\xi)|^2 a^{-3} \mathrm{d}s\,\mathrm{d}a \lesssim \frac{|\xi_1|^2}{(1+|\nu_s\xi_1|)^4}. \quad (2.29)$$

If $|\nu_s\xi_1|/2 \le |\xi_2|$, then since $|\widehat{\gamma}(\xi_2)| \lesssim (1+|\xi_2|)^4$, we can estimate

$$|\widehat{\gamma}(\xi_2)| \le (1+|\xi_2|)^2 \left(1+\nu_s \frac{|\xi_1|}{2}\right)^2. \quad (2.30)$$

The form of the Fourier transform in (2.25) and the estimates (2.29), (2.28) and (2.30), imply that

$$\frac{1}{N}\int_0^\Gamma \int_{\max\{-\Delta+\theta,-\nu_s\}}^{\min\{\Delta+\theta,\nu_s\}} \int_{\mathbb{R}^2} |\langle f_N, \mu_{a,s,t,1}\rangle|^2 a^{-3}\,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t = \frac{1}{N}\int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^2} |\langle f_N, \mu_{a,s,t,1}\rangle|^2 a^{-3}\,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t + \mathcal{O}(\nu_s^{-1}),$$

and thus

$$\mathrm{B}_\varepsilon = \frac{1}{N}\int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^2} |\langle f_N, \mu_{a,s,t,1}\rangle|^2 a^{-3}\,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t + \frac{1}{N} O\left(\frac{1}{\varepsilon}\right) + \mathcal{O}\left(\frac{1}{(r-w\varepsilon(\Delta+2))^4}\right) + \mathcal{O}\left(\varepsilon^{-\frac{1}{11}}\right).$$

Step 5: (Use of Parseval property of homogeneous systems)

By (2.1) we obtain that for every $\varepsilon > 0$

$$\frac{1}{N}\|f_N\|_{L^2}^2 \longrightarrow \mathrm{B}_\varepsilon + \mathcal{O}\left(\frac{1}{(r-w\varepsilon(\Delta+2))^4}\right) + \mathcal{O}\left(\varepsilon^{-\frac{1}{11}}\right) \text{ for } N \to \infty.$$

Moreover, we also observe that

$$\frac{1}{N}\|f_N\|_{L^2}^2 \longrightarrow \left\|h^1_{\varepsilon,\Xi,w/|\vec{n}_1|}\right\|^2_{L^2((-r,r)^2)} \text{ for } N \to \infty.$$

Hence, we conclude that

$$\begin{aligned}\mathrm{B}_\varepsilon =& \left\|h^1_{\varepsilon,\Xi,w/|\vec{n}_1|}\right\|^2_{L^2((-r,r)^2)} + \mathcal{O}\left(\frac{1}{(r-w\varepsilon(\Delta+2))^4}\right) + \mathcal{O}\left(\varepsilon^{-\frac{1}{11}}\right)\\ =& \left|H^{(1,0)}_{\varepsilon,\Xi,w/|\vec{n}_1|}\right|^2_{H^1((-r,r)^2)} + \mathcal{O}\left(\frac{1}{(r-w\varepsilon(\Delta+2))^4}\right) + \mathcal{O}\left(\varepsilon^{-\frac{1}{11}}\right).\end{aligned}$$

We have that

$$\left(1+\left|\min\left\{\frac{\vec{n}_2}{\vec{n}_1},\frac{\vec{n}_1}{\vec{n}_2}\right\}\right|^2\right)^{-1/2} = \max\{|\vec{n}_1|,|\vec{n}_2|\}.$$

Using (2.17) and (2.18) we conclude that

$$\begin{aligned}\mathrm{B}_\varepsilon =& \ |\vec{n}_1|\left(1+\left|\min\left\{\frac{\vec{n}_2}{\vec{n}_1},\frac{\vec{n}_1}{\vec{n}_2}\right\}\right|^2\right)^{-1/2} \left|H^{\vec{n}}_{\varepsilon,\Xi,w}\right|^2_{H^1((-r,r)^2)} + \mathcal{O}\left(\frac{1}{(r-w\varepsilon(\Delta+2))^4}\right) + \mathcal{O}\left(\varepsilon^{-\frac{1}{11}}\right)\\ =& \ \vec{n}_1 \max\{|\vec{n}_1|,|\vec{n}_2|\} \left|H^{\vec{n}}_{\varepsilon,\Xi,w}\right|^2_{H^1((-r,r)^2)} + \mathcal{O}\left(\frac{1}{(r-w\varepsilon(\Delta+2))^4}\right) + \mathcal{O}\left(\varepsilon^{-\frac{1}{11}}\right).\end{aligned}$$

This demonstrates that

$$\begin{aligned}\mathrm{I}_1^{\varepsilon,\omega,\vec{n}} + \mathrm{I}_{-1}^{\varepsilon,\omega,\vec{n}} =& \max\{|\vec{n}_1|,|\vec{n}_2|\}\left(\vec{n}_1\omega\left(1,\frac{\vec{n}_2}{\vec{n}_1}\right)^2 + \vec{n}_2\omega\left(-1,\frac{\vec{n}_1}{\vec{n}_2}\right)^2\right)\left|H^{\vec{n}}_{\varepsilon,\Xi,w}\right|^2_{H^1((-r,r)^2)}\\ &+ r\, o\left(\frac{1}{\varepsilon}\log_2\left(\frac{1}{\varepsilon}\right)^{-1}\right) + \mathcal{O}\left(\frac{1}{(r-w\varepsilon(\Delta+2))^4}\right) + o\left(\frac{1}{\sqrt{\varepsilon}}\right),\end{aligned}$$

completing the proof. □

The factor of $|\vec{n}_1|^2$ or $|\vec{n}_2|^2$ in (2.20) counteracts a certain stretching effect of the shearing operation. Indeed, as we have seen in the proof above, the shearlet-based seminorm is, apart from the weight, invariant to shearing. On the other hand, the H^1 seminorm clearly is not, since shearing stretches one direction stronger than the other.

2.6 Construction of a recovery sequence for polygons

We proceed by analysing the asymptotic behavior of the shearlet-based Ginzburg–Landau energy for functions that admit a smooth phase-transition over a polygonal curve. We start by giving a precise definition of polygons, then we introduce functions that admit a smooth phase-transition over a polygonal curve. After that we describe the asymptotic behavior of the shearlet-based Besov seminorm of these functions. At the end of the section we shall describe how this yields a sequence of functions with a smooth phase-transition over a polygonal curve such that the associated shearlet-based Ginzburg–Landau energy evaluated on these functions converges to an anisotropic perimeter functional.

Definition 2.12. *For $N \in \mathbb{N}$, a closed set $P \subset (0,1)^2$ such that ∂P is a piecewise affine, non-self-intersecting curve with N pieces is called N*-gon*. The N-points where the boundary curve is not affine are called vertices. We denote them by $x_1^P, \dots, x_N^P$ and we will always assume that they are ordered so that the line between x_i^P and x_{i+1}^P is in ∂P for all $i = 1, \dots, N$, where $x_{N+1}^P \coloneqq x_1^P$. The identification $x_{N+1}^P = x_1^P$ will also be used in the sequel without additional comments.*

The length of the boundary curve of P is then

$$\ell(P) \coloneqq \sum_{i=1,\dots,N} |x_i^P - x_{i+1}^P|.$$

If there is no need to explicitly specify the number N, then we shall simply call such a set a polygon.

To make precise what we mean by functions with transitions along polygonal curves, we first need to introduce a more general notion of transition profiles. A continuously differentiable function $W : \mathbb{S}^1 \to \mathbb{R}^+$ is called *directional transition width*. A continuously differentiable function $\Xi : \mathbb{S}^1 \times [-1/2, 1/2] \to [-1/2, 1/2]$, such that, for each $\eta \in \mathbb{S}^1$, $\Xi(\eta, \cdot)$ is a transition profile, is called *directional transition profile*.

For a polygon $P \subset (0,1)^2$ with vertices $x_1, \dots, x_N$ there exists a normal map

$$\partial P \ni x \mapsto \vec{n}_x \in \mathbb{S}^1,$$

which is well-defined everywhere except at $x_1, \dots, x_N$. We replace this normal map by an auxiliary map which is well-defined everywhere and coincides with $\vec{n}_x$ at every point, except in a neighbourhood of $x_1, \dots, x_N$.

For $\varepsilon > 0$ and a directional transition width W, we pick a function $\vec{n}_x^{\varepsilon,W}$ such that

$$\partial P \ni x \mapsto \vec{n}_x^{\varepsilon,W} \in \mathbb{S}^1,$$

is smooth with derivative bounded by π/ε, $\vec{n}_x = \vec{n}_x^{\varepsilon,W}$ whenever $|x - x_i| > \varepsilon \max\{1, \|W\|_\infty\}$ for all $i = 1, \dots, N$.

We then consider the projection operator

$$\pi : (0,1)^2 \to \partial P : x \mapsto \operatorname*{arg\,min}_{y \in \partial P} W\left(\frac{x-y}{|x-y|}\right)|x-y|,$$

which is well-defined almost everywhere. The reason we focus on polygons with directional transition profiles is that precisely these functions form recovery sequences for the classical anisotropic Ginzburg–Landau energy as analysed in [2, Proposition 4.10].

For a polygon P, a directional transition profile Ξ, and a directional transition width W we now define

$$P_{\varepsilon,\Xi,W} \coloneqq \begin{cases} \frac{1}{2} - \Xi\left(\vec{n}_{\pi(x)}^{\varepsilon,W},\ \frac{\pi(x)-x}{\varepsilon}/W\left(\vec{n}_{\pi(x)}^{\varepsilon,W}\right)\right) & \text{if } |x-\pi(x)| < \frac{1}{2}\varepsilon W\left(\vec{n}_{\pi(x)}^{\varepsilon,W}\right), x \in P, \\ \frac{1}{2} - \Xi\left(\vec{n}_{\pi(x)}^{\varepsilon,W},\ \frac{x-\pi(x)}{\varepsilon}/W\left(\vec{n}_{\pi(x)}^{\varepsilon,W}\right)\right) & \text{if } |x-\pi(x)| < \frac{1}{2}\varepsilon W\left(\vec{n}_{\pi(x)}^{\varepsilon,W}\right), x \notin P, \\ 1 & \text{if } |x-\pi(x)| > \varepsilon W\left(\vec{n}_{\pi(x)}^{\varepsilon,W}\right), x \in P, \\ 0 & \text{if } |x-\pi(x)| > \varepsilon W\left(\vec{n}_{\pi(x)}^{\varepsilon,W}\right), x \notin P. \end{cases}$$

We are now able to analyse the asymptotic behavior of $|P_{\varepsilon,\Xi,W}|_{B,p}$ for $\varepsilon \to 0$ and fixed Ξ, W. Moreover, we will see, that the estimates even hold independently of Ξ and W if $\|\Xi'\|_\infty$ and $\|W\|_\infty$ grow slowly compared to ε^{-1}.

Proposition 2.13. *Let $\psi \in L^2(\mathbb{R}^2)$ satisfy the assumptions of Proposition 2.11. Let Ω be a norm on $\mathbb{R}^2$ and let ω be the associated directional weight. Assume that P is a polygon, Ξ a directional transition profile, and W a directional weight. We then have that*

$$|P_{\varepsilon,\Xi,W}|_{B,p}^2 = \int_{(0,1)^2} \Omega(\nabla P_{\varepsilon,\Xi,W})^2 \, \mathrm{d}x + o\left(\frac{1}{\varepsilon} \log_2\left(\frac{1}{\varepsilon}\right)^{-1}\right).$$

The implicit constant is independent of Ξ and W, if

$$\|\Xi'\|_\infty \leq \varepsilon^{-\frac{1}{128}}, \qquad \|W\|_\infty \leq \frac{1}{2}\varepsilon^{-\frac{1}{128}}, \qquad \textit{and} \qquad \|\Xi'\|_\infty \left\|\frac{1}{W}\right\|_\infty \leq 2. \tag{2.31}$$

Before we can prove the result above, we first need to show an auxiliary result, producing a covering of ∂P.

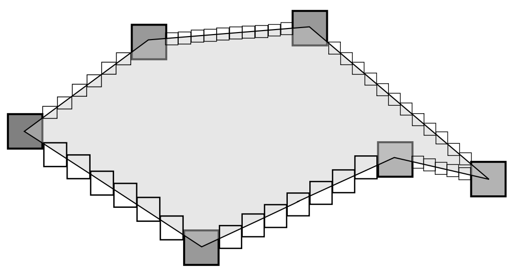

Figure 1: Covering of the boundary of a polygon as in Proposition 2.14

Proposition 2.14. *Let $N \in \mathbb{N}$ and let P be an N-gon with vertices $x_1^P, \ldots, x_N^P$. Then, there exists an $r_0 > 0$ and $c_0 \in (0, 1/2]$ such that for all $r \in (0, r_0)$ there are $L_1, \ldots, L_N \in \mathbb{N}$ with $\ell(P)/r \geq \sum_{i=1}^N L_i \geq \ell(P)/(4r)$, and $(z_{k,i})_{k=1,\ldots,L_i, i=1,\ldots,N} \subset \partial P$, and $c_0 r/4 \leq s_1, \ldots, s_N \leq c_0 r/2$ such that*

$$\begin{aligned} \partial P &\subset \dot{\bigcup}_{i=1,\ldots,N} \left(\dot{\bigcup}_{k=1,\ldots,L_i} C(z_{k,i}, s_i) \,\dot\cup\, C(x_i, r)\right), \ \textit{and} \\ g_i &\subset \dot{\bigcup}_{k=1,\ldots,L_i} C(z_{k,i}, s_i) \,\dot\cup\, C(x_i, r) \,\dot\cup\, C(x_{i+1}, r), \ \textit{for all } i = 1, \ldots, N, \end{aligned} \tag{2.32}$$

where $C(z, s) = z + (-s, s]^2$ is the semi-open cube of radius s with center z and g_i is the line segment between x_i and x_{i+1}. Moreover, $\chi_{P \cap C(z_{k,i}, s_k + c_0 r/4)}(\cdot - z_{k,i})$ is a rotated Heaviside function on $[-s_k - c_0 r/4, s_k + c_0 r/4]^2$ with the same normal direction as ∂P at $z_{k,i}$ for all $k = 1, \ldots, L$.

