STABLE LIFTINGS OF POLYNOMIAL TRACES ON TETRAHEDRA *

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Abstract. On the reference tetrahedron K, we construct, for each $k \in \mathbb{N}_0$, a right inverse for the trace operator $u \mapsto (u, \partial_{\mathbf{n}} u, \ldots, \partial_{\mathbf{n}}^k u)|_{\partial K}$. The operator is stable as a mapping from the trace space of $W^{s,p}(K)$ to $W^{s,p}(K)$ for all $p \in (1, \infty)$ and $s \in (k + 1/p, \infty)$. Moreover, if the data is the trace of a polynomial of degree $N \in \mathbb{N}_0$, then the resulting lifting is a polynomial of degree N. One consequence of the analysis is a novel characterization for the range of the trace operator.

Key words. trace lifting, polynomial extension, polynomial lifting

AMS subject classifications. 46E35, 65N30

1. Introduction.

The numerical analysis of high-order finite element and spectral element methods heavily rely on the existence of stable polynomial liftings – bounded operators mapping suitable piecewise polynomials on the boundary of the element to polynomials defined over the entire element. A number of operators have been constructed on the reference triangle and square, beginning with the pioneering work of Babuška et al. [14, 15]. Their lifting maps $H^{\frac{1}{2}}(\partial E)$ boundedly into $H^1(E)$, where E is either a suitable reference triangle or square, with the additional property that if the datum is continuous and its restriction to each edge is a polynomial of degree $N \geq 0$, then the lifting is a polynomial of degree N. Other constructions for continuous piecewise polynomials on ∂E are stable from a discrete trace space to $L^2(E)$ [4], from $L^2(\partial E)$ to $H^{\frac{1}{2}}(E)$ [3], from $W^{1-\frac{1}{p},p}(\partial E)$ to $W^{1,p}(E)$ for $1 < p < \infty$ [45], and from $W^{s-\frac{1}{p},p}(\partial E)$ to $W^{s,p}(E)$ for $s \geq 1$ and $1 < p < \infty$ [48]. Liftings for other types of traces are also available; e.g. lifting the normal trace of $H(\mathrm{div}; E)$ [3], lifting the trace and normal derivative simultaneously into $H^2(E)$ [2], and lifting an arbitrary number of normal derivatives simultaneously into $W^{s,p}(E)$ [48].

Many of the above results have been extended to three space dimensions. Muñoz-Sola [46] generalized the construction of Babuška et al. [14, 15] to the tetrahedron, while Belgacem [16] gave a different construction for the cube using orthogonal polynomials. Commuting lifting operators for the spaces appearing in the de Rham complex on tetrahedra [30, 31, 32] and hexahedra [27] have also been constructed. These operators, among others, have been used extensively in a priori error analysis [7, 15, 36, 39, 45, 46], a posteriori error analysis [22, 25, 26, 37], the analysis of preconditioners [4, 5, 8, 9, 11, 14, 49], the analysis of sprectral element methods, particularly in weighted Sobolev spaces [18, 19, 20, 21], and in the stability analysis of mixed finite element methods [6, 13, 28, 29, 33, 43]. Nevertheless, two types of operators are notably missing from the currently available results in three dimensions: (i) lifting operators stable in L^p-based Sobolev spaces, crucial in the analysis of high-order finite element methods for nonlinear problems; and (ii) lifting operators for the simultaneous lifting of the trace and normal derivative (and higher-order normal derivatives) which appear in the analysis of fourth-order (and higher-order) problems and in the analysis of mixed finite element methods for problems in electromagnetism and incompressible flow.

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We address both of the above problems; namely, for each $k \in \mathbb{N}_0$, we construct a right inverse for the trace operator $u \mapsto (u, \partial_{\mathbf{n}} u, \ldots, \partial_{\mathbf{n}}^k u)|_{\partial K}$ on the reference tetrahedron K that is stable from the trace of $W^{s,p}(K)$ to $W^{s,p}(K)$ for all $p \in (1, \infty)$ and $s \in (k + 1/p, \infty)$. Additionally, if the data is the trace of a polynomial of degree $N \in \mathbb{N}_0$, then the resulting lifting is a polynomial of degree N. A precise statement appears at the end of section 2, which also contains a characterization for the trace space that appears to be novel and some potential applications of the results. These results generalize our construction on the reference triangle [48] to the reference tetrahedron and to Sobolev spaces with minimal regularity.

The remainder of the manuscript is organized as follows. In section 3, we detail an explicit construction of the lifting operator in a sequence of four steps, each consisting of an intermediate single-face lifting operator. The remainder of the manuscript is devoted to the analysis of the intermediate single-face operators: sections 4 and 5 characterize the continuity of a related operator defined on all of $\mathbb{R}^3$, while section 6 concludes with the proofs of the continuity properties of the intermediate operators.

2. The traces of $W^{s,q}(K)$ functions and statement of main result. We begin by reviewing the regularity properties of the traces of a function u defined on a tetrahedron. Here, we will work in the setting of Sobolev spaces defined on an open Lipschitz domain $\mathcal{O} \subseteq \mathbb{R}^d$. Let $s = m + \sigma$ be a nonnegative real number with $m \in \mathbb{N}_0$ and $\sigma \in [0, 1)$. We denote by $W^{s,p}(\mathcal{O})$, $p \in [1, \infty)$, the standard fractional Sobolev(-Slobodeckij) space [1] equipped with norm defined by

$$\|v\|_{s,p,\mathcal{O}}^p := \sum_{n=1}^{m} |v|_{n,p,\mathcal{O}}^p + \begin{cases} \sum_{|\alpha|=m} |D^\alpha v|_{\sigma,p,\mathcal{O}}^p & \text{if } \sigma > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where the integer-valued seminorms and fractional seminorms are given by

$$|v|_{n,p,\mathcal{O}}^p := \sum_{|\alpha|=n} \int_{\mathcal{O}} |D^\alpha v(\boldsymbol{x})|^p \, \mathrm{d}\boldsymbol{x} \quad \text{and} \quad |v|_{\sigma,p,\mathcal{O}}^p := \iint_{\mathcal{O}\times\mathcal{O}} \frac{|v(\boldsymbol{x}) - v(\boldsymbol{y})|^p}{|\boldsymbol{x} - \boldsymbol{y}|^{\sigma p + d}} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{y},$$

with the usual modification for $p = \infty$. When $s = 0$, the Sobolev space $W^{0,p}(\mathcal{O})$ coincides with the standard Lebesgue space $L^p(\mathcal{O})$, and we denote the norm by $\|\cdot\|_{p,\mathcal{O}}$. We also require fractional Sobolev spaces defined on domain boundaries. Given a $C^{k,1}$, $k \in \mathbb{N}_0$, $(d-1)$-dimensional manifold $\Gamma \subseteq \partial\mathcal{O}$, the surface gradient D_Γ is well-defined a.e. on Γ, and we define $W^{s,p}(\Gamma)$, $0 \leq s \leq k+1$, analogously (see e.g. [47, §2.5.2]) with the norm

$$\|v\|_{s,p,\Gamma}^p := \sum_{|\beta|\leq m} \int_\Gamma |D_\Gamma^\beta v(\boldsymbol{x})|^p \, \mathrm{d}\boldsymbol{x} + \sum_{|\beta|=m} \iint_{\Gamma\times\Gamma} \frac{|D_\Gamma^\beta v(\boldsymbol{x}) - D_\Gamma^\beta v(\boldsymbol{y})|^p}{|\boldsymbol{x} - \boldsymbol{y}|^{\sigma p + d - 1}} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{y},$$

where the sums are over multi-indices $\beta \in \mathbb{N}_0^{d-1}$. The seminorms $|\cdot|_{s,p,\Gamma}$ are defined similarly.

2.1. Elementary trace results. When the domain is the reference tetrahedron $K := \{(x, y, z) \in \mathbb{R}^3 : 0 < x, y, z, x + y + z < 1\}$ depicted in Figure 1a, the space $W^{r,p}(\partial K)$, $0 \leq r < 1$, may be equipped with an equivalent norm that is more amenable to the analysis of traces. Let Γ_i and Γ_j, $1 \leq i < j \leq 4$, be two faces of K and let $\gamma_{ij} = \gamma_{ji}$ denote the shared edge with vertices $\boldsymbol{a}$ and $\boldsymbol{b}$. Then, the vertices of Γ_i are denoted by $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}_i$, while the vertices of Γ_j are denoted by $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}_j$. Since Γ_i and Γ_j are both triangles, there exist unique affine mappings $\boldsymbol{F}_{ij} : T \to \Gamma_i$

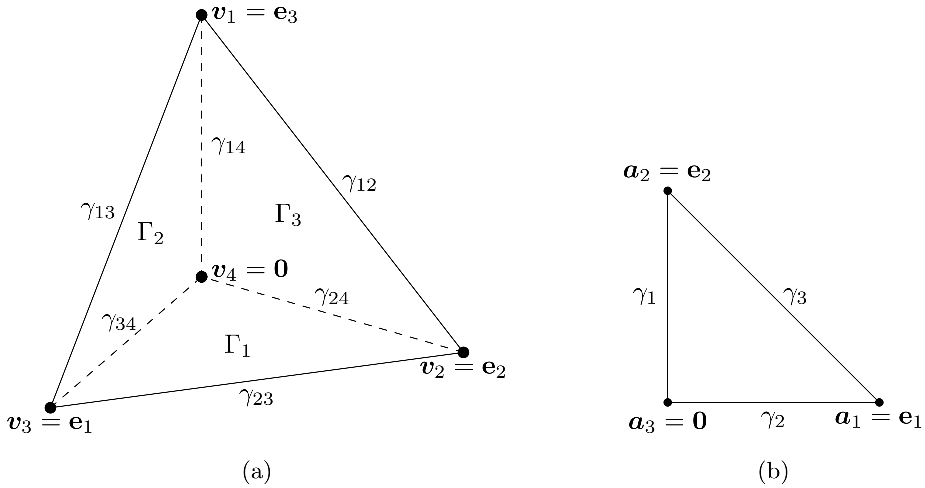

Fig. 1: Reference (a) tetrahedron and (b) triangle, where $\mathbf{e}_i$ are the standard unit vectors. Note that the label for $\Gamma_4 = \{(x, y, z) \in \bar{K} : x + y + z = 1\}$ is omitted in (a).

and $\boldsymbol{F}_{ji} : T \to \Gamma_j$ from the reference triangle $T := \{(x, y) \in \mathbb{R}^2 : 0 < x, y, x + y < 1\}$, labeled as in Figure 1b, satisfying

$$\boldsymbol{F}_{ij}(0,0) = \boldsymbol{a}, \qquad \boldsymbol{F}_{ij}(1,0) = \boldsymbol{b}, \quad \text{and} \quad \boldsymbol{F}_{ij}(0,1) = \boldsymbol{c}_i, \tag{2.1a}$$

$$\boldsymbol{F}_{ji}(0,0) = \boldsymbol{a}, \qquad \boldsymbol{F}_{ji}(1,0) = \boldsymbol{b}, \quad \text{and} \quad \boldsymbol{F}_{ji}(0,1) = \boldsymbol{c}_j, \tag{2.1b}$$

and we define the following norm:

$$|||f|||^p_{r,p,\partial K} := \sum_{i=1}^{4} \|f_i\|^p_{r,p,\Gamma_i} + \begin{cases} \displaystyle\sum_{1 \le i < j \le 4} \mathcal{I}^p_{ij}(f_i, f_j) & \text{if } rp = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where f_i denotes the restriction of f to Γ_i and $\mathcal{I}^p_{ij}(f, g)$ is defined by the rule

$$\mathcal{I}^p_{ij}(f, g) := \int_T |f \circ \boldsymbol{F}_{ij}(\boldsymbol{x}) - g \circ \boldsymbol{F}_{ji}(\boldsymbol{x})|^p \frac{\mathrm{d}\boldsymbol{x}}{x_2}. \tag{2.2}$$

Thanks to Corollary B.2, $|||\cdot|||_{r,p,\partial K}$ is an equivalent norm on $W^{r,p}(\partial K)$; i.e.

$$\|f\|_{r,p,\partial K} \approx_{r,p} |||f|||_{r,p,\partial K} \qquad \forall f \in W^{r,p}(\partial K), \tag{2.3}$$

and we shall use the two norms interchangeably with the common notation $\|\cdot\|_{r,p,\partial K}$. Here, and in what follows, we use the standard notation $a \lesssim_c b$ to indicate $a \le Cb$ where C is a constant depending only on c, and $a \approx_c b$ if $a \lesssim_c b$ and $b \lesssim_c a$.

Now let $u \in W^{s,p}(K)$, $1 < p < \infty$, $s = m + \sigma > 1/p$ with $m \in \mathbb{N}_0$ and $\sigma \in [0, 1)$ (so that the trace operator is well-defined), be a function defined on the reference tetrahedron. The presence of edges and corners on the boundary of K limits the regularity of the trace of u. Nevertheless, we can iteratively apply the standard $W^{s,p}(K)$ trace theorem (e.g. [41, Theorem 3.1] or [42, p. 208 Theorem 1]): $W^{s,p}(K)$

embeds continuously into $W^{s-\frac{1}{p},p}(\partial K)$ for $1/p < s < 1 + 1/p$. In particular, for $k \in \mathbb{N}_0$, the kth-order derivative tensor given by

$$(D^k u)_{i_1 i_2 \dots i_k} = \partial_{x_{i_1}} \partial_{x_{i_2}} \cdots \partial_{x_{i_k}} u$$

satisfies $D^k u \in W^{s-k,p}(K) \subset W^{1+\sigma,p}(K)$, $0 \le k \le m-1$, and $D^m u \in W^{\sigma,p}(K)$; thus, the traces satisfy

$$\begin{cases} D^k u|_{\partial K} \in W^{1-\frac{1}{p},p}(\partial K) & \text{for } 0 \le k < s - \frac{1}{p}, \\ D^{m-1} u|_{\partial K} \in W^{1+\sigma-\frac{1}{p},p}(\partial K) & \text{if } m \ge 1 \text{ and } \sigma p < 1, \\ D^m u|_{\partial K} \in W^{\sigma-\frac{1}{p},p}(\partial K) & \text{if } \sigma p > 1. \end{cases}$$