Proof of Proposition 2.13. Assume that P is an N-gon, Ξ is a directional transition profile, and set $c_\Xi := \max_{\vec{n} \in \mathbb{S}^1} |\Xi(\vec{n})| < \infty$. Let W be a directional weight such that $W(\mathbb{S}^1) \subset [a, b]$, $0 < a < b < \infty$.

Let $r = 4\varepsilon^{1/16}/c_0$ and $\varepsilon > 0$ so small that $r < \min\{r_0, \mathrm{dist}(P, \mathbb{R}^2 \setminus (0,1)^2)\}$. Let $L_1, \ldots, L_N \in \mathbb{N}$, and $(z_{k,i})_{k=1,\ldots,L_i, i=1,\ldots,N} \subset \partial P$, and $c_0 r/4 \leq s_1, \ldots, s_N \leq c_0 r/2$ as in Proposition 2.14. We define

$$G^r := [0,1]^2 \setminus \left(\dot{\bigcup}_{i=1,\ldots,N} \left(\dot{\bigcup}_{k=1,\ldots,L_i} C(z_{k,i}, s_i) \,\dot\cup\, C(x_i, r)\right)\right).$$

We then have that

$$
\begin{aligned}
|P_{\varepsilon,\Xi,W}|_{B,p}^2 &= \sum_{i=1}^{N}\sum_{k=1}^{L_i}\sum_{\iota=-1,1}\int_0^{\Gamma} a^{-2}\int_{-\Delta}^{\Delta}\int_{C(z_{k,i},s_i)} \left|\left\langle P_{\varepsilon,\Xi,W}, \psi_{a,s,t,\iota}^{\omega,per}\right\rangle\right|^2 a^{-3}\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a \\
&\quad + \sum_{i=1}^{N}\sum_{\iota=-1,1}\int_0^{\Gamma} a^{-2}\int_{-\Delta}^{\Delta}\int_{C(x_i,r)} \left|\left\langle P_{\varepsilon,\Xi,W}, \psi_{a,s,t,\iota}^{\omega,per}\right\rangle\right|^2 a^{-3}\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a \\
&\quad + \sum_{\iota=-1,1}\int_0^{\Gamma} a^{-2}\int_{-\Delta}^{\Delta}\int_{G^r} \left|\left\langle P_{\varepsilon,\Xi,W}, \psi_{a,s,t,\iota}^{\omega,per}\right\rangle\right|^2 a^{-3}\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a \\
&\geq \sum_{i=1}^{N}\sum_{k=1}^{L_i}\sum_{\iota=-1,1}\int_0^{\Gamma} a^{-2}\int_{-\Delta}^{\Delta}\int_{C(z_{k,i},s_i)} \left|\left\langle P_{\varepsilon,\Xi,W}, \psi_{a,s,t,\iota}^{\omega,per}\right\rangle\right|^2 a^{-3}\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a.
\end{aligned}
$$

Since $\varepsilon^{1/16} \leq c_0 r/4$, we conclude that all shearlets $\psi_{a,s,t,\iota}^{\omega,per}$ with $t \in C(z_{k,i}, s_i)$ such that $P_{\varepsilon,\Xi,W} \neq H_{\varepsilon,\Xi(\vec{n}_i),W(\vec{n}_i)}^{\vec{n}_i}(\cdot - z_{k,i})$ on supp $\psi_{a,s,t,\iota}^{\omega,per}$ have scale a such that $\sqrt{a} \geq \varepsilon^{1/16} - \|W\|_\infty \varepsilon$. We shall see that we can neglect the associated part of the sum above because of the following argument: we have that

$$
\begin{aligned}
&\sum_{i=1}^{N}\sum_{k=1}^{L_i}\sum_{\iota=-1,1}\int_0^{\Gamma} a^{-2}\int_{-\Delta}^{\Delta}\int_{C(z_{k,i},s_i)} \left|\left\langle P_{\varepsilon,\Xi,W}, \psi_{a,s,t,\iota}^{\omega,per}\right\rangle\right|^2 a^{-3}\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a \\
&= \sum_{i=1}^{N}\sum_{k=1}^{L_i}\sum_{\iota=-1,1}\int_0^{\Gamma} a^{-2}\int_{-\Delta}^{\Delta}\int_{C(z_{k,i},s_i)} \left|\left\langle H_{\varepsilon,\Xi(\vec{n}_i),W(\vec{n}_i)}^{\vec{n}_i}(\cdot - z_{k,i}), \psi_{a,s,t,\iota}^{\omega,per}\right\rangle\right|^2 a^{-3}\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a \\
&\quad + \sum_{i=1}^{N}\sum_{k=1}^{L_i}\sum_{\iota=-1,1}\int_{(\varepsilon^{1/16}-\|W\|_\infty\varepsilon)^2}^{\Gamma} a^{-2}\int_{-\Delta}^{\Delta}\int_{C(z_{k,i},s_i)} \left|\left\langle P_{\varepsilon,\Xi,W}, \psi_{a,s,t,\iota}^{\omega,per}\right\rangle\right|^2 \\
&\qquad - \left|\left\langle H_{\varepsilon,\Xi(\vec{n}_i),W(\vec{n}_i)}^{\vec{n}_i}(\cdot - z_{k,i}), \psi_{a,s,t,\iota}^{\omega,per}\right\rangle\right|^2 a^{-3}\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a.
\end{aligned}
$$

Assumption 2.19 implies that the underlying shearlet system satisfies a Bessel inequality, see Appendix D. Hence

$$
\begin{aligned}
&\sum_{i=1}^{N}\sum_{k=1}^{L_i}\sum_{\iota=-1,1}\int_{(\varepsilon^{1/16}-\|W\|_\infty\varepsilon)^2}^{\Gamma} a^{-2}\int_{-\Delta}^{\Delta}\int_{C(z_{k,i},s_i)} \left|\left\langle P_{\varepsilon,\Xi,W}, \psi_{a,s,t,\iota}^{\omega,per}\right\rangle\right|^2 \\
&\qquad - \left|\left\langle H_{\varepsilon,\Xi(\vec{n}_i),W(\vec{n}_i)}^{\vec{n}_i}(\cdot - z_{k,i}), \psi_{a,s,t,\iota}^{\omega,per}\right\rangle\right|^2 a^{-3}\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a \\
&\lesssim \sum_{i=1}^{N}\sum_{k=1}^{L_i}\varepsilon^{-\frac{1}{4}} \left\|P_{\varepsilon,\Xi,W} - H_{\varepsilon,\Xi(\vec{n}_i),W(\vec{n}_i)}^{\vec{n}_i}(\cdot - z_{k,i})\right\|_{L^2((0,1)^2)}^2 \\
&\lesssim \sum_{i=1}^{N}\sum_{k=1}^{L_i}\varepsilon^{-\frac{1}{4}} = \mathcal{O}\left(\varepsilon^{-\frac{1}{16}}\varepsilon^{-\frac{1}{4}}\right) = \mathcal{O}\left(\varepsilon^{-\frac{1}{2}}\right),
\end{aligned}
$$

where the implicit constants are independent of $\|W\|_\infty$ if $\|W\|_\infty \varepsilon \leq \varepsilon^{1/16}/2$. This is always satisfied if (2.31) holds. Thus, we get by the properties of the covering and Proposition 2.11 that

$$
\begin{aligned}
|P_{\varepsilon,\Xi,W}|_{B,p}^2 &\geq \sum_{i=1}^{N}\sum_{k=1}^{L_i}\sum_{\iota=-1,1}\int_0^{\Gamma} a^{-2}\int_{-\Delta}^{\Delta}\int_{C(z_{k,i},s_i)} \left|\left\langle H_{\varepsilon,\Xi(\vec{n}_i),W(\vec{n}_i)}^{\vec{n}_i}, \psi_{a,s,t,\iota}^{\omega}\right\rangle\right|^2 a^{-3}\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a + \mathcal{O}\left(\varepsilon^{-\frac{1}{2}}\right) \\
&\geq \sum_{i-1}^{N}\sum_{k=1}^{L_i}\Omega\left(\vec{n}_i\right)^2 \left|H_{\varepsilon,\Xi(\vec{n}_i),W(\vec{n}_i)}^{\vec{n}_i}\right|_{H^1((-r,r)^2)}^2
\end{aligned}
$$

$$
+\sum_{i=1}^{N}\sum_{k=1}^{L_i}\left(r\, o\left(\frac{1}{\varepsilon}\log_2\left(\frac{1}{\varepsilon}\right)^{-1}\right)+o\left(\frac{1}{\sqrt{\varepsilon}}\right)+\mathcal{O}\left(\frac{1}{(\varepsilon^{\frac{1}{16}}-w\varepsilon(\Delta+2))^4}\right)\right).
$$

As $\sum_{i=1}^{N}\sum_{k=1}^{L_i} r=\mathcal{O}(1)$ and $\sum_{i=1}^{N}\sum_{k=1}^{L_i}=\mathcal{O}(\varepsilon^{-1/16})$ we have that

$$
\begin{aligned}
|P_{\varepsilon,\Xi,W}|_{B,p}^2 &\geq \sum_{i-1}^{N}\sum_{k=1}^{L_i}\Omega\left(\vec{n}_i\right)^2 |H^{\vec{n}_i}_{\varepsilon,\Xi(\vec{n}_i)W(\vec{n}_i)}|^2_{H^1((-r,r)^2)}+o\left(\frac{1}{\varepsilon}\log_2\left(\frac{1}{\varepsilon}\right)^{-1}\right)\\
&=\int_{(0,1)^2\setminus\bigcup_{i=1}^{N}C(x_i,r)}\Omega(\nabla P_{\varepsilon,\Xi,W})(x)^2\,\mathrm{d}x+o\left(\frac{1}{\varepsilon}\log_2\left(\frac{1}{\varepsilon}\right)^{-1}\right).
\end{aligned}
$$

By construction, the support of $\nabla P_{\varepsilon,\Xi,W}$ is located in an $\varepsilon\|W\|_\infty$ neighbourhood of ∂P. Moreover, $\|\nabla P_{\varepsilon,\Xi,W}\|_\infty^2 \lesssim \|\Xi\|_\infty^2\varepsilon^2$. Since $r=\mathcal{O}(\varepsilon^{1/16})$ this yields that

$$
\int_{\bigcup_{i=1}^{N}C(x_i,r)}\Omega(\nabla P_{\varepsilon,\Xi,W})(x)^2\,\mathrm{d}x\leq\mathcal{O}\left(N\frac{1}{\varepsilon^{31/32}}\right) \tag{2.33}
$$

as soon as $\|W\|_\infty\|\Xi\|_\infty^2 r\varepsilon^{-1}\leq\varepsilon^{-31/32}$, which is the case for sufficiently large ε and fixed Ξ, W and always if (2.31) is satisfied. Hence,

$$
|P_{\varepsilon,\Xi,W}|_{B,p}^2\geq\int_{(0,1)^2}\Omega(\nabla P_{\varepsilon,\Xi,W})(x)^2\,\mathrm{d}x+o\left(\frac{1}{\varepsilon}\log_2\left(\frac{1}{\varepsilon}\right)^{-1}\right).
$$

To establish the lower bound, we define

$$
\widetilde{G}^r\coloneqq[0,1]^2\setminus\left(\dot{\bigcup}_{i=1,\dots,N}\left(\dot{\bigcup}_{k=1,\dots,L_i}C\left(z_{k,i},s_i(1+r)\right)\ \dot\cup\ C(x_i,r)\right)\right).
$$