Additionally, the trace of $W^{m+\frac{1}{2},2}(\mathbb{R}^3)$, $m \ge 1$, on the plane $\mathbb{R}^2 \times \{0\}$ belongs to $W^{m,2}(\mathbb{R}^2)$ (see e.g. [1, Chapter 7] or [42, p. 20 Theorem 4]), and so standard arguments show that the trace of $W^{m+\frac{1}{2},2}(K)$ on the face Γ_i, $1 \le i \le 4$, belongs to $W^{m,2}(\Gamma_i)$. Thanks to the norm-equivalence (2.3), we arrive at the following conditions:

(2.4)
$$\begin{cases} \displaystyle\sum_{i=1}^{4} \|D^k u\|_{p,\Gamma_i}^p < \infty & \text{for } 0 \le k < s - \dfrac{1}{p}, \\ \displaystyle\sum_{i=1}^{4} \|D^{m-1} u\|_{1+\sigma-\frac{1}{p},p,\Gamma_i}^p < \infty & \text{if } m \ge 1 \text{ and either } \sigma p < 1 \text{ or } (\sigma, p) = \left(\dfrac{1}{2}, 2\right), \\ \displaystyle\sum_{i=1}^{4} \|D^m u\|_{\sigma-\frac{1}{p},p,\Gamma_i}^p < \infty & \text{if } \sigma p > 1, \\ \displaystyle\sum_{1 \le i < j \le 4} \mathcal{I}_{ij}^p(D^m u, D^m u) < \infty & \text{if } \sigma p = 2. \end{cases}$$

Remark 2.1. The case $\sigma p = 1$ for $p \neq 2$, which is not included in conditions (2.4), is beyond the scope of this paper since the trace of a $W^{m+\frac{1}{p},p}(\mathbb{R}^3)$, $m \in \mathbb{N}$, function on the plane $\mathbb{R}^2 \times \{0\}$ belongs to a Besov space, which cannot be identified with an integer-order Sobolev space [42, p. 20 Theorem 4].

When $s > 2/p$, we obtain additional conditions since the trace of a $W^{s,p}(K)$ function on the edge γ_{ij}, $1 \le i < j \le 4$ is well-defined. This can be seen from standard arguments owing to the fact that the trace of $W^{s,p}(\mathbb{R}^3)$ on the line $\mathbb{R} \times \{0\}^2$ is well-defined. In particular, the traces of the k-th derivative tensor, $0 \le k < s - 2/p$, on Γ_i and Γ_j, $1 \le i < j \le 4$, must agree on the shared edge γ_{ij}:

$$D^k u|_{\Gamma_i}(\boldsymbol{x}) = D^k u|_{\Gamma_j}(\boldsymbol{x}) \qquad \text{for a.e. } \boldsymbol{x} \in \gamma_{ij} \text{ and all } 0 \le k < s - \frac{2}{p}, \tag{2.5}$$

where (2.5) is to be interpreted in the trace sense.

2.2. Trace operators. We now turn to the consequences of (2.4) and (2.5) for various trace operators.

2.2.1. Zeroth-order operator. First consider the 0th-order "boundary-derivative" operator $\mathfrak{D}_i^0$ on Γ_i, $1 \leq i \leq 4$, defined formally by the rule

$$\mathfrak{D}_i^0(f) := f \qquad \text{on } \Gamma_i. \tag{2.6}$$

Then, (2.4) and (2.5) show that for $u \in W^{s,p}(K)$, $(s,q) \in \mathcal{A}_0$, where

$$\mathcal{A}_k := \left\{ (s,p) \in \mathbb{R}^2 : 1 < p < \infty,\ (s-k)p > 1,\ \text{and } s - \frac{1}{p} \notin \mathbb{Z} \text{ if } p \neq 2 \right\}, \quad k \in \mathbb{N}_0, \tag{2.7}$$

the trace $f = u|_{\partial K}$ satisfies the following conditions:

1. $W^{s-\frac{1}{p},p}$ regularity on each face:

$$\mathfrak{D}_i^0(f) \in W^{s-\frac{1}{p},p}(\Gamma_i), \qquad 1 \leq i \leq 4. \tag{2.8}$$

2. Compatible traces along edges: For $1 \leq i < j \leq 4$, there holds

$$\mathfrak{D}_i^0(f)|_{\gamma_{ij}} - \mathfrak{D}_j^0(f)|_{\gamma_{ij}} = 0 \qquad \text{if } sp > 2, \tag{2.9a}$$

$$\mathcal{I}_{ij}^p(\mathfrak{D}_i^0(f), \mathfrak{D}_j^0(f)) < \infty \qquad \text{if } sp = 2. \tag{2.9b}$$

If $(s-n)p = 2$ for some $n \in \mathbb{N}$, then we obtain an additional condition since the n-th derivative tensor satisfies $\mathcal{I}_{ij}^p(D^n u, D^n u) < \infty$ for $1 \leq i < j \leq 4$. To describe this condition we define the following notation for a d-dimensional tensor S and vector $v \in \mathbb{R}^3$:

$$v^{\otimes 0} \cdot S := S \quad \text{and} \quad v^{\otimes j} \cdot S := S_{i_1 i_2 \cdots i_d} v_{i_1} v_{i_2} \cdots v_{i_j}, \qquad 1 \leq j \leq d.$$

In particular, for $1 \leq i < j \leq 4$, denoting by $\mathbf{t}_{ij}$ a unit vector tangent to γ_{ij}, we can differentiate $\mathfrak{D}_i^0(u)$ and $\mathfrak{D}_j^0(u)$ in the direction $\mathbf{t}_{ij}$ to obtain the following identity.

$$\frac{\partial^n \mathfrak{D}_i^0(u)}{\partial \mathbf{t}_{ij}^n} = \mathbf{t}_{ij}^{\otimes n} \cdot D^n u|_{\Gamma_i} \quad \text{and} \quad \frac{\partial^n \mathfrak{D}_j^0(u)}{\partial \mathbf{t}_{ij}^n} = \mathbf{t}_{ij}^{\otimes n} \cdot D^n u|_{\Gamma_j}.$$

Consequently, the trace $f = u|_{\partial K}$ also satisfies the following property:

3. Compatible tangential derivatives: For $1 \leq i < j \leq 4$ and $n \in \mathbb{N}$, there holds

$$\mathcal{I}_{ij}^p \left(\frac{\partial^n \mathfrak{D}_i^0(u)}{\partial \mathbf{t}_{ij}^n}, \frac{\partial^n \mathfrak{D}_j^0(u)}{\partial \mathbf{t}_{ij}^n} \right) < \infty \qquad \text{if } (s-n)p = 2. \tag{2.10}$$

2.2.2. First-order operator. For $(s,p) \in \mathcal{A}_1$, we turn to the regularity of the trace of the gradient of $u \in W^{s,p}(K)$. To this end, on each face Γ_i, $1 \leq i \leq 4$, let $\{\boldsymbol{\tau}_{i,1}, \boldsymbol{\tau}_{i,2}\}$ be orthonormal vectors tangent to Γ_i and let $\mathbf{n}_i$ denote the outward unit normal vector on Γ_i. We define the 1st-order "boundary-derivative" operator $\mathfrak{D}_i^1$ on Γ_i, $1 \leq i \leq 4$, by the rule

$$\mathfrak{D}_i^1(f,g) := \sum_{j=1}^{2} \frac{\partial f}{\partial \boldsymbol{\tau}_{i,j}} \boldsymbol{\tau}_{i,j} + g\mathbf{n}_i \qquad \text{on } \Gamma_i, \tag{2.11}$$

so that $\mathfrak{D}_i^1(u, \partial_{\mathbf{n}} u) = Du|_{\Gamma_i}$. Again applying (2.4) and (2.5), we obtain analogues of (2.8) and (2.9) stated below in (2.13) and (2.14) with $k = 0$. However, if $(s-2)p > 2$, then the second derivative tensor has matching traces on edges (i.e. (2.5) holds with $k = 2$). In particular, for $1 \le i < j \le 4$, we define the vectors

$$\mathbf{b}_{ij} := \mathbf{t}_{ij} \times \mathbf{n}_i \quad \text{and} \quad \mathbf{b}_{ji} := \mathbf{t}_{ij} \times \mathbf{n}_j, \tag{2.12}$$

where we recall that $\mathbf{t}_{ij}$ is a unit vector tangent to γ_{ij}, so that on γ_{ij}, there holds

$$\mathbf{b}_{ji} \cdot \frac{\partial \mathfrak{D}_i^1(u, \partial_{\mathbf{n}} u)}{\partial \mathbf{b}_{ij}} = \mathbf{b}_{ji} \cdot \frac{\partial Du}{\partial \mathbf{b}_{ij}} = \frac{\partial^2 u}{\partial \mathbf{b}_{ij} \partial \mathbf{b}_{ji}} = \mathbf{b}_{ij} \cdot \frac{\partial Du}{\partial \mathbf{b}_{ji}} = \mathbf{b}_{ij} \cdot \frac{\partial \mathfrak{D}_j^1(u, \partial_{\mathbf{n}} u)}{\partial \mathbf{b}_{ji}}$$

in the sense of traces. As a consequence, the operator $\mathfrak{D}_i^1$ satisfies the additional condition (2.15) below with $n = 0$ thanks to (2.4) and (2.5). Finally, we can differentiate in the direction tangent to each edge to obtain the analogue of (2.10) stated in (2.14b) and (2.15b) below. To summarize, the traces $f = u|_{\partial K}$ and $g = \partial_{\mathbf{n}} u|_{\partial K}$ satisfy the following for all $(s, p) \in \mathcal{A}_1$:

1. $W^{s-1-\frac{1}{p},p}$ regularity on each face:

$$\mathfrak{D}_i^1(f, g) \in W^{s-1-\frac{1}{p},p}(\Gamma_i), \qquad 1 \le i \le 4. \tag{2.13}$$

2. Compatible traces along edges: For $1 \le i < j \le 4$ and $n \in \mathbb{N}_0$, there holds

$$\mathfrak{D}_i^1(f, g)|_{\gamma_{ij}} - \mathfrak{D}_j^1(f, g)|_{\gamma_{ij}} = 0 \qquad \text{if } (s-1)p > 2, \tag{2.14a}$$

$$\mathcal{I}_{ij}^p \left(\frac{\partial^n \mathfrak{D}_i^1(f, g)}{\partial \mathbf{t}_{ij}^n}, \frac{\partial^n \mathfrak{D}_j^1(f, g)}{\partial \mathbf{t}_{ij}^n} \right) < \infty \qquad \text{if } (s-n-1)p = 2. \tag{2.14b}$$

3. Compatible traces of higher derivatives along edges: For $1 \le i < j \le 4$ and $n \in \mathbb{N}_0$, there holds

(2.15a)

$$\mathbf{b}_{ji} \cdot \left. \frac{\partial \mathfrak{D}_i^1(f, g)}{\partial \mathbf{b}_{ij}} \right|_{\gamma_{ij}} - \mathbf{b}_{ij} \cdot \left. \frac{\partial \mathfrak{D}_j^1(f, g)}{\partial \mathbf{b}_{ji}} \right|_{\gamma_{ij}} = 0 \qquad \text{if } (s-2)p > 2,$$

(2.15b)

$$\mathcal{I}_{ij}^p \left(\mathbf{b}_{ji} \cdot \frac{\partial^{n+1} \mathfrak{D}_i^1(f, g)}{\partial \mathbf{t}_{ij}^n \partial \mathbf{b}_{ij}}, \mathbf{b}_{ij} \cdot \frac{\partial^{n+1} \mathfrak{D}_j^1(f, g)}{\partial \mathbf{t}_{ij}^n \partial \mathbf{b}_{ji}} \right) < \infty \qquad \text{if } (s-n-2)p = 2.$$

Remark 2.2. For smooth enough functions, conditions (2.14a) and (2.15a) may be interpreted as the application of the vertex compatibility conditions for traces on the triangle (see e.g. [48, eqs. (2.11a) and (2.12a)]) at every point on the edge γ_{ij}.

2.2.3. mth-order operator. We now turn to the general case of the trace of the m-th derivative tensor of a function $u \in W^{s,p}(K)$, where $m \ge 2$ and $(s, p) \in \mathcal{A}_m$. Given a collection of functions $F = (f^0, f^1, \dots, f^m)$ defined on ∂K, we define the m-th order "boundary-derivative" operator $\mathfrak{D}_i^m$ on Γ_i, $1 \le i \le 4$, by the rule

$$\mathfrak{D}_i^m(F) := \sum_{\substack{\alpha \in \mathbb{N}_0^3 \\ |\alpha| = m}} \frac{\partial^{\alpha_1 + \alpha_2} f^{\alpha_3}}{\partial \boldsymbol{\tau}_{i,1}^{\alpha_1} \partial \boldsymbol{\tau}_{i,2}^{\alpha_2}} \sum_{\phi \in \mathfrak{M}_i(\alpha)} \phi(1) \otimes \phi(2) \otimes \cdots \otimes \phi(m) \qquad \text{on } \Gamma_i, \tag{2.16}$$

where the set $\mathfrak{M}_i(\alpha)$ consists of the following mappings.