Then we have that

$$
\begin{aligned}
|P_{\varepsilon,\Xi,W}|_{B,p}^2\leq&\sum_{i=1}^{N}\sum_{k=1}^{L_i}\sum_{\iota=-1,1}\int_0^\Gamma a^{-2}\int_{-\Delta}^{\Delta}\int_{C(0,s_i+r^{3/2})}\left|\left\langle P_{\varepsilon,\Xi,W},\psi^{\omega,per}_{a,s,t,\iota}\right\rangle\right|^2a^{-3}\,\mathrm{d}s\ \mathrm{d}t\,\mathrm{d}a\\
&+\sum_{i=1}^{N}\sum_{\iota=-1,1}\int_0^\Gamma a^{-2}\int_{-\Delta}^{\Delta}\int_{C(x_i,r)}\left|\left\langle P_{\varepsilon,\Xi,W},\psi^{\omega,per}_{a,s,t,\iota}\right\rangle\right|^2a^{-3}\,\mathrm{d}s\ \mathrm{d}t\,\mathrm{d}a\\
&+\sum_{\iota=-1,1}\int_0^\Gamma a^{-2}\int_{-\Delta}^{\Delta}\int_{\widetilde{G}^r}\left|\left\langle P_{\varepsilon,\Xi,W},\psi^{\omega,per}_{a,s,t,\iota}\right\rangle\right|^2a^{-3}\,\mathrm{d}s\ \mathrm{d}t\,\mathrm{d}a.
\end{aligned}
$$

By construction, all shearlets with $m\in\widetilde{G}^r$ intersecting $\partial P+B_{\varepsilon\|W\|_\infty}(0)$ must satisfy $\sqrt{a}>s_ir/2-\|W\|_\infty\varepsilon\geq\varepsilon^{1/8}/2-\|W\|_\infty\varepsilon$, assuming ε to be so small that $\varepsilon^{1/8}/2-\|W\|_\infty\varepsilon\geq\varepsilon^{1/8}/4$. This is always satisfied independent of W if (2.31) holds. Thus, we have $a^2\leq 8\varepsilon^{-1/2}$. Invoking again the Bessel inequality of the periodised shearlet system, see Appendix D, this implies that

$$
\sum_{\iota=-1,1}\int_0^\Gamma a^{-2}\int_{-\Delta}^{\Delta}\int_{\widetilde{G}^r}\left|\left\langle P_{\varepsilon,\Xi,W},\psi^{\omega,per}_{a,s,t,\iota}\right\rangle\right|^2a^{-3}\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a=\mathcal{O}\left(\frac{1}{\sqrt{\varepsilon}}\right)=o\left(\frac{1}{\varepsilon}\log_2\left(\frac{1}{\varepsilon}\right)^{-1}\right).
$$

Moreover, using Proposition 2.11 and (2.33) we obtain

$$
\begin{aligned}
&|P_{\varepsilon,\Xi,W}|_{B,p}^2 \\
&\leq \sum_{i=1}^{N}\sum_{k=1}^{L_i}\sum_{\iota=-1,1}\int_0^\Gamma a^{-2}\int_{-\Delta}^{\Delta}\int_{C(0,s_i+r^{3/2})}\left|\left\langle P_{\varepsilon,\Xi,W},\psi_{a,s,t,\iota}^{\omega,per}\right\rangle\right|^2 a^{-3}\,\mathrm{d}s\;\mathrm{d}t\,\mathrm{d}a + o\left(\frac{1}{\varepsilon}\log_2\left(\frac{1}{\varepsilon}\right)^{-1}\right) \\
&\leq \sum_{i=1}^{N}\sum_{k=1}^{L_i}\sum_{\iota=-1,1}\int_0^\Gamma a^{-2}\int_{-\Delta}^{\Delta}\int_{C(0,s_i(1+\frac{4}{c_0}r^{1/2}))}\left|\left\langle H_{\varepsilon,\Xi(\vec{n}_i),W(\vec{n}_i)}^{\vec{n}_i},\psi_{a,s,t,\iota}^{\omega}\right\rangle\right|^2 a^{-3}\,\mathrm{d}s\;\mathrm{d}t\,\mathrm{d}a + o\left(\frac{1}{\varepsilon}\log_2\left(\frac{1}{\varepsilon}\right)^{-1}\right) \\
&= \left(1+\frac{4}{c_0}r^{\frac{1}{2}}\right)\int_{(0,1)^2\setminus\bigcup_{i=1}^N C(x_i,r)}\Omega(\nabla P_{\varepsilon,\Xi,W})(x)^2\;\mathrm{d}x + o\left(\frac{1}{\varepsilon}\log_2\left(\frac{1}{\varepsilon}\right)^{-1}\right).
\end{aligned}
$$

Invoking (2.33) one last time, we observe that

$$|P_{\varepsilon,\Xi,W}|_{B,p}^2 = \int_{(0,1)^2}\Omega(\nabla P_{\varepsilon,\Xi,W})(x)^2\;\mathrm{d}x + o\left(\frac{1}{\varepsilon}\log_2\left(\frac{1}{\varepsilon}\right)^{-1}\right).$$

We observed in the course of the proof that (2.31) implies that the implicit constants are independent of Ξ and W if (2.31) is satisfied. ☐

Proposition 2.13 shows that for functions of the form $P_{\varepsilon,\Xi,W}$ the shearlet-based seminorm $|P_{\varepsilon,\Xi,W}|_{B,p}$ converges to the anisotropic Sobolev seminorm $\int_{(0,1)^2}\Omega(\nabla P_{\varepsilon,\Xi,W})(x)\;\mathrm{d}x$. Hence,

$$\left|\mathrm{SGL}_\varepsilon^\omega(P_{\varepsilon,\Xi,W}) - \mathrm{GL}_\varepsilon^\Omega(P_{\varepsilon,\Xi,W})\right| = o\left(\frac{1}{\varepsilon}\log_2\left(\frac{1}{\varepsilon}\right)^{-1}\right),$$

with constants independent of Ξ, W if (2.31) holds. It was demonstrated in [2, Proposition 4.10] that there exists a sequence of functions u_ε of the form $P_{\varepsilon,\Xi_\varepsilon,W_\varepsilon}$, with $\min W_\varepsilon \leq \|\Xi_\varepsilon'\|_\infty \leq \max W_\varepsilon$ such that

$$\lim_{\varepsilon\to 0}\mathrm{GL}_\varepsilon^\Omega(u_\varepsilon) \to P_\Omega(\chi_P).$$

Moreover, $\|\Xi_\varepsilon'\|_\infty \to \infty$ at any sufficiently slow rate. In particular, we can choose $\|\Xi_\varepsilon'\|_\infty \leq \varepsilon^{-\frac{1}{128}}$. This demonstrates existence of a recovery sequence.

2.7 Recovery sequences for characteristic functions of sets with finite perimeter

The standard method for the construction of recovery sequences in Γ-convergence arguments is to establish lower semicontinuity of the limit functional and then reduce the problem to finding a recovery sequence for functions from a dense, but more accessible space, only. We will follow precisely this argument. Indeed the anisotropic perimeter functional is lower semi-continuous [16, Theorem 20.1] and the set of polygons is dense in the set of sets of finite perimeter. This yields the following proposition.

Proposition 2.15. *Let $\psi \in L^2(\mathbb{R}^2)$ satisfy the assumptions of Proposition 2.11. Let Ω be a norm on $\mathbb{R}^2$ and let ω be the associated directional weight. Assume that $D \subset (0,1)^2$ is a set of finite perimeter. We then have that there exists a sequence $(u_\varepsilon)_{\varepsilon>0} \subset H^1((0,1)^2)$ with $u_\varepsilon \to \chi_D$:*

$$\limsup_{\varepsilon>0}\mathrm{SGL}_\varepsilon^\omega(u_\varepsilon) \leq P_\Omega(\chi_D).$$

2.8 The set $\mathcal{B}_\Omega$

The set $\mathcal{B}_\Omega$ describes the feasible functions for the shearlet-based Ginzburg–Landau energy. It is somewhat unsatisfying that the shearlet-based Ginzburg–Landau energy still depends—at least indirectly—on the anisotropic Sobolev seminorm. Nonetheless, most of the relevant functions appearing in a phase-field problem are contained in $\mathcal{B}_\Omega$. Indeed, it was demonstrated in Proposition 2.13 that all functions of the form $P_{\varepsilon,\Xi,W}$ for any phase-transition function Ξ and any directional width W are elements of $\mathcal{B}_\Omega$, as long as ε is sufficiently small. Additionally, the definition of $\mathcal{B}_\Omega$ demonstrates that for sufficiently small $\varepsilon > 0$ also smooth pertubations of functions of the form $P_{\varepsilon,\Xi,W}$ are contained in $\mathcal{B}_\Omega$. Indeed, if $f_\varepsilon \in H^1((0,1)^2)$ with

$$|f_\varepsilon|_{H^1}^2 = o\left(\varepsilon^{-1}/\log_2(2+\varepsilon^{-1})\right),$$

then, for sufficiently small ε, we have that $P_{\varepsilon,\Xi,W} + f_\varepsilon \in \mathcal{B}_\Omega$. This is because,

$$||f_\varepsilon + P_{\varepsilon,\Xi,W}|_{B,p} - |P_{\varepsilon,\Xi,W}|_{B,p}| \leq |f_\varepsilon|_{B,p} \leq |f_\varepsilon|_{H^1},$$

by the reverse triangle inequality and by Proposition 2.5, particularly (2.7).

We observe with equation (A.2) below (after adding weights), that we have an alternative representation of $|f|_B$ as $\|\mathcal{S}(f)\|_{L^2}$ where, for $f \in L^2(\mathbb{R}^2)$,

$$\mathcal{S}(f) := \mathcal{F}^{-1}\left(\xi \mapsto \hat{f}(\xi) \cdot \left(|\widehat{K}(\xi)|^2 + |\xi_1|^2 \cdot \int_{-\Delta}^{\Delta}\int_0^{\Gamma} |\omega(1,s)|^2 \left|\widehat{\mu}(a\xi_1, \sqrt{a}(\xi_2 + s\xi_1))\right|^2 a^{-\frac{3}{2}} \,\mathrm{d}a\,\mathrm{d}s \right.\right.$$
$$\left.\left. + \ |\xi_2|^2 \cdot \int_{\mathbb{R}^2}\int_{-\Delta}^{\Delta}\int_0^{\Gamma} |\omega(-1,s)|^2 \left|\widehat{\widetilde{\mu}}(\sqrt{a}(\xi_1 + s\xi_2), a\xi_2)\right|^2 a^{-\frac{3}{2}} \,\mathrm{d}a\,\mathrm{d}s\right)^{\frac{1}{2}}\right).$$

If $\mathfrak{d}(\mathrm{supp}\ u_\varepsilon, \mathbb{R}^2 \setminus (0,1)^2)$ is sufficiently large then there exists a Γ_0 such that $\langle u_\varepsilon, \psi_{a,s,t,\iota}\rangle = 0$ for all $t \notin (0,1)^2$ and $a < \Gamma_0$. In this case, we have that $|u_\varepsilon|_{B,p} = \|\mathcal{S}_p(u_\varepsilon)\|_{L^2} + o(\|u_\varepsilon\|_{L^2})$, where

$$\mathcal{S}_p(f) := \mathcal{F}^{-1}\left(\xi \mapsto \widehat{u_\varepsilon}(\xi) \cdot \left(|\xi_1|^2 \cdot \int_{-\Delta}^{\Delta}\int_0^{\Gamma_0} |\omega(1,s)|^2 \left|\widehat{\mu}(a\xi_1, \sqrt{a}(\xi_2 + s\xi_1))\right|^2 a^{-\frac{3}{2}} \,\mathrm{d}a\,\mathrm{d}s \right.\right.$$
$$\left.\left. + \ |\xi_2|^2 \cdot \int_{\mathbb{R}^2}\int_{-\Delta}^{\Delta}\int_0^{\Gamma_0} |\omega(-1,s)|^2 \left|\widehat{\widetilde{\mu}}(\sqrt{a}(\xi_1 + s\xi_2), a\xi_2)\right|^2 a^{-\frac{3}{2}} \,\mathrm{d}a\,\mathrm{d}s\right)^{\frac{1}{2}}\right).$$

For a sequence $u_\varepsilon \to \chi_D$, where $D \subset (0,1)^2$, it appears to be reasonable to study the action of the pseudo-differential operator $\mathcal{S}_p$ on u_ε to understand the behavior of the $|u_\varepsilon|_{B,p}$ norm. Due to these considerations, a promising direction to remove the restriction to be a member of the set B_Ω could be to study the microlocal behavior of the sequences $u_\varepsilon \to \chi_D$, where $D \subset (0,1)^2$ and such that $\sup_\varepsilon SGL_\varepsilon(u_\varepsilon) < \infty$.

3 The discrete shearlet-based Ginzburg–Landau energy

The construction of the shearlet-based Ginzburg–Landau energy of Definition 2.12 is based on the continuous shearlet transform. In practice, it is more appropriate to work with a discrete energy, where the integrals over the parameters in the shearlet-based Besov seminorm are replaced by sums. Indeed, we will show that the shearlet-based Ginzburg–Landau energy can be uniformly approximated by a discrete variant such that both energies have the same Γ-limit.

We start by defining a discrete shearlet-based Besov seminorm. Let $\Gamma > 1, \Delta > 0$; then, for $c > 0$ we define $J_c := c\mathbb{N}_0 - \log_2(\Gamma)$ and for all $j \in J_c$ we set $K_{j,c} := \left\{k \in -\Delta + c\mathbb{Z} \colon |k| \leq 2^{\frac{j}{2}}\Delta\right\}$. Moreover, we define the following *rounding operators*: for $x \in \mathbb{R}$

$$[x]_c := \max\left\{x^* \in c\mathbb{N}_0 - \log_2(\Gamma), x^* \leq \max\{-\log_2(\Gamma), x\}\right\} \quad (3.1)$$

and, for $x \in [-2^{\frac{j}{2}}\Delta, 2^{\frac{j}{2}}\Delta]$,

$$[x]_{j,c} \coloneqq \max\left\{x^* \in K_{j,c}, x^* \leq x\right\}. \tag{3.2}$$

Next, we define the *discrete shearlet-based Besov seminorm*: let $\psi \in L^2(\mathbb{R}^2)$, let ω be a directional weight, $c > 0$, and define $A_{1,2^{-j}} \coloneqq A_{2^{-j}}$ and $A_{-1,2^{-j}} \coloneqq \widetilde{A}_{2^{-j}}$; then,

$$|f|^2_{DB,c} \coloneqq c^4 \sum_{\iota=-1,1} \sum_{j\in J_c} \sum_{k\in K_{c,j}} \sum_{\substack{m\in cA_{\iota,2^{-j}}\mathbb{Z}^2,\\ m\in[0,1]^2}} 2^{2j} \left|\left\langle f, \psi^{\omega,per}_{2^{-j},2^{-j/2}k,m,\iota}\right\rangle\right|^2 \in [0,\infty], \quad \text{for } f \in L^2(\mathbb{R}^2).$$

Let $T : (0,1] \to (0,1]$ be a map such that $T(x) \to 0$ for $x \to 0$. We then define the associated *discrete Ginzburg–Landau energy* by

$$\mathrm{DSGL}^\omega_{\varepsilon,T}(u) \coloneqq \begin{cases} \varepsilon|u|^2_{DB,T(\varepsilon)} + \frac{1}{4\varepsilon}\int_{(0,1)^2} \mathcal{W}(u)(x)\,\mathrm{d}x, & \text{if } u \in \mathcal{B}_\Omega,\\ \infty, & \text{if } u \in BV \setminus \mathcal{B}_\Omega. \end{cases} \tag{3.3}$$

We shall now study under what conditions on the map T and a sequence $(u_\varepsilon)_{\varepsilon>0} \subset H^1((0,1)^2)$ we have that $|\mathrm{DSGL}^\omega_{\varepsilon,T}(u_\varepsilon) - \mathrm{SGL}^\omega_\varepsilon(u_\varepsilon)| \to 0$ for $\varepsilon \to 0$. This will turn out to be the case if $\|u_\varepsilon\|_{H^1} = o(T(\varepsilon)^{-1})$ for $\varepsilon \to 0$. Moreover, we will see that if $T(\varepsilon) = o(\sqrt{\varepsilon})$ then for any sequence $(u_\varepsilon)_{\varepsilon>0} \subset H^1((0,1)^2)$ either $|\mathrm{DSGL}^\omega_{\varepsilon,T}(u_\varepsilon) - \mathrm{SGL}^\omega_\varepsilon(u_\varepsilon)| \to 0$ or $\mathrm{DSGL}^\omega_{\varepsilon,T}(u_\varepsilon) \to \infty$ and $\mathrm{SGL}^\omega_\varepsilon(u_\varepsilon) \to \infty$.

Theorem 3.1. *Let $\psi \in L^2(\mathbb{R}^2)$ satisfy the assumptions of Proposition 2.5 and be such that*

$$|\widehat{\psi}(\xi)| \lesssim \frac{\min\{|\xi_1|^M, 1\}}{(1+|\xi_1|^2)^{L/2}(1+|\xi_2|^2)^{L/2}}, \quad \textit{for all } \xi \in \mathbb{R}^2,$$

for $M > 3, N > 4$. Let $T : (0,1] \to (0,1]$, and $(u_\varepsilon)_{\varepsilon>0} \subset H^1_0((0,1)^2)$ be such that $|u_\varepsilon|_{H^1} = o(T(\varepsilon)^{-1})$; then, $|\mathrm{DSGL}^\omega_{\varepsilon,T}(u_\varepsilon) - \mathrm{SGL}^\omega_\varepsilon(u_\varepsilon)| \to 0$ for $\varepsilon \to 0$. If $\sup_{\varepsilon>0}\|u_\varepsilon\|_{L^2} < \infty$ and $\liminf_{\varepsilon>0}\sqrt{\varepsilon}\|u_\varepsilon\|_{H^1} = \infty$ then $\liminf_{\varepsilon>0}\mathrm{DSGL}^\omega_{\varepsilon,T}(u_\varepsilon) = \infty$.

In particular, if $T(\varepsilon) = o(\sqrt{\varepsilon})$ for $\varepsilon \to 0$, then $\mathrm{DSGL}^\omega_{\varepsilon,T}$ and $\mathrm{SGL}^\omega_\varepsilon$ have the same Γ-limit.

Proof. In view of (3.3), it is sufficient to show that, if $(u_\varepsilon)_{\varepsilon>0} \subset H^1_0((0,1)^2)$ such that $|u_\varepsilon|_{H^1} = o(T(\varepsilon)^{-1})$, then

$$\left||u_\varepsilon|^2_{B,p} - |u_\varepsilon|^2_{DB,T(\varepsilon)}\right| \to 0 \quad \text{for } \varepsilon \to 0,$$

and if $\liminf_{\varepsilon>0}\varepsilon|u_\varepsilon|^2_{H^1} = \infty$ then $\liminf_{\varepsilon>0}\varepsilon|u_\varepsilon|^2_{DB,T(\varepsilon)} = \infty$.

We observe that there exists an $R \coloneqq R(\Gamma,\Delta) \in \mathbb{N}$ such that $\operatorname{supp}\psi_{a,s,t,\iota} \subset B_R(t)$ for all $(a,s,t,\iota) \in (0,\Gamma] \times [-\Delta,\Delta] \times \mathbb{R}^2 \times \{-1,1\}$.

By similar arguments to those following equation (2.10), we observe that, for all $f \in H^1_0((0,1)^2)$,

$$\begin{aligned} |f|^2_{B,p} &= \frac{1}{N^2}\sum_{\iota=-1,1}\int_0^\Gamma\int_{-\Delta}^\Delta a^{-2}\int_{[0,N]^2}\left|\left\langle f_{[-R,NR]}, \psi^\omega_{a,s,t,\iota}\right\rangle_{L^2(\mathbb{R}^2)}\right|^2 a^{-3}\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a\\ &= \frac{1}{N^2}\sum_{\iota=-1,1}\int_0^\Gamma\int_{-\Delta}^\Delta a^{-2}\int_{\mathbb{R}^2}\left|\left\langle f_{[-R,N+R]}, \psi^\omega_{a,s,t,\iota}\right\rangle_{L^2(\mathbb{R}^2)}\right|^2 a^{-3}\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a\\ &\quad - \frac{1}{N^2}\sum_{\iota=-1,1}\int_0^\Gamma\int_{-\Delta}^\Delta a^{-2}\int_{\mathbb{R}^2\setminus[0,N]^2}\left|\left\langle f_{[-R,N+R]}, \psi^\omega_{a,s,t,\iota}\right\rangle_{L^2(\mathbb{R}^2)}\right|^2 a^{-3}\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a. \end{aligned} \tag{3.4}$$

We further compute that

$$
\begin{aligned}
&\frac{1}{N^2}\sum_{\iota=-1,1}\int_0^\Gamma\int_{-\Delta}^{\Delta} a^{-2}\int_{\mathbb{R}^2\setminus[0,N]^2}\left|\left\langle f_{[-R,N+R]},\psi^\omega_{a,s,t,\iota}\right\rangle_{L^2(\mathbb{R}^2)}\right|^2 a^{-3}\ \mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a\\
&= \frac{1}{N^2}\sum_{\iota=-1,1}\int_0^\Gamma\int_{-\Delta}^{\Delta} a^{-2}\int_{\mathbb{R}^2\setminus[0,N]^2}\left|\left\langle f_{[-R,N+R]}-f_{[R,N-R]},\psi^\omega_{a,s,t,\iota}\right\rangle_{L^2(\mathbb{R}^2)}\right|^2 a^{-3}\ \mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a\\
&\le \frac{1}{N^2}\sum_{\iota=-1,1}\int_0^\Gamma\int_{-\Delta}^{\Delta} a^{-2}\int_{\mathbb{R}^2}\left|\left\langle f_{[-R,N+R]}-f_{[R,N-R]},\psi^\omega_{a,s,t,\iota}\right\rangle_{L^2(\mathbb{R}^2)}\right|^2 a^{-3}\ \mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a\\
&\lesssim \frac{1}{N^2}\mathcal{O}\left(|f_{[-R,N+R]}-f_{[R,N-R]}|^2_{H^1}\right)\lesssim \frac{1}{N}\mathcal{O}\left(|f|^2_{H^1}\right),
\end{aligned}
$$

where the last inequality follows by equation (2.5) and equation (2.9). Denoting $c_\varepsilon := T(\varepsilon)$, we compute similarly to the continuous case

$$
\begin{aligned}
|f|^2_{DB,c_\varepsilon} &= c_\varepsilon^4\sum_{\iota=-1,1}\sum_{j\in J_{c_\varepsilon}}\sum_{k\in K_{c_\varepsilon,j}}\sum_{\substack{m\in c_\varepsilon A_{\iota,2^{-j}}\mathbb{Z}^2,\\ m\in[0,N]^2}} 2^{2j}\left|\left\langle f,\psi^{\omega,per}_{2^{-j},2^{-j/2}k,m,\iota}\right\rangle_{L^2(\mathbb{R}^2)}\right|^2\\
&= \frac{c_\varepsilon^4}{N^2}\sum_{\iota=-1,1}\sum_{j\in J_{c_\varepsilon}}\sum_{k\in K_{c_\varepsilon,j}}\sum_{m\in c_\varepsilon A_{\iota,2^{-j}}\mathbb{Z}^2} 2^{2j}\left|\left\langle f_{[-R,N+R]},\psi^{\omega}_{2^{-j},2^{-j/2}k,m,\iota}\right\rangle_{L^2(\mathbb{R}^2)}\right|^2\\
&\quad -\frac{c_\varepsilon^4}{N^2}\sum_{\iota=-1,1}\sum_{j\in J_{c_\varepsilon}}\sum_{k\in K_{c_\varepsilon,j}}\sum_{\substack{m\in c_\varepsilon A_{\iota,2^{-j}}\mathbb{Z}^2\\ m\notin[0,N]^2}} 2^{2j}\left|\left\langle f_{[-R,N+R]},\psi^{\omega}_{2^{-j},2^{-j/2}k,m,\iota}\right\rangle_{L^2(\mathbb{R}^2)}\right|^2.
\end{aligned}
\tag{3.5}
$$

By assumption, we have that supp $\psi^\omega_{2^{-j},2^{-j/2}k,m,\iota}\subset B_R(m)$. Thus we have that

$$
\begin{aligned}
&\frac{c_\varepsilon^4}{N^2}\sum_{\iota=-1,1}\sum_{j\in J_{c_\varepsilon}}\sum_{k\in K_{c_\varepsilon,j}}\sum_{\substack{m\in c_\varepsilon A_{\iota,2^{-j}}\mathbb{Z}^2,\\ m\notin[0,N]^2}} 2^{2j}\left|\left\langle f_{[-R,N+R]},\psi^{\omega}_{2^{-j},2^{-j/2}k,m,\iota}\right\rangle_{L^2(\mathbb{R}^2)}\right|^2\\
&= \frac{c_\varepsilon^4}{N^2}\sum_{\iota=-1,1}\sum_{j\in J_{c_\varepsilon}}\sum_{k\in K_{c_\varepsilon,j}}\sum_{\substack{m\in c_\varepsilon A_{\iota,2^{-j}}\mathbb{Z}^2,\\ m\notin[0,N]^2}} 2^{2j}\left|\left\langle f_{[-R,N+R]}-f_{[R,N-R]},\psi^{\omega}_{2^{-j},2^{-j/2}k,m,\iota}\right\rangle_{L^2(\mathbb{R}^2)}\right|^2\\
&\le \frac{c_\varepsilon^4}{N^2}\sum_{\iota=-1,1}\sum_{j\in J_{c_\varepsilon}}\sum_{k\in K_{c_\varepsilon,j}}\sum_{m\in c_\varepsilon A_{\iota,2^{-j}}\mathbb{Z}^2} 2^{2j}\left|\left\langle f_{[-R,N+R]}-f_{[R,N-R]},\psi^{\omega}_{2^{-j},2^{-j/2}k,m,\iota}\right\rangle_{L^2(\mathbb{R}^2)}\right|^2 =: \mathrm{I}.
\end{aligned}
$$