$$\mathfrak{M}_i(\alpha) := \{\phi : \{1, 2, \dots, |\alpha|\} \to \{\boldsymbol{\tau}_{i,1}, \boldsymbol{\tau}_{i,2}, \mathbf{n}_i\} \text{ s.t. } |\phi^{-1}(\boldsymbol{\tau}_{i,j})| = \alpha_j, \ j = 1, 2\},$$

where we recall that $\{\boldsymbol{\tau}_{i,1}, \boldsymbol{\tau}_{i,2}\}$ are orthonormal vectors tangent to Γ_i. For notational convenience, we set

$$\mathfrak{D}_i^l(F) := \mathfrak{D}_i^l(f^0, f^1, \ldots, f^l), \qquad 0 \le l < m.$$

Then, one may readily verify that $\mathfrak{D}_i^m(u, \partial_{\mathbf{n}} u, \ldots, \partial_{\mathbf{n}}^m u) = D^m u$ on Γ_i. Let $f^l = \partial_{\mathbf{n}}^l u$ on ∂K. As before, we obtain $W^{s-m-\frac{1}{p},p}$ regularity of $\mathfrak{D}_i^m(F)$ (2.17) below on each face thanks to (2.4) and the edge compatibility conditions (2.18) below with $l = 0$ from (2.5).

As was the case with the first-order operator, there are additional edge compatibility conditions. In particular, if $(s - m - l)p > 0$ for some $1 \le l \le m$, then (2.5) shows that the $(m + l)$th derivative tensor has matching traces on edges. Some components of the $(m + l)$th derivative tensor can be expressed in terms of $\mathfrak{D}_l^m(F)$. In particular, on the edge γ_{ij}, $1 \le i < j \le 4$, there holds

$$\begin{aligned}\mathbf{b}_{ji}^{\otimes l} \cdot \frac{\partial^l \mathfrak{D}_i^m(F)}{\partial \mathbf{b}_{ij}^l} = \mathbf{b}_{ji}^{\otimes l} \cdot \frac{\partial^l D^m u}{\partial \mathbf{b}_{ij}^l} &= \mathbf{b}_{ji}^{\otimes l} \cdot \left(\mathbf{b}_{ij}^{\otimes l} \cdot D^{m+l} u\right) \\ &= \mathbf{b}_{ij}^{\otimes l} \cdot \left(\mathbf{b}_{ji}^{\otimes l} \cdot D^{m+l} u\right) = \mathbf{b}_{ij}^{\otimes l} \cdot \frac{\partial^l D^m u}{\partial \mathbf{b}_{ji}^l} = \mathbf{b}_{ij}^{\otimes l} \cdot \frac{\partial^l \mathfrak{D}_j^m(F)}{\partial \mathbf{b}_{ji}^l}\end{aligned}$$

in the sense of traces, where we used symmetry of the derivative tensor $D^{m+l}u$. We can also differentiate in the direction tangent to each edge to obtain similar conditions. Consequently, $\mathfrak{D}_i^m(F)$ satisfies (2.18) below. In summary, for $m \in \mathbb{N}_0$, the traces $F = (u, \partial_{\mathbf{n}} u, \ldots, \partial_{\mathbf{n}}^m u)$ satisfy the following for all $(s,p) \in \mathcal{A}_m$:

1. $W^{s-m-\frac{1}{p},p}$ regularity on each face:

   $$\mathfrak{D}_i^m(F) \in W^{s-m-\frac{1}{p},p}(\Gamma_i), \qquad 1 \le i \le 4, \tag{2.17}$$

   where $\mathfrak{D}_i^0$, $\mathfrak{D}_i^1$, and $\mathfrak{D}_i^l$, $l \ge 2$, are defined in (2.6), (2.11), and (2.16).
2. Compatible traces along edges: For $1 \le i < j \le 4$ and $0 \le l \le m$ and $n \in \mathbb{N}_0$, there holds

   (2.18a)
   $$\left.\mathbf{b}_{ji}^{\otimes l} \cdot \frac{\partial^l \mathfrak{D}_i^m(F)}{\partial \mathbf{b}_{ij}^l}\right|_{\gamma_{ij}} - \left.\mathbf{b}_{ij}^{\otimes l} \cdot \frac{\partial^l \mathfrak{D}_j^m(F)}{\partial \mathbf{b}_{ji}^l}\right|_{\gamma_{ij}} = 0 \qquad \text{if } (s-m-l)p > 2,$$

   (2.18b)
   $$\mathcal{I}_{ij}^p\left(\mathbf{b}_{ji}^{\otimes l} \cdot \frac{\partial^{l+n} \mathfrak{D}_i^m(F)}{\partial \mathbf{t}_{ij}^n \partial \mathbf{b}_{ij}^l}, \mathbf{b}_{ij}^{\otimes l} \cdot \frac{\partial^{l+n} \mathfrak{D}_j^m(F)}{\partial \mathbf{t}_{ij}^n \partial \mathbf{b}_{ji}^l}\right) < \infty \qquad \text{if } (s-m-l-n)p = 2.$$

Remark 2.3. As was the case in Remark 2.2, condition (2.18a) is simply the application of the vertex compatibility conditions for traces on the triangle [48, eq. (7.2)] at every point on the edge γ_{ij}, provided that u is smooth enough.

2.3. The trace theorem on a tetrahedron. Motivated by the conditions derived in the previous section, we define trace spaces as follows. Given a set of indices $\mathcal{S} \subseteq \{1,2,3,4\}$ with $|\mathcal{S}| \ge 1$, let $\Gamma_{\mathcal{S}} := \cup_{i \in \mathcal{S}} \Gamma_i$. We define the trace space on part of the boundary $\mathrm{Tr}_k^{s,p}(\Gamma_{\mathcal{S}})$ for $k \in \mathbb{N}_0$ and $(s,p) \in \mathcal{A}_k$ as follows.

$$\begin{aligned}\mathrm{Tr}_k^{s,p}(\Gamma_{\mathcal{S}}) := \{F = (f^0, f^1, \ldots, f^k) \in L^p(\Gamma_{\mathcal{S}})^{k+1} : &\text{ For } 0 \le m \le k, \\ &F \text{ satisfies (2.17) for } i \in \mathcal{S} \text{ and} \\ &\text{(2.18) for } i,j \in \mathcal{S} \text{ with } i<j,\ 0 \le l \le m \text{ and } n \in \mathbb{N}_0\},\end{aligned}$$

equipped with the norm

$$\|(f^0, f^1, \dots, f^k)\|^p_{\mathrm{Tr}^{s,p}_k,\Gamma_{\mathcal{S}}} := \sum_{m=0}^{k} \sum_{i\in\mathcal{S}} \|f_i^m\|^p_{s-m-\frac{1}{p},p,\Gamma_i}$$
$$+ \sum_{\substack{i,j\in\mathcal{S} \\ i<j \\ 0\le l\le k \\ n\in\mathbb{N}_0}} \begin{cases} \mathcal{I}^p_{ij}\left(\mathbf{b}_{ji}^{\otimes l}\cdot \frac{\partial^{l+n}\mathfrak{D}_i^k(F)}{\partial\mathbf{t}^n_{ij}\partial\mathbf{b}^l_{ij}}, \mathbf{b}_{ij}^{\otimes l}\cdot \frac{\partial^{l+n}\mathfrak{D}_j^k(F)}{\partial\mathbf{t}^n_{ij}\partial\mathbf{b}^l_{ji}}\right) & \text{if } (s-k-l-n)p = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the sum in the definition contains only finitely many nonzero terms, and hence is well defined. When $\mathcal{S} = \{1,2,3,4\}$, we set $\mathrm{Tr}^{s,p}_k(\partial K) := \mathrm{Tr}^{s,p}_k(\Gamma_{\mathcal{S}})$ and $\|\cdot\|_{\mathrm{Tr}^{s,p}_k,\partial K} := \|\cdot\|_{\mathrm{Tr}^{s,p}_k,\Gamma_{\mathcal{S}}}$. The following trace theorem is a consequence of the discussion in the previous section.