We define $i_\iota := 1$ if $\iota = 1$ and $i_\iota := 2$ if $\iota=-1$ and $\gamma' := \psi$ and obtain by partial integration that

$$
\begin{aligned}
\mathrm{I} &= \frac{c_\varepsilon^4}{N^2}\sum_{\iota=-1,1}\sum_{j\in J_{c_\varepsilon}}\sum_{k\in K_{c_\varepsilon,j}}\sum_{m\in c_\varepsilon A_{\iota,2^{-j}}\mathbb{Z}^2}\left|\left\langle \frac{\partial}{\partial x_{i_\iota}}\left(f_{[-R,N+R]}-f_{[R,N-R]}\right),\gamma^\omega_{2^{-j},2^{-j/2}k,S_{2^{-j/2}k}A_{2^{-j}}m,\iota}\right\rangle_{L^2(\mathbb{R}^2)}\right|^2\\
&= \frac{1}{N^2}\mathcal{O}\left(|f_{[-R,N+R]}-f_{[R,N-R]}|^2_{H^1}\right) = \frac{1}{N}\mathcal{O}\left(|f|^2_{H^1}\right),
\end{aligned}
$$

where the second to last step follows from a long but simple computation which, with minor changes, is performed in [13, Section 5.1.1] and the last inequality follows with (2.9). Let now $(u_\varepsilon)_{\varepsilon>0}\subset H^1_0((0,1)^2)$. We then define

$$
N(u_\varepsilon) := \left\lceil \max\left\{|u_\varepsilon|^3_{H^1},\frac{1}{\varepsilon}\right\}\right\rceil.
$$

It is not hard to see that $|(u_\varepsilon)_{[-R,N(u_\varepsilon)+R]}/N(u_\varepsilon)|_{H^1(\mathbb{R}^2)} \sim |u_\varepsilon|_{H^1((0,1)^2)}$ and by (3.4) and (3.5) we have that, for $\varepsilon \to 0$,

$$\left| |u_\varepsilon|^2_{B,p} - \sum_{\iota=-1,1} \int_0^\Gamma \int_{-\Delta}^{\Delta} a^{-2} \int_{\mathbb{R}^2} \left| \left\langle \frac{(u_\varepsilon)_{[-R,N+R]}}{N(u_\varepsilon)^2}, \psi^\omega_{a,s,t,\iota} \right\rangle_{L^2(\mathbb{R}^2)} \right|^2 a^{-3} \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}a \right| \to 0$$

and

$$\left| |u_\varepsilon|^2_{DB,c_\varepsilon} - c_\varepsilon^4 \sum_{\iota=-1,1} \sum_{j\in J_{c_\varepsilon}} \sum_{k\in K_{c_\varepsilon,j}} \sum_{m\in c_\varepsilon\mathbb{Z}^2} 2^{2j} \left| \left\langle \frac{(u_\varepsilon)_{[-R,N+R]}}{N(u_\varepsilon)}, \psi^\omega_{2^{-j},2^{-j/2}k,m,\iota} \right\rangle_{L^2(\mathbb{R}^2)} \right|^2 \right| \to 0.$$

Thus, the result follows if for all $(h_\varepsilon)_{\varepsilon>0} \subset H^1(\mathbb{R}^2)$ with $|h_\varepsilon|_{H^1} \in \mathcal{O}(T(\varepsilon)^{-1})$:

$$\begin{aligned}\Bigg| &\int_{\mathbb{R}^2} \int_{-\Delta}^{\Delta} \int_0^\Gamma a^{-2} |\langle h_\varepsilon, \psi^\omega_{a,s,t,\iota}\rangle_{L^2(\mathbb{R}^2)}|^2 a^{-3} \,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t \\ &- c_\varepsilon^4 \sum_{\iota=-1,1} \sum_{j\in J_{c_\varepsilon}} \sum_{k\in K_{c_\varepsilon,j}} \sum_{m\in c_\varepsilon A_{\iota,2^{-j}}\mathbb{Z}^2} 2^{2j} \left| \left\langle h_\varepsilon, \psi^\omega_{2^{-j},2^{-j/2}k,m,\iota} \right\rangle_{L^2(\mathbb{R}^2)} \right|^2 \Bigg| \to 0 \quad \text{for } \varepsilon \to 0, \end{aligned} \tag{3.6}$$

and for all $(h_\varepsilon)_{\varepsilon>0} \subset H^1(\mathbb{R}^2)$ with $\sup_{\varepsilon>0} \|u_\varepsilon\|_{L^2} < \infty$ and $\liminf_{\varepsilon>0} \varepsilon |h_\varepsilon|^2_{H^1} = \infty$:

$$\liminf_{\varepsilon>0} \varepsilon \cdot c_\varepsilon^4 \sum_{\iota=-1,1} \sum_{j\in J_{c_\varepsilon}} \sum_{k\in K_{c_\varepsilon,j}} \sum_{m\in c_\varepsilon A_{\iota,2^{-j}}\mathbb{Z}^2} 2^{2j} \left| \left\langle h_\varepsilon, \psi^\omega_{2^{-j},2^{-j/2}k,m,\iota} \right\rangle_{L^2(\mathbb{R}^2)} \right|^2 = \infty. \tag{3.7}$$

We will now verify the statements (3.6) and (3.7). Let $h \in H^1(\mathbb{R}^2)$; then, by partial integration, we compute that

$$\begin{aligned} &\sum_{\iota=-1,1} \int_{\mathbb{R}^2} \int_{-\Delta}^{\Delta} \int_0^\Gamma a^{-2} |\langle h, \psi^\omega_{a,s,t,\iota}\rangle_{L^2(\mathbb{R}^2)}|^2 a^{-3} \,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t \\ &= \sum_{\iota=-1,1} \int_{\mathbb{R}^2} \int_{-\Delta}^{\Delta} \int_0^\Gamma \left| \left\langle \frac{\partial}{\partial x_{i_\iota}} h, \gamma^\omega_{a,s,t,\iota} \right\rangle_{L^2(\mathbb{R}^2)} \right|^2 a^{-3} \,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t. \end{aligned}$$

By the co-area formula, we can rewrite (**??**) as

$$\begin{aligned} &\sum_{\iota=-1,1} \int_{\mathbb{R}^2} \int_{-\Delta}^{\Delta} \int_0^\Gamma \left| \left\langle \frac{\partial}{\partial x_{i_\iota}} h, \gamma^\omega_{a,s,t,\iota} \right\rangle_{L^2(\mathbb{R}^2)} \right|^2 a^{-3} \,\mathrm{d}a\,\mathrm{d}s\,\mathrm{d}t \\ &= \sum_{\iota=-1,1} \int_{-\log_2(\Gamma)}^{\infty} \int_{-2^{j/2}\Delta}^{2^{j/2}\Delta} \int_{\mathbb{R}^2} \left| \left\langle \frac{\partial}{\partial x_{i_\iota}} h, \gamma^\omega_{2^{-j},2^{-j/2}k,m,\iota} \right\rangle_{L^2(\mathbb{R}^2)} \right|^2 2^{3j/2} \,\mathrm{d}m\,\mathrm{d}k\,\mathrm{d}j \\ &= \sum_{\iota=-1,1} \int_{-\log_2(\Gamma)}^{\infty} \int_{-2^{j/2}\Delta}^{2^{j/2}\Delta} \int_{\mathbb{R}^2} \left| \left\langle \mathcal{F}\left(\frac{\partial}{\partial x_{i_\iota}} h\right), \mathcal{F}\left(\gamma^\omega_{2^{-j},2^{-j/2}k,m,\iota}\right) \right\rangle_{L^2(\mathbb{R}^2)} \right|^2 2^{3j/2} \,\mathrm{d}m\,\mathrm{d}k\,\mathrm{d}j \\ &= \sum_{\iota=-1,1} \int_{-\log_2(\Gamma)}^{\infty} \int_{-2^{j/2}\Delta}^{2^{j/2}\Delta} \int_{\mathbb{R}^2} \omega(2^{-j/2}k,\iota)^2 \left| \mathcal{F}\left(\frac{\partial}{\partial x_{i_\iota}} h\right)(\xi) \right|^2 \left| \mathcal{F}\left(\widehat{\gamma}\left(A_{2^{-j}} S^T_{-2^{-j/2}k}\xi\right)\right) \right|^2 \mathrm{d}\xi\,\mathrm{d}k\ \mathrm{d}j \\ &= \sum_{\iota=-1,1} \int_{\mathbb{R}^2} \left| \mathcal{F}\left(\frac{\partial}{\partial x_{i_\iota}} h\right)(\xi) \right|^2 \int_{-\log_2(\Gamma)}^{\infty} \int_{-2^{j/2}\Delta}^{2^{j/2}\Delta} \omega(2^{-j/2}k,\iota)^2 \left| \mathcal{F}\left(\widehat{\gamma}\left(A_{2^{-j}} S^T_{-2^{-j/2}k}\xi\right)\right) \right|^2 \mathrm{d}k\,\mathrm{d}j\ \mathrm{d}\xi \\ &=: S(h). \end{aligned}$$

We shall now perform a similar computation for the discrete shearlet-based Besov seminorm. We have by Plancherel's identity that, for $c > 0$,

$$
\begin{aligned}
& c^4 \sum_{j \in c\mathbb{Z}} \sum_{k \in K_{c,j}} \sum_{m \in c_\varepsilon A_{\iota, 2^{-j}} \mathbb{Z}^2} 2^{2j} \left| \left\langle h, \psi^{\omega, per}_{2^{-j}, 2^{-j/2}k, m, \iota} \right\rangle_{L^2(\mathbb{R}^2)} \right|^2 \\
&= c^4 \sum_{j \in c\mathbb{Z}} \sum_{k \in K_{c,j}} \sum_{m \in c_\varepsilon A_{\iota, 2^{-j}} \mathbb{Z}^2} \left| \left\langle \mathcal{F}\left(\frac{\partial}{\partial x_{i_\iota}} h \right), \mathcal{F}\left(\gamma^{\omega}_{2^{-j}, 2^{-j/2}k, m, \iota} \right) \right\rangle_{L^2(\mathbb{R}^2)} \right|^2 .
\end{aligned}
\tag{3.8}
$$

Following once again the computation from [13, Section 5.1.1] shows that (3.8) equals:

$$
\begin{aligned}
& W_1^\iota(h,c) + W_2^\iota(h,c) \\
&\coloneqq c^2 \sum_{j \in c\mathbb{Z}} \sum_{k \in K_{c,j}} \int_{\mathbb{R}^2} \omega(2^{-j/2}k, \iota)^2 \left| \mathcal{F}\left(\frac{\partial}{\partial x_{i_\iota}} h \right) \right|^2 \left| \widehat{\gamma}\left(A_{2^{-j}} S^T_{-2^{j/2}k} \xi \right) \right|^2 \mathrm{d}\xi + W_2^\iota\left(\frac{\partial}{\partial x_{i_\iota}} h, c \right),
\end{aligned}
$$

where for a function $\mathcal{R} : (0,1] \to (0,1]$:

$$
\left| W_2^\iota\left(\frac{\partial}{\partial x_{i_\iota}} h, c \right) \right| \le \mathcal{R}(c) \left\| \frac{\partial}{\partial x_{i_\iota}} h \right\|^2_{L^2(\mathbb{R}^2)} ,
\tag{3.9}
$$

and for sufficiently small $c > 0$

$$
W_1^\iota(h,c) \ge \left\| |\xi_{i_\iota}| \cdot \hat{h} \cdot \chi_{\mathcal{C}^\iota} \right\|^2_{L^2(\mathbb{R}^2)} ,
$$

where $\mathcal{C}^1 \coloneqq \{x \colon |x_1|, |x_2| \ge 1, |x_1| \ge |x_2|\}$, and $\mathcal{C}^{-1} \coloneqq \{x \colon |x_1|, |x_2| \ge 1, |x_1| \le |x_2|\}$. Hence, we conclude that

$$
W_1^\iota(h,c) + W_2^\iota(h,c) \gtrsim \left\| |\xi| \hat{h}_{|\mathcal{C}^1 \cup \mathcal{C}^{-1}} \right\|^2_{L^2(\mathbb{R}^2)} - \mathcal{R}(c) |h|^2_{H^1(\mathbb{R}^2)}.
$$

We have that

$$
\left\| |\xi| \hat{h}_{|\mathcal{C}^1 \cup \mathcal{C}^{-1}} \right\|^2_{L^2(\mathbb{R}^2)} \gtrsim |h|^2_{H^1(\mathbb{R}^2)} - \|h\|^2_{L^2(\mathbb{R}^2)}.
$$

If we now insert u_ε for h we deduce that

$$
\varepsilon \big(W_1^\iota(u_\varepsilon, c_\varepsilon) + W_2^\iota(u_\varepsilon, c_\varepsilon) \big) \gtrsim \varepsilon (1 - \mathcal{R}(c_\varepsilon)) |u_\varepsilon|^2_{H^1(\mathbb{R}^2)} - \varepsilon \|u_\varepsilon\|^2_{L^2(\mathbb{R}^2)}.
\tag{3.10}
$$

Applying [13, Proposition 3.3] to [13, Equation (38)] yields that $\mathcal{R}(c_\varepsilon) \lesssim c_\varepsilon$ and thus (3.10) implies (3.7). To obtain (3.6), we now estimate with (3.9):

$$
\left| S(h) - \sum_{\iota = -1, 1} \big(W_1^\iota(h,c) + W_2^\iota(h,c) \big) \right| \le \left| S(h) - W_1^1(h,c) + W_1^{-1}(h,c) \right| + \mathcal{O}\left(\mathcal{R}(c) |h|^2_{H^1} \right).
\tag{3.11}
$$

We rewrite W_1^ι as a triple integral by using the rounding operators (3.1) and (3.2):

$$
\begin{aligned}
W_1^\iota(h,c) = & \int_{-\log_2(\Gamma)}^{\infty} \int_{-2^{[j]_c/2}\Delta}^{[2^{[j]_c/2}\Delta]_{j,c} + c} \int_{\mathbb{R}^2} \omega\left(2^{-[j]_c/2} [k]_{j,c}, \iota \right)^2 \\
& \cdot \left| \mathcal{F}\left(\frac{\partial}{\partial x_{i_\iota}} h \right) \right|^2 \left| \widehat{\gamma}\left(A_{2^{-[j]_c}} S^T_{-2^{[j]_c/2}[k]_{j,c}} \xi \right) \right|^2 \mathrm{d}\xi \, \mathrm{d}k \, \mathrm{d}j.
\end{aligned}
$$

We proceed by estimating $|S(h) - W_1^1(h,c) - W_1^{-1}(h,c)|$:

$$
\begin{aligned}
&|S(h) - W_1^1(h,c) - W_1^{-1}(h,c)| \\
\le\ & \int_{\mathbb{R}^2} \left|\mathcal{F}\left(\frac{\partial}{\partial x_{i_\iota}} h\right)\right|^2 \int_{-\log_2(\Gamma)}^{\infty} \int_{-2^{j/2}\Delta}^{2^{j/2}\Delta} \left|\omega\left(2^{-[j]_c/2}[k]_{j,c},\iota\right)^2 \left|\widehat{\gamma}\left(A_{2^{-[j]_c}} S^T_{-2^{[j]_c/2}[k]_{j,c}}\xi\right)\right|^2\right. \\
&- \left.\omega\left(2^{-j/2}k,\iota\right)^2 \left|\widehat{\gamma}\left(A_{2^{-j}} S^T_{-2^{j/2}k}\xi\right)\right|^2\right| \mathrm{d}k \,\mathrm{d}j \,\mathrm{d}\xi \\
&+ \int_{\mathbb{R}^2} \left|\mathcal{F}\left(\frac{\partial}{\partial x_{i_\iota}} h\right)\right|^2 \int_{-\log_2(\Gamma)}^{\infty} \int_{2^{j/2}\Delta}^{[2^{[j]_c/2}\Delta]_{j,c}+c} \omega\left(2^{-[j]_c/2}[k]_{j,c},\iota\right)^2 \left|\widehat{\gamma}\left(A_{2^{-[j]_c}} S^T_{-2^{[j]_c/2}[k]_{j,c}}\xi\right)\right|^2 \mathrm{d}k \,\mathrm{d}j \,\mathrm{d}\xi \\
\le\ & |h|^2_{H^1(\mathbb{R}^2)} \sup_{\xi\in\mathbb{R}^2} \left(\int_{-\log_2(\Gamma)}^{\infty} \int_{-2^{j/2}\Delta}^{2^{j/2}\Delta} \left|\omega\left(2^{-[j]_c/2}[k]_{j,c},\iota\right)^2 \left|\widehat{\gamma}\left(A_{2^{-[j]_c}} S^T_{-2^{[j]_c/2}[k]_{j,c}}\xi\right)\right|^2\right.\right. \\
&- \left.\omega(2^{-j/2}k,\iota)^2 \left|\widehat{\gamma}\left(A_{2^{-j}} S^T_{-2^{j/2}k}\xi\right)\right|^2\right| \mathrm{d}k \,\mathrm{d}j \\
&+ \left.\int_{-\log_2(\Gamma)}^{\infty} \int_{2^{j/2}\Delta}^{[2^{[j]_c/2}\Delta]_{j,c}+c} \omega\left(2^{-[j]_c/2}[k]_{j,c},\iota\right)^2 \left|\widehat{\gamma}\left(A_{2^{-[j]_c}} S^T_{-2^{[j]_c/2}[k]_{j,c}}\xi\right)\right|^2 \mathrm{d}k \,\mathrm{d}j\right) \\
=:\ & |h|^2_{H^1(\mathbb{R}^2)} \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^2} \big(U_1(\xi,c) + U_2(\xi,c)\big),
\end{aligned}
\tag{3.12}
$$

where

$$
\begin{aligned}
U_1(\xi,c) &:= \int_{-\log_2(\Gamma)}^{\infty} \int_{-2^{j/2}\Delta}^{2^{j/2}\Delta} \left|\omega\left(2^{-[j]_c/2}[k]_{j,c},\iota\right)^2 \left|\widehat{\gamma}\left(A_{2^{-[j]_c}} S^T_{-2^{[j]_c/2}[k]_{j,c}}\xi\right)\right|^2\right. \\
&\qquad \left. -\omega\left(2^{-j/2}k,\iota\right)^2 \left|\widehat{\gamma}\left(A_{2^{-j}} S^T_{-2^{j/2}k}\xi\right)\right|^2\right| \mathrm{d}k \,\mathrm{d}j \quad \text{and} \\
U_2(\xi,c) &:= \int_{-\log_2(\Gamma)}^{\infty} \int_{2^{j/2}\Delta}^{[2^{[j]_c/2}\Delta]_{j,c}+c} \omega\left(2^{-[j]_c/2}[k]_{j,c},\iota\right)^2 \left|\widehat{\gamma}\left(A_{2^{-[j]_c}} S^T_{-2^{[j]_c/2}[k]_{j,c}}\xi\right)\right|^2 \mathrm{d}k \,\mathrm{d}j.
\end{aligned}
$$

We estimate $U_1(\xi,c)$ and $U_2(\xi,c)$ individually and start with $U_2(\xi,c)$. We have that, for all $\xi \in \mathbb{R}^2$,

$$
|\widehat{\gamma}(\xi)| \lesssim \frac{\min\{|\xi_1|^M, 1\}}{(1+|\xi_1|^2)^{\frac{L}{2}}(1+|\xi_2|^2)^{\frac{L}{2}}}. \tag{3.13}
$$

Hence

$$
|U_2(\xi,c)| \lesssim \int_{-\log_2(\Gamma)}^{\infty} \int_{2^{j/2}\Delta}^{[2^{[j]_c/2}\Delta]_{j,c}+c} \frac{\min\{|2^{-j}\xi_1|^{2M}, 1\}}{(1+|2^{-j}\xi_1|^2)^L(1+|2^{-j/2}(\xi_2 - 2^{-[j]_c/2}[k]_{j,c}\xi_1|^2))^L} \mathrm{d}k \,\mathrm{d}j.
$$

Since $[k]_{j,c} = [2^{[j]_c/2}\Delta]_{j,c}$ on the domain of integration we have that, if $\xi_1 \neq 0$, then

$$
\begin{aligned}
|U_2(\xi,c)| &\lesssim c \int_{-\log_2(\Gamma)}^{\infty} \frac{\min\{|2^{-j}\xi_1|^{2M}, 1\}}{(1+|2^{-j}\xi_1|^2)^L(1+|2^{-j/2}(\xi_2 - 2^{-[j]_c/2}[2^{[j]_c/2}\Delta]_{j,c}\xi_1|^2))^L} \mathrm{d}j \\
&\le c \int_{-\log_2(\Gamma)}^{\infty} \frac{\min\{|2^{-j}\xi_1|^{2M}, 1\}}{(1+|2^{-j}\xi_1|^2)^L} \mathrm{d}j \\
&\le c \int_{-\infty}^{\infty} \frac{\min\{|2^{-j+\log_2(|\xi_1|)}|^{2M}, 1\}}{(1+|2^{-j+\log_2(|\xi_1|)}|^2)^L} \mathrm{d}j = c \int_{-\infty}^{\infty} \frac{\min\{|2^{-j}|^{2M}, 1\}}{(1+|2^{-j}|^2)^L} \mathrm{d}j \lesssim c.
\end{aligned}
\tag{3.14}
$$

We continue by estimating $U_1(\xi,c)$. We compute with the binomial formula that, for all $\xi\in\mathbb{R}^2$,

$$\begin{aligned}
&U_1(\xi,c)\\
&= \int_{-\log_2(\Gamma)}^{\infty}\int_{-2^{j/2}\Delta}^{2^{j/2}\Delta}\Big|\omega\left(2^{-[j]_c/2}[k]_{j,c},\iota\right)^2\left|\widehat{\gamma}\left(A_{2^{-[j]_c}}S^T_{-2^{[j]_c/2}[k]_{j,c}}\xi\right)\right|^2\\
&\quad -\omega(2^{-j/2}k,\iota)^2\left|\widehat{\gamma}\left(A_{2^{-j}}S^T_{-2^{j/2}k}\xi\right)\right|^2\Big|\,\mathrm{d}k\,\mathrm{d}j\\
&= \int_{-\log_2(\Gamma)}^{\infty}\int_{-2^{j/2}\Delta}^{2^{j/2}\Delta}\left|\omega\left(2^{-[j]_c/2}[k]_{j,c},\iota\right)\widehat{\gamma}\left(A_{2^{-[j]_c}}S^T_{-2^{[j]_c/2}[k]_{j,c}}\xi\right)-\ \omega(2^{-j/2}k,\iota)\widehat{\gamma}\left(A_{2^{-j}}S^T_{-2^{j/2}k}\xi\right)\right|\\
&\quad\cdot\left|\omega\left(2^{-[j]_c/2}[k]_{j,c},\iota\right)\widehat{\gamma}\left(A_{2^{-[j]_c}}S^T_{-2^{[j]_c/2}[k]_{j,c}}\xi\right)+\ \omega(2^{-j/2}k,\iota)\widehat{\gamma}\left(A_{2^{-j}}S^T_{-2^{j/2}k}\xi\right)\right|\mathrm{d}k\,\mathrm{d}j. \qquad (3.15)
\end{aligned}$$

Using the decay estimate (3.13), we obtain that if $\iota=1$, then, for all $\xi\in\mathbb{R}^2$,

$$\begin{aligned}
&\left|\omega\left(2^{-[j]_c/2}[k]_{j,c},\iota\right)\widehat{\gamma}\left(A_{2^{-[j]_c}}S^T_{-2^{[j]_c/2}[k]_{j,c}}\xi\right)+\ \omega(2^{-j/2}k,\iota)\widehat{\gamma}\left(A_{2^{-j}}S^T_{-2^{j/2}k}\xi\right)\right|\\
&\lesssim\ \frac{\min\{|2^{-j}\xi_1|,1\}^M}{(1+|2^{-j}\xi_1|^2)^{\frac{L}{2}}(1+|2^{-j/2}(\xi_2+2^{-j/2}k\xi_1)|^2)^{\frac{L}{2}}}. \qquad (3.16)
\end{aligned}$$

If $\iota=-1$, then the roles of ξ_1 and ξ_2 are reversed. Additionally, by the Lipschitz continuity of ω, $\widehat{\gamma}$ and the Lipschitz continuity of $x\mapsto 2^x$ and $x\mapsto 2^{x/2}$ for x in a neighbourhood of 0, we conclude that, for all $\xi\in\mathbb{R}^2$,

$$\begin{aligned}
&\left|\omega\left(2^{-[j]_c/2}[k]_{j,c},\iota\right)\widehat{\gamma}\left(A_{2^{-[j]_c}}S^T_{-2^{[j]_c/2}[k]_{j,c}}\xi\right)-\ \omega(2^{-j/2}k,\iota)\widehat{\gamma}\left(A_{2^{-j}}S^T_{-2^{j/2}k}\xi\right)\right|\\
&\lesssim\ \left|A_{2^{-j}}S^T_{-2^{j/2}k}\xi-A_{2^{-[j]_c}}S^T_{-2^{[j]_c/2}[k]_{j,c}}\xi\right|\\
&\le\ \left|\left(2^{-j}-2^{-[j]_c}\right)\xi_1,\left(2^{-j/2}-2^{-[j]_c/2}\right)\xi_2-\left(2^{-j}k-2^{-[j]_c}[k]_{j,c}\right)\xi_1\right|\\
&\le\ \left|\left(c2^{-j}\xi_1,\left(2^{-j/2}-2^{-[j]_c/2}\right)\xi_2-\left(2^{-j}k-2^{-[j]_c/2}2^{-j/2}k\right)\xi_1-\left(2^{-[j]_c/2}2^{-j/2}k-2^{-[j]_c}[k]_{j,c}\right)\xi_1\right)\right|\\
&\le\ \left|\left(c2^{-j}\xi_1,\left(2^{-j/2}-2^{-[j]_c/2}\right)\left(\xi_2-2^{-j/2}k\xi_1\right)-\left(2^{-[j]_c/2}2^{-j/2}k-2^{-[j]_c}[k]_{j,c}\right)\xi_1\right)\right|\\
&\le\ c\left|2^{-j}\xi_1,\left(2^{-j/2}\right)\left(\xi_2-2^{-j/2}k\xi_1\right)\right|+c\left|2^{-[j]_c}\xi_1\right|\\
&\lesssim\ c\left(1+\left|2^{-j}\xi_1\right|^2\right)^{\frac{1}{2}}\left(1+\left|2^{-j/2}\left(\xi_2+2^{-j/2}k\xi_1\right)\right|^2\right)^{\frac{1}{2}}. \qquad (3.17)
\end{aligned}$$

Combining (3.15), (3.16) and (3.17) yields that

$$U_1(\xi,c)\lesssim c\int_{-\log_2(\Gamma)}^{\infty}\int_{-2^{j/2}\Delta}^{2^{j/2}\Delta}\frac{\min\{|2^{-j}\xi_1|,1\}^M}{(1+|2^{-j}\xi_1|^2)^{\frac{L}{2}-1}(1+|2^{-j/2}(\xi_2+2^{-j/2}k\xi_1)|^2)^{\frac{L}{2}-1}}\,\mathrm{d}k\,\mathrm{d}j\lesssim c. \qquad (3.18)$$

Additionally, applying (3.18), (3.14) to (3.12) and (3.11) shows that

$$\left|S(h)-\sum_{\iota=-1,1}\left(W_1^\iota(h,c)+W_2^\iota(h,c)\right)\right|=\mathcal{O}\left((c+R(c))\,|h|^2_{H^1}\right)=\mathcal{O}\left(c|h|^2_{H^1}\right),$$

which after replacing h by u_ε and c by c_ε implies (3.6). This completes the proof.