THEOREM 2.4. *Let $\mathcal{S}\subseteq\{1,2,3,4\}$, $k\in\mathbb{N}_0$, and $(s,p)\in\mathcal{A}_k$ be given. Then, for every $u\in W^{s,p}(K)$, the traces satisfy $(u,\partial_{\mathbf{n}}u,\dots,\partial^k_{\mathbf{n}}u)|_{\Gamma_{\mathcal{S}}}\in\mathrm{Tr}^{s,p}_k(\Gamma_{\mathcal{S}})$ and*

$$\|(u,\partial_{\mathbf{n}}u,\dots,\partial^k_{\mathbf{n}}u)\|_{\mathrm{Tr}^{s,p}_k,\Gamma_{\mathcal{S}}} \lesssim_{k,s,p} \|u\|_{s,p,K}. \tag{2.19}$$

2.4. The trace of polynomials. Given $N\in\mathbb{N}_0$, let $\mathcal{P}_N(K)$ denote the set of all polynomials of total degree at most N, while $\mathcal{P}_{-M} := \{0\}$ for $M>0$. If $u\in\mathcal{P}_N(K)$, then $u\in W^{s,p}(K)$ for all $s\ge 0$ and $p\ge 1$. Consequently, for each $k\in\mathbb{N}_0$ and $\mathcal{S}\subseteq\{1,2,3,4\}$, the traces $F=(u,\partial_{\mathbf{n}}u,\dots,\partial^k_{\mathbf{n}}u)|_{\Gamma_{\mathcal{S}}}\in\mathrm{Tr}^{s,p}_k(\Gamma_{\mathcal{S}})$ for all $(s,p)\in\mathcal{A}_k$. In particular, s may be taken to be arbitrarily large in (2.18a). Thus, the traces satisfy

$$f_i^m \in \mathcal{P}_{N-m}(\Gamma_i), \qquad 0\le m\le k,\ i\in\mathcal{S}, \tag{2.20a}$$
$$\mathfrak{D}_i^m(F)|_{\gamma_{ij}} = \mathfrak{D}_j^m(F)|_{\gamma_{ij}}, \qquad 0\le m\le k,\ i,j\in\mathcal{S},\ i<j, \tag{2.20b}$$
$$\mathbf{b}_{ji}^{\otimes l}\cdot\left.\frac{\partial^l\mathfrak{D}_i^k(F)}{\partial\mathbf{b}^l_{ij}}\right|_{\gamma_{ij}} = \mathbf{b}_{ij}^{\otimes l}\cdot\left.\frac{\partial^l\mathfrak{D}_j^k(F)}{\partial\mathbf{b}^l_{ji}}\right|_{\gamma_{ij}}, \qquad 0\le l\le k,\ i,j\in\mathcal{S},\ i<j. \tag{2.20c}$$

Note that we have not included the integral condition (2.18b) in the list (2.20) above. The following lemma shows that if a tuple of functions defined on ∂K satisfy (2.20), then (2.18b) is automatically satisfied.

LEMMA 2.5. *Let $\mathcal{S}\subseteq\{1,2,3,4\}$ and $k\in\mathbb{N}_0$. If $F:\Gamma_{\mathcal{S}}\to\mathbb{R}^{k+1}$ satisfies (2.20), then $F\in\mathrm{Tr}^{s,p}_k(\Gamma_{\mathcal{S}})$ for all $(s,p)\in\mathcal{A}_k$.*

Proof. Let $(s,p)\in\mathcal{A}_k$, $0\le l\le m\le k$, be given. Thanks to (2.20b) and (2.20c), the difference

$$H_{ij} := \mathbf{b}_{ji}^{\otimes l}\cdot\frac{\partial^l\mathfrak{D}_i^m(F)}{\partial\mathbf{b}^l_{ij}}\circ\boldsymbol{F}_{ij} - \mathbf{b}_{ij}^{\otimes l}\cdot\frac{\partial^l\mathfrak{D}_j^m(F)}{\partial\mathbf{b}^l_{ji}}\circ\boldsymbol{F}_{ji}, \qquad \text{on } T, i,j\in\mathcal{S},$$

vanishes on the edge γ_2 of the reference triangle T and H_{ij} has entries $\mathcal{P}_{N-m-l}(T)$. Thus, $H_{ij} = x_2 G_{ij}$, where G_{ij} has entries in $\mathcal{P}_{N-m-l-1}(T)$. Consequently, for all $n\in\mathbb{N}_0$, there holds

$$\mathcal{I}^p_{ij}\left(\mathbf{b}_{ji}^{\otimes l}\cdot\frac{\partial^{l+n}\mathfrak{D}_i^m(F)}{\partial\mathbf{t}^n_{ij}\partial\mathbf{b}^l_{ij}}, \mathbf{b}_{ij}^{\otimes l}\cdot\frac{\partial^{l+n}\mathfrak{D}_j^m(F)}{\partial\mathbf{t}^n_{ij}\partial\mathbf{b}^l_{ji}}\right) \approx_p \int_T |\partial^n_{x_1}H_{ij}(\boldsymbol{x})|^p\frac{\mathrm{d}\boldsymbol{x}}{x_2}$$
$$= \int_T |\partial^n_{x_1}G_{ij}(\boldsymbol{x})|^p x_2^{p-1}\,\mathrm{d}\boldsymbol{x},$$

which is finite since G_{ij} has polynomial entries. The inclusion $F \in \mathrm{Tr}_k^{s,p}(\Gamma_{\mathcal{S}})$ now follows from (2.20). ◻

2.5. Statement of the main result. The aim of the current work is to construct a right inverse $\mathcal{L}_k$ of the operator $u \mapsto (u, \partial_{\mathbf{n}} u, \ldots, \partial_{\mathbf{n}}^k u)|_{\partial K}$ for each $k \in \mathbb{N}_0$ that is bounded from $\mathrm{Tr}_k^{s,p}(\partial K)$ into $W^{s,p}(K)$ for all $(s,p) \in \mathcal{A}_k$ *and* preserves polynomials in the following sense: if $F = (f^0, f^1, \ldots, f^k)$ is the trace of some degree N polynomial, then $\mathcal{L}_k(F)$ is a polynomial of degree N. In particular, the main result is as follows.