□

A Equivalence of the shearlet-based Besov seminorm and the H^1 seminorm

We first compute that

$$\mathcal{F}(\psi_{a,s,t})(\xi) = a^{\frac{3}{4}} \widehat{\psi}\left(a\xi_1, \sqrt{a}(\xi_2 + s\xi_1)\right) \quad \text{for all } \xi \in \mathbb{R}^2. \tag{A.1}$$

Now let $f \in L^2(\mathbb{R}^2)$, $\Gamma, \Delta > 0$; then by Parseval's identity and (A.1) we have that

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\langle f, K(\cdot - t)\rangle|^2 \, \mathrm{d}t + \int_{\mathbb{R}^2} \int_{-\Delta}^{\Delta} \int_0^{\Gamma} a^{-2} |\langle f, \psi_{a,s,t}\rangle|^2 a^{-3} \, \mathrm{d}a \, \mathrm{d}s \, \mathrm{d}t \\
&\quad + \int_{\mathbb{R}^2} \int_{-\Delta}^{\Delta} \int_0^{\Gamma} a^{-2} \left|\left\langle f, \widetilde{\psi}_{a,s,t}\right\rangle\right|^2 a^{-3} \, \mathrm{d}a \, \mathrm{d}s \, \mathrm{d}t \\
&= \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \Bigg(|\widehat{K}(\xi)|^2 + \int_{-\Delta}^{\Delta} \int_0^{\Gamma} \left|\widehat{\psi}(a\xi_1, \sqrt{a}(\xi_2 + s\xi_1))\right|^2 a^{-\frac{7}{2}} \, \mathrm{d}a \, \mathrm{d}s \\
&\quad + \int_{\mathbb{R}^2} \int_{-\Delta}^{\Delta} \int_0^{\Gamma} \left|\widehat{\widetilde{\psi}}(\sqrt{a}(\xi_1 + s\xi_2), a\xi_2)\right|^2 a^{-\frac{7}{2}} \, \mathrm{d}a \, \mathrm{d}s \Bigg) \, \mathrm{d}\xi =: \mathrm{I}(f).
\end{aligned}$$

Let $\widehat{\mu}(\xi) := \widehat{\psi}(\xi)/\xi_1$ and $\widehat{\widetilde{\mu}}(\xi) := \widehat{\widetilde{\psi}}(\xi)/\xi_2$; then,

$$\begin{aligned}
\mathrm{I}(f) = \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \Bigg(&|\widehat{K}(\xi)|^2 + |\xi_1|^2 \cdot \int_{-\Delta}^{\Delta} \int_0^{\Gamma} \left|\widehat{\mu}(a\xi_1, \sqrt{a}(\xi_2 + s\xi_1))\right|^2 a^{-\frac{3}{2}} \, \mathrm{d}a \, \mathrm{d}s \\
&+ |\xi_2|^2 \cdot \int_{\mathbb{R}^2} \int_{-\Delta}^{\Delta} \int_0^{\Gamma} \left|\widehat{\widetilde{\mu}}(\sqrt{a}(\xi_1 + s\xi_2), a\xi_2)\right|^2 a^{-\frac{3}{2}} \, \mathrm{d}a \, \mathrm{d}s \Bigg) \, \mathrm{d}\xi.
\end{aligned} \tag{A.2}$$

It was shown in [10, Theorem 4.3] and [10, Lemma 4.2] that there exist Δ^*, Γ^* such that for all $\Gamma > \Gamma^*, \Delta > \Delta^*$ there exist $A, B \in (0, \infty)$ such that

$$\begin{aligned}
A &\le \int_{-\Delta}^{\Delta} \int_0^{\Gamma} \left|\widehat{\mu}(\sqrt{a}(\xi_1 + s\xi_2), a\xi_2)\right|^2 a^{-\frac{3}{2}} \, \mathrm{d}a \, \mathrm{d}s, && \text{for all } \xi \text{ such that } |\xi_1| \ge |\xi_2| \text{ and } |\xi_1| \ge 1; \\
A &\le \int_{-\Delta}^{\Delta} \int_0^{\Gamma} \left|\widehat{\widetilde{\mu}}(\sqrt{a}(\xi_1 + s\xi_2), a\xi_2)\right|^2 a^{-\frac{3}{2}} \, \mathrm{d}a \, \mathrm{d}s, && \text{for all } \xi \text{ such that } |\xi_2| \ge |\xi_1| \text{ and } |\xi_2| \ge 1; \\
&\int_{-\Delta}^{\Delta} \int_0^{\Gamma} \left|\widehat{\mu}(\sqrt{a}(\xi_1 + s\xi_2), a\xi_2)\right|^2 a^{-\frac{3}{2}} \, \mathrm{d}a \, \mathrm{d}s \le B, && \text{for all } \xi \in \mathbb{R}^2; \\
&\int_{-\Delta}^{\Delta} \int_0^{\Gamma} \left|\widehat{\widetilde{\mu}}(\sqrt{a}(\xi_1 + s\xi_2), a\xi_2)\right|^2 a^{-\frac{3}{2}} \, \mathrm{d}a \, \mathrm{d}s \le B, && \text{for all } \xi \in \mathbb{R}^2.
\end{aligned}$$

We have that, if $|\xi_1| \ge |\xi_2|$, then $|\xi|^2/2 \le |\xi_1|^2 \le |\xi|^2$ and if $|\xi_2| \ge |\xi_1|$, then $|\xi|^2/2 \le |\xi_2|^2 \le |\xi|^2$. Additionally, by assumption, we have that there exist $0 < \widetilde{A} < \widetilde{B}$ such that $|K(\xi)|/|\xi| \in [\widetilde{A}, \widetilde{B}]$ for all $\xi \in [-1, 1]^2$. Hence, for $\Gamma > \Gamma^*, \Delta > \Delta^*$ we have that

$$\mathrm{I}(f) = \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 |S(\xi)|^2 \, \mathrm{d}\xi,$$

where $S : \mathbb{R}^2 \to \mathbb{R}$ is such that

$$\min\left\{\widetilde{A}, \frac{A}{2}\right\} |\xi|^2 \le |S(\xi)|^2 \le \max\left\{B, \widetilde{B}\right\} |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^2.$$

Since $|f|_{H^1} \sim \int_{\mathbb{R}^2} |\xi|^2 |\hat{f}(\xi)|^2 \, \mathrm{d}\xi$, this shows that $|f|_{H^1} \sim I(f)$ for all $f \in H^1(\mathbb{R}^2)$.

B Proof of Proposition 2.14

Proof. Let α be the smallest interior or exterior angle of P and define $c_0 \coloneqq \min\{\sin(\alpha), 1\}/2$. We denote for $i = 1, \dots, N$ by g_i, the line segment with start point x_i and end point x_{i+1}. Let $r_1 \coloneqq \min\{\mathfrak{d}(g_i, g_j) : i, j \in \{1, \dots, N\},\ i > j,\ i - j > 1,\ (i,j) \neq (N,1)\}$ be the smallest distance between each line segment and the closest non-neighbouring line segment. Since ∂P is non-self-intersecting, we have that $r_1 > 0$. Let $r_2 \coloneqq \min_{i=1,\dots,N} |x_i - x_{i+1}|$, where $x_{N+1} = x_1$. Again, we have that $r_2 > 0$ since otherwise two vertices would coincide. We set $r_0 \coloneqq \min\{r_1, r_2\}/8$.

Let $r < r_0$ and let $i \in \{1, \dots, N\}$; then, by the Pythagorean Theorem, there is a coordinate $t \in \{1, 2\}$ such that

$$\kappa_i \coloneqq |(x_i)_t - (x_{i+1})_t| \geq |x_i - x_{i+1}|/\sqrt{2} \geq |x_i - x_{i+1}|/4 + r_2/4 \geq |x_i - x_{i+1}|/4 + 2r. \tag{B.1}$$

Define $L_i \coloneqq \lceil (\kappa_i - 2r)/(rc_0) \rceil$ and $s_i \coloneqq (\kappa_i - 2r)/(2L_i)$. We have that $L_i \geq |x_i - x_{i+1}|/(4r)$ by equation (B.1). Moreover, we have that

$$s_i = \frac{\kappa_i - 2r}{2\lceil (\kappa_i - 2r)/(c_0 r) \rceil} \leq c_0 r \frac{\kappa_i - 2r}{2(\kappa_i - 2r)} = \frac{c_0 r}{2}$$

and

$$s_i \geq \frac{\kappa_i - 2r}{(2 + 2(\kappa_i - 2r))/(c_0 r)} = c_0 r \frac{\kappa_i - 2r}{2 + 2(\kappa_i - 2r)} \geq \frac{c_0 r}{4},$$

where the last estimate follows since $0 < \kappa_i - 2r < 1$ and hence $(2 + 2(\kappa_i - 2r)) \leq 4$. It is not hard to see that g_i can be parametrised by

$$g_i(w) = x_i + \begin{pmatrix} \lambda_i^{t-1} w \\ \lambda_i^{2-t} w \end{pmatrix}, \quad w \in [0, \kappa_i],$$

for a $|\lambda| = 1$. We define, for $k = 1, \dots, L_i$, $w_k \coloneqq r + 2s_i k - s_i$ and

$$z_{k,i} \coloneqq x_i + \begin{pmatrix} \lambda_i^{t-1} w_k \\ \lambda_i^{2-t} w_k \end{pmatrix}.$$

Since $|\lambda| = 1$ it follows that $|(z_{k,i})_t - (z_{k,i+1})_t| = 2s_i$ for $i = 1, \dots, L_i - 1$ and $|(z_{k,i})_{t'} - (z_{k,i+1})_{t'}| \leq 2s_i$ for $i = 1, \dots, L_i - 1$, where $t' = 2$ if $t = 1$ and $t' = 1$ if $t = 2$. Moreover, $|(x_i)_t - (z_{1,i})_t| = r$ and $|(x_i)_{t'} - (z_{1,i})_{t'}| \leq r$ and $|(x_{i+1})_t - (z_{L_i,1})_t| = r$ and $|(x_{i+1})_{t'} - (z_{L_i,i})_{t'}| \leq r$. This shows that

$$g_i \subset \dot{\bigcup}_{k=1,\dots,L_i} C(z_{k,i}, s_i) \,\dot{\cup}\, \dot{\bigcup}_{\ell=i,i+1} C(x_\ell, r). \tag{B.2}$$

Next, we compute for any $i \leq N$ and $k_i \leq L_i$ the distance between $z_{k_i,i}$ and any line segment g_j, $j \neq i$. If $|j - i| > 1$ and $(i,j) \neq (1,N), (N,1)$, then we know by assumption that $\mathfrak{d}(z_{k_i,i}, g_j) \geq r_1 \geq 8r$. We also obtain that $\mathfrak{d}(x_i, g_j) \geq 8r$ and $\mathfrak{d}(x_{i+1}, g_j) \geq 8r$. Hence, for any $k_j \leq L_j$ we have

$$C(z_{k_i,i}, s_i) \cap C(z_{k_j,j}, s_j) = \varnothing \quad \text{and} \quad C(z_{k_i,i}, s_i) \cap C(x_j, r) = \varnothing. \tag{B.3}$$

If $j = i + 1$ or $j = 1$, $i = N$, then let $z^* \in g_j$ be such that $\mathfrak{d}(z_{k,i}, z^*) = \mathfrak{d}(z_{k,i}, g_{i+1})$. This choice is possible since g_j is compact. If $z^* = x_j$ then we have by construction $\mathfrak{d}(z_{k,i}, z^*) \geq s_i + r$. If the smallest angle between g_i and g_j is larger than or equal to $\pi/2$ then it is clear that $z^* = x_j$, so we can proceed by assuming that the smallest angle β between g_i and g_j satisfies $\alpha \leq \beta < \pi/2$. By construction, the triangle with vertices $z_{k,i}, z^*, x_j$ has a right angle at z^*. By the law of sines, we conclude $\mathfrak{d}(z_{k,i}, z^*)/\sin(\beta) = |z_{k,i} - x_j|$. Using that $|z_{k,i} - x_j| > r + s_i$ we conclude that $\mathfrak{d}(z_{k,i}, z^*) \geq 2(r + s_i)\sin(\alpha) = 2(r + s_i)c_0$.