THEOREM 2.6. *Let $k \in \mathbb{N}_0$. There exists a linear operator*

$$\mathcal{L}_k : \bigcup_{(s,p)\in\mathcal{A}_k} \mathrm{Tr}_k^{s,p}(\partial K) \to L^1(K)$$

satisfying the following properties: for all $(s,p) \in \mathcal{A}_k$ and $F = (f^0, f^1, \ldots, f^k) \in \mathrm{Tr}_k^{s,p}(\partial K)$, $\mathcal{L}_k(F) \in W^{s,p}(K)$,

$$\partial_{\mathbf{n}}^l \mathcal{L}_k(F)|_{\partial K} = f^l, \qquad 0 \le l \le k, \quad \text{and} \quad \|\mathcal{L}_k(F)\|_{s,p,K} \lesssim_{k,s,p} \|F\|_{\mathrm{Tr}_k^{s,p},\partial K}.$$

Moreover, if F is a piecewise polynomial of degree $N \in \mathbb{N}_0$ satisfying (2.20) with $\mathcal{S} = \{1,2,3,4\}$, then $\mathcal{L}_k(F) \in \mathcal{P}_N(K)$.

The construction of the lifting operator $\mathcal{L}_k$ in Theorem 2.6 is the focus of the next section, and the proof of Theorem 2.6 appears in subsection 3.5. An immediate consequence is the following characterization of the range of the trace operator.

COROLLARY 2.7. *For each $k \in \mathbb{N}_0$, the operator $u \mapsto (u, \partial_{\mathbf{n}} u, \ldots, \partial_{\mathbf{n}}^k u)|_{\partial K}$ is surjective from $W^{s,p}(K)$ onto $\mathrm{Tr}_k^{s,p}(\partial K)$ for all $(s,p) \in \mathcal{A}_k$.*

2.6. Potential applications. Theorem 2.6 has many potential applications, particularly in the analysis of high-order finite element methods. For brevity, we discuss three applications. Firstly, the extension operator may be used analogously to the constructions in [7, 15, 45] to establish optimal (with respect to mesh size and polynomial degree) a priori error estimates for $W^{s,p}$-conforming finite element spaces for all $p \in (1,\infty)$ and $s > 1/p$. Secondly, the lifting operator will be crucial to obtain bounds explicit in polynomial degree for preconditioners for high-order finite element discretizations of fourth-order (and higher-order) elliptic problems similar to H^1-stable extensions for second-order problems in 2D and 3D [14, 49] and H^2-stable extensions for fourth-order problems in 2D [8]. Finally, in a similar vein to [6, 10, 13, 29, 43], the extension operator may be helpful in constructing a polynomial-preserving right inverse of the curl operator that preserves some trace properties (e.g. vanishing tangential trace, vanishing trace, etc.) and in proving discrete Friedrichs inequalities. These results have applications to the stability, convergence theory, and preconditioning of high-order discretizations of mixed and parameter-dependent problems (see also e.g. [11, 28, 33]).

3. Construction of the lifting operator. The construction of the lifting operator $\mathcal{L}_k$, $k \in \mathbb{N}_0$, proceeds face-by-face using similar techniques to [46, 48]. The main idea is to perform a sequence of liftings and corrections using a fundamental convolution operator (see e.g. [12, eq. (4.2)], [14], [18], [19, p. 56, eq. (2.1)], [47, §2.5.5]) and subsequent modifications to it. Given a nonnegative integer $k \in \mathbb{N}_0$, a smooth compactly supported function $b \in C_c^\infty(T)$, and a function $f : T \to \mathbb{R}$, we define the operator $\mathcal{E}_k^{[1]}$ formally by the rule

$$\mathcal{E}_k^{[1]}(f)(\boldsymbol{x}, z) := \frac{(-z)^k}{k!} \int_T b(\boldsymbol{y}) f(\boldsymbol{x} + z\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} \qquad \forall (\boldsymbol{x}, z) \in K, \tag{3.1}$$

and we use the notation $\mathcal{E}_k^{[1]}[b]$ when we want to make the dependence on b explicit. Note that for $(\boldsymbol{x}, z) \in K$ and $\boldsymbol{y} \in T$, there holds $\boldsymbol{x} + z\boldsymbol{y} \in T$, and so (3.1) is well-defined for e.g. $f \in C^\infty(\bar{T})$. For functions $f : \Gamma_1 \to \mathbb{R}$ we define

$$\mathcal{E}_k^{[1]}(f) := \mathcal{E}_k^{[1]}(f \circ \mathfrak{I}_1), \qquad \text{where} \quad \mathfrak{I}_1(\boldsymbol{x}) := (\boldsymbol{x}, 0) \qquad \forall \boldsymbol{x} \in T. \tag{3.2}$$

3.1. Lifting from one face. The first result concerns the interpolation and continuity properties of $\mathcal{E}_k^{[1]}$.

LEMMA 3.1. *Let $b \in C_c^\infty(T)$, $k \in \mathbb{N}_0$, and $(s, p) \in \mathcal{A}_k$. Then, for all $f \in W^{s-k-\frac{1}{p},p}(\Gamma_1)$, there holds*

$$\partial_{\mathbf{n}}^m \mathcal{E}_k^{[1]}(f)|_{\Gamma_1} = \delta_{mk} \left(\int_T b(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right) f, \qquad 0 \le m \le k, \tag{3.3}$$

and

$$\|\mathcal{E}_k^{[1]}(f)\|_{s,p,K} \lesssim_{b,k,s,p} \|f\|_{s-k-\frac{1}{p},p,\Gamma_1}. \tag{3.4}$$

Moreover, if $f \in \mathcal{P}_N(\Gamma_1)$, $N \in \mathbb{N}_0$, then $\mathcal{E}_k^{[1]}(f) \in \mathcal{P}_{N+k}(K)$.

The proof appears in subsection 6.1. We now construct a lifting operator from Γ_1.

LEMMA 3.2. *Let $b \in C_c^\infty(T)$ with $\int_T b(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = 1$ and $k \in \mathbb{N}_0$. We formally define the following operators for $F = (f^0, f^1, \ldots, f^k) \in L^p(\Gamma_1)^{k+1}$:*

$$\mathcal{L}_0^{[1]}(F) := \mathcal{E}_0^{[1]}(f^0), \tag{3.5a}$$

$$\mathcal{L}_m^{[1]}(F) := \mathcal{E}_m^{[1]}(f^m - \partial_{\mathbf{n}}^m \mathcal{L}_{m-1}^{[1]}(F)|_{\Gamma_1}), \qquad 1 \le m \le k. \tag{3.5b}$$

Then, for all $(s, p) \in \mathcal{A}_k$ and $F \in \mathrm{Tr}_k^{s,p}(\Gamma_1)$, $\mathcal{L}_k^{[1]}(F)$ is well-defined and there holds

$$\partial_{\mathbf{n}}^m \mathcal{L}_k^{[1]}(F)|_{\Gamma_1} = f^m, \qquad 0 \le m \le k, \quad \textit{and} \quad \|\mathcal{L}_k^{[1]}(F)\|_{s,p,K} \lesssim_{b,k,s,p} \|F\|_{\mathrm{Tr}_k^{s,p},\Gamma_1}. \tag{3.6}$$

Moreover, if $f^m \in \mathcal{P}_{N-m}(\Gamma_1)$, $0 \le m \le k$, for some $N \in \mathbb{N}_0$, then $\mathcal{L}_k^{[1]}(F) \in \mathcal{P}_N(K)$.