We conclude, for all $i \leq N$, $k_1 \leq L_i$ and $k_2 \leq L_{i+1}$, that

$$C(z_{k_1,i}, s_i) \cap C(z_{k_2,i}, s_{i+1}) \subset B_{\sqrt{2}s_i}(z_{k_1,i}) \cap B_{\sqrt{2}s_{i+1}}(z_{k_2,i+1}) = \varnothing, \tag{B.4}$$

since $2\sqrt{2}s_i < rc_0/\sqrt{2} < |z_{k_2,j} - z_{k_1,i}|$. From (B.2), (B.3), and (B.4) we conclude that for any $i, j \leq N$, $k_i \leq L_i$, $k_j \leq L_j$ we have that

$$\begin{aligned} C(z_{k_i,i}, s_i) \cap C(z_{k_j,j}, s_j) &= \varnothing, \text{ if } (i, k_i) \neq (j, k_j), \text{ and} \\ C(x_i, r) \cap C(x_i, r) &= \varnothing, \text{ if } j \neq i, \\ C(z_{k_i,i}, s_i) \cap C(x_j, r) &= \varnothing. \end{aligned} \tag{B.5}$$

We set $L := \sum_{i=1}^N L_i$ and get that $L \geq \sum_{i=1}^N |x_i - x_{i+1}|/4 = \ell(P)/4$. Next, we conclude from (B.2) and (B.5) that (2.32) holds. Finally, it is not hard to see that, $C(z_{k,i}, s_i + c_0 r/4) \cap g_j = \varnothing$ for all $j \neq i$ and therefore $\chi_{P \cap C(z_{k,i}, s_i + c_0 r/4)}(\cdot - z_{k,i})$ is a rotated Heaviside function on $[-s_i - c_0 r/4, s_i + c_0 r/4]^2$ with the same normal direction as ∂P at $z_{k,i}$. $\square$

C Some integral estimates

We start by showing equation (2.28). We have that

$$\begin{aligned} \int_\Gamma^\infty \int_{\mathbb{R}} |\widehat{\mu}_{a,s,0,1}(\xi)|^2 a^{-3} \mathrm{d}s\, \mathrm{d}a &= \int_\Gamma^\infty \int_{\mathbb{R}} \left| a^{\frac{3}{4}} \frac{\min\{|a\xi_1|^3, 1\}}{(1 + |a\xi_1|)^4 (1 + |\sqrt{a}(s\xi_1 + \xi_2)|)^4} \right|^2 a^{-3} \mathrm{d}s\, \mathrm{d}a \\ &\leq \left(\frac{1}{(1 + |\Gamma\xi_1|)^2} \right) \int_\Gamma^\infty \int_{\mathbb{R}} \left| \frac{\min\{|a\xi_1|^3, 1\}}{(1 + |a\xi_1|)^3 (1 + |\sqrt{a}(s\xi_1 + \xi_2)|)^4} \right|^2 a^{-\frac{3}{2}} \mathrm{d}s\, \mathrm{d}a \\ &\lesssim \frac{1}{|\Gamma|^2} \left(\frac{1}{(1 + |\xi_1|)^2} \right) \int_{\mathbb{R}^+} \int_{\mathbb{R}} \left| \frac{\min\{|a\xi_1|^3, 1\}}{(1 + |a\xi_1|)^3 (1 + |\sqrt{a}(s\xi_1 + \xi_2)|)^4} \right|^2 a^{-\frac{3}{2}} \mathrm{d}s\, \mathrm{d}a. \end{aligned}$$

By [10, Equation 5], we have that

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}} \left| \frac{\min\{|a\xi_1|^3, 1\}}{(1 + |a\xi_1|)^3 (1 + |\sqrt{a}(s\xi_1 + \xi_2)|)^4} \right|^2 a^{-\frac{3}{2}} \mathrm{d}s\, \mathrm{d}a < \infty, \tag{C.1}$$

which yields (2.28).

Next, we demonstrate (2.29). If $\nu_s |\xi_1|/2 \geq |\xi_2|$ then

$$\begin{aligned} &\int_0^\Gamma \int_{-\infty}^{-\nu_s} |\widehat{\mu}_{a,s,0,1}(\xi)|^2 a^{-3} \mathrm{d}s\, \mathrm{d}a \\ &= \int_0^\Gamma \int_{\mathbb{R}} \left| a^{\frac{3}{4}} \frac{\min\{|a\xi_1|^3, 1\}}{(1 + |a\xi_1|)^4 (1 + |\sqrt{a}(s\xi_1 + \xi_2)|)^4} \right|^2 a^{-3} \mathrm{d}s\, \mathrm{d}a \\ &\leq \int_0^\Gamma \int_{\mathbb{R}} \left| a^{\frac{3}{4}} \frac{\min\{|a\xi_1|^3, 1\}}{(1 + |\sqrt{a}(s\xi_1)|)^2 (1 + |a\xi_1|)^4 (1 + |\sqrt{a}(s\xi_1 + \xi_2)|)^2} \right|^2 a^{-3} \mathrm{d}s\, \mathrm{d}a \\ &\leq \int_0^\Gamma \frac{\min\{|a\xi_1|^2, 1\}}{(1 + |\sqrt{a}(s\xi_1)|)^4} \int_{\mathbb{R}} \left| a^{\frac{3}{4}} \frac{\min\{|a\xi_1|^3, 1\}}{(1 + |a\xi_1|)^4 (1 + |\sqrt{a}(s\xi_1 + \xi_2)|)^2} \right|^2 a^{-3} \mathrm{d}s\, \mathrm{d}a \\ &\leq \frac{|\xi_1|^2}{(1 + |(s\xi_1)|)^4} \int_0^\Gamma \int_{\mathbb{R}} \left| a^{\frac{3}{4}} \frac{\min\{|a\xi_1|^3, 1\}}{(1 + |a\xi_1|)^4 (1 + |\sqrt{a}(s\xi_1 + \xi_2)|)^2} \right|^2 a^{-3} \mathrm{d}s\, \mathrm{d}a \\ &\lesssim \frac{|\xi_1|^2}{(1 + |(s\xi_1)|)^4}, \end{aligned}$$

where we again invoked [10, Equation 5] to estimate the integral over s and a. This completes the estimate of (2.29).

D The Bessel inequality

Let $f \in L^2(\mathbb{R}^2)$; we then compute, by Plancherel's theorem, that

$$\int_0^\Gamma \int_{-\Delta}^{\Delta} \int_{\mathbb{R}^2} |\langle f, \psi_{a,s,t,\iota}\rangle|^2 a^{-3} \; \mathrm{d}t \, \mathrm{d}s \, \mathrm{d}a = \int_0^\Gamma \int_{-\Delta}^{\Delta} \int_{\mathbb{R}^2} \left|\left\langle \hat{f}\widehat{\psi}_{a,s,0,\iota}, \mathrm{e}^{-2\pi i \langle \cdot, m\rangle} \right\rangle\right|^2 a^{-3} \; \mathrm{d}t \, \mathrm{d}s \, \mathrm{d}a.$$

Using Parseval's identity and Fubini's theorem, we obtain that

$$\begin{aligned} \int_0^\Gamma \int_{-\Delta}^{\Delta} \int_{\mathbb{R}^2} \left|\left\langle \hat{f} \; \widehat{\psi}_{a,s,0,\iota}, \mathrm{e}^{-2\pi i \langle \cdot, m\rangle} \right\rangle\right|^2 a^{-3} \; \mathrm{d}t \, \mathrm{d}s \, \mathrm{d}a &= \int_0^\Gamma \int_{-\Delta}^{\Delta} \left\| \hat{f}\widehat{\psi}_{a,s,0,\iota} \right\|_{L^2}^2 a^{-3} \mathrm{d}s \, \mathrm{d}a \\ &= \int_{\mathbb{R}^2} \left|\hat{f}(\xi)\right|^2 \int_0^\Gamma \int_{-\Delta}^{\Delta} \left|\widehat{\psi}_{a,s,0,\iota}\right|^2 a^{-3} \mathrm{d}s \, \mathrm{d}a \, d\xi. \end{aligned}$$

Equation (C.1) and

$$\left|\widehat{\phi^1}(\xi)\right| \leq (1+|\xi|)^{-4} \quad \text{ and } \quad \left|\widehat{\psi^1}(\xi)\right| \leq \frac{\min\{|\xi|, 1\}^4}{(1+|\xi|)^4}$$

imply that $\int_0^\Gamma \int_{-\Delta}^{\Delta} |\widehat{\psi}_{a,s,0,\iota}|^2 a^{-3} \mathrm{d}s \, \mathrm{d}a$ is uniformly bounded by a constant $C > 0$, which yields that

$$\int_0^\Gamma \int_{-\Delta}^{\Delta} \int_{\mathbb{R}^2} |\langle f, \psi_{a,s,t,\iota}\rangle|^2 a^{-3} \; \mathrm{d}t \, \mathrm{d}s \, \mathrm{d}a \leq C \|f\|_{L^2(\mathbb{R}^2)}^2.$$

If $f \in L^2((0,1)^2)$, then

$$\int_0^\Gamma \int_{-\Delta}^{\Delta} \int_{(0,1)^2} \left|\left\langle f, \psi_{a,s,t,\iota}^{per}\right\rangle\right|^2 a^{-3} \; \mathrm{d}t \, \mathrm{d}s \, \mathrm{d}a \leq \int_0^\Gamma \int_{-\Delta}^{\Delta} \int_{\mathbb{R}^2} \left|\left\langle f^{per}, \psi_{a,s,t,\iota}\right\rangle\right|^2 a^{-3} \; \mathrm{d}t \, \mathrm{d}s \, \mathrm{d}a,$$

where $f^{per}(x) = f(x - \lfloor x \rfloor)$ for all $x \in [-2,2]^2$, where $\lfloor \cdot \rfloor$ is applied componentwise, and $f^{per}(x) = 0$ on $\mathbb{R}^2 \setminus [-2,2]^2$. Hence,

$$\int_0^\Gamma \int_{-\Delta}^{\Delta} \int_{(0,1)^2} \left|\left\langle f, \psi_{a,s,t,\iota}^{per}\right\rangle\right|^2 a^{-3} \; \mathrm{d}t \, \mathrm{d}s \, \mathrm{d}a \leq \|f^{per}\|_{L^2(\mathbb{R}^2)}^2 \lesssim \|f\|_{L^2((0,1)^2)}^2.$$

Performing the same argument for the second cone yields that

$$\sum_{\iota=-1,1} \int_0^\Gamma \int_{-\Delta}^{\Delta} \int_{(0,1)^2} |\langle f, \psi_{a,s,t,\iota}\rangle|^2 a^{-3} \; \mathrm{d}t \, \mathrm{d}s \, \mathrm{d}a \lesssim \|f\|_{L^2((0,1)^2)}^2, \qquad \text{for all } f \in L^2((0,1)^2).$$

E Counterexample to the convergence result of Bertozzi and Dobrosotskaya

In [7, Theorem 1.1] it is claimed, that for any sequence $u_n \in H^1((0,1)^2)$ converging to χ_D in $L^1((0,1)^2)$, where D is a set of finite perimeter, the quotient

$$\frac{|u_n|_{B,p}}{|u_n|_{H^1((0,1)^2)}}$$

converges to $\mathcal{R}(\chi_D)$, and $\mathcal{R}(\chi_D)$ is a nonconstant function of ∂D. This theorem cannot hold as the following counterexample demonstrates.

For a contradiction, we assume that [7, Theorem 1.1] holds. Pick $D \subset (0,1/2)^2$ with finite perimeter and a sequence $(u_n)_{n\in\mathbb{N}} \subset H^1((0,1)^2)$, such that $u_n \to \chi_D$ in $L^1((0,1)^2)$, and $|u_n|_{H^1((0,1)^2)} \sim |u_n|_{B,p} \to \infty$ for $n \to$

∞, and supp $u_n \subset (0,1/2)^2$. Let g_n be a sequence in $H^1((0,1)^2)$ with $\|g_n\|_{L^1((0,1)^2)} \to 0$, supp $g_n \subset (1/2,1)^2$ and $|g_n|_{H^1((0,1)^2)} \sim |u_n|^2_{H^1((0,1)^2)}$. Since the $H^1((0,1)^2)$ seminorm of g_n is asymptotically much larger than the $H^1((0,1)^2)$ seminorm of u_n, one can easily observe that

$$\frac{|u_n + g_n|_{B,p}}{|u_n + g_n|_{H^1((0,1)^2)}} - \frac{|g_n|_{B,p}}{|g_n|_{H^1((0,1)^2)}} \to 0,$$

for $n \to \infty$. Hence, by [7, Theorem 1.1],

$$\frac{|u_n|_{B,p}}{|u_n|_{H^1((0,1)^2)}}, \quad \frac{|u_n + g_n|_{B,p}}{|u_n + g_n|_{H^1((0,1)^2)}}, \text{ and } \frac{|g_n|_{B,p}}{|g_n|_{H^1((0,1)^2)}},$$

all converge to the same limit $\mathcal{R}(\chi_D)$ for $n \to \infty$. The limit of $|g_n|_{B,p}/|g_n|_{H^1((0,1)^2)}$, however, is independent of ∂D, completing the asserted contradiction.

The main mistake in [7] causing the incorrect conclusion of [7, Theorem 1.1] can be found in [7, Lemma 2.6], where in the second to last line an upper bound of 2^J is mistaken for a lower bound. The best achievable lower bound from the results in [7] appears to be $2^{J/2}$.

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