Proof. We proceed by induction on k. The case $k = 0$ follows immediately from Lemma 3.1. Now assume that the lemma is true for some $k \in \mathbb{N}_0$ and let $(s, p) \in \mathcal{A}_{k+1}$ and $F \in \mathrm{Tr}_{k+1}^{s,p}(\Gamma_1)$ be as in the statement of the lemma. Then, we may apply the lemma to $\tilde{F} = (f^0, f^1, \ldots, f^k) \in \mathrm{Tr}_k^{s,p}(\Gamma_1)$ to conclude that for $0 \le m \le k$ and

$$\partial_{\mathbf{n}}^m \mathcal{L}_k^{[1]}(\tilde{F})|_{\Gamma_1} = f^m, \quad 0 \le m \le k, \quad \text{and} \quad \|\mathcal{L}_k^{[1]}(\tilde{F})\|_{s,p,K} \lesssim_{b,k,s,p} \|\tilde{F}\|_{\mathrm{Tr}_k^{s,p},\Gamma_1}.$$

Thanks to the trace theorem (Theorem 2.4), there holds $f^{k+1} - \partial_{\mathbf{n}}^{k+1} \mathcal{L}_k^{[1]}(F)|_{\Gamma_1} \in W^{s-k-1-\frac{1}{p},p}(\Gamma_1)$ with

$$\|f^{k+1} - \partial_{\mathbf{n}}^{k+1} \mathcal{L}_k^{[1]}(\tilde{F})\|_{s-k-1-\frac{1}{p},p,\Gamma_1} \lesssim_{b,k,s,p} \|F\|_{\mathrm{Tr}_{k+1}^{s,p},\Gamma_1},$$

and so (3.6) follows from Lemma 3.1. Additionally, if F satisfies $f^m \in \mathcal{P}_{N-m}(\Gamma_1)$, $0 \le m \le k+1$, for some $N \in \mathbb{N}_0$, then $\tilde{F}$ satisfies the same condition, where the upper bound of m is restricted to k. Consequently, $\mathcal{L}_k^{[1]}(\tilde{F}) \in \mathcal{P}_N(K)$ and so $f^{k+1} - \partial_{\mathbf{n}}^{k+1} \mathcal{L}_k^{[1]}(\tilde{F})|_{\Gamma_1} \in \mathcal{P}_{N-k-1}(\Gamma_1)$. Thus, $\mathcal{L}_{k+1}^{[1]}(F) \in \mathcal{P}_N(K)$ thanks to Lemma 3.1. ∎

3.2. Lifting from two faces. We now seek a lifting operator from Γ_2 that has zero trace on Γ_1. The operator will be a generalization of the form introduced in [46]. We first define an operator that lifts traces from Γ_1 and has zero trace on Γ_2, and then define the lifting operator from Γ_2 in terms of this operator. To this end, denote by ω_i the barycentric coordinates of T defined as follows.

$$\omega_i(\boldsymbol{x}) := x_i, \quad 1 \le i \le 2, \quad \text{and} \quad \omega_3(\boldsymbol{x}) := 1 - x_1 - x_2 \quad \forall \boldsymbol{x} \in T. \tag{3.7}$$

Given nonnegative integers $k, r \in \mathbb{N}_0$, a smooth compactly supported function $b \in C_c^\infty(T)$, and a function $f : T \to \mathbb{R}$, we define the operator $\mathcal{M}_{k,r}^{[1]}$ formally by the rule

$$\begin{aligned} \mathcal{M}_{k,r}^{[1]}(f)(\boldsymbol{x}, z) &:= x_2^r \mathcal{E}_k^{[1]}(\omega_2^{-r} f)(\boldsymbol{x}, z) \\ &= x_2^r \frac{(-z)^k}{k!} \int_T \frac{b(\boldsymbol{y}) f(\boldsymbol{x} + z\boldsymbol{y})}{(x_2 + z y_2)^r} \, \mathrm{d}\boldsymbol{y} \qquad \forall (\boldsymbol{x}, z) \in K. \end{aligned} \tag{3.8}$$

Note that when $r = 0$, we have $\mathcal{M}_{k,0}^{[1]} = \mathcal{E}_k^{[1]}$. For functions $f : \Gamma_1 \to \mathbb{R}$, we again abuse notation and set $\mathcal{M}_{k,r}^{[1]}(f) := \mathcal{M}_{k,r}^{[1]}(f \circ \mathfrak{I}_1)$.

The presence of the weight ω_2^{-r} in the operator $\mathcal{M}_{k,r}^{[1]}$ means that derivatives of $f : \Gamma_1 \to \mathbb{R}$ up to order r have to vanish on edge γ_{12} in an appropriate sense. To this end, let $s = m + \sigma$ with $m \in \mathbb{N}_0$ and $\sigma \in [0, 1)$ and $1 < p < \infty$. Given a face Γ_j, $1 \le j \le 4$, and $\mathfrak{E}$ a subset of the edges of Γ_j, we define the following subspaces of $W^{s,p}(\Gamma_j)$ with vanishing traces on the edges in $\mathfrak{E}$:

$$W_{\mathfrak{E}}^{s,p}(\Gamma_j) := \left\{ f \in W^{s,p}(\Gamma_j) : D_\Gamma^\beta f|_\gamma = 0 \text{ for all } 0 \le |\beta| < s - \frac{1}{p} \text{ and } \gamma \in \mathfrak{E} \right. \\ \left. \text{and } {}_{\mathfrak{E}}\|f\|_{s,p,T} < \infty \right\}, \tag{3.9}$$

where the norm on $W_{\mathfrak{E}}^{s,p}(\Gamma_j)$ is given by

$${}_{\mathfrak{E}}\|f\|_{s,p,\Gamma_j}^p := \|f\|_{s,p,\Gamma_j}^p + \begin{cases} \|\operatorname{dist}(\cdot, \bigcup_{\gamma \in \mathfrak{E}} \gamma)^{-\sigma} D_\Gamma^m f\|_{p,\Gamma_j}^p & \text{if } \sigma p = 1 \text{ and } \mathfrak{E} \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

and we recall that D_Γ is the surface gradient operator. When $\mathfrak{E}$ consists of only one edge γ, we set $W_\gamma^{s,p}(\Gamma_j) := W_{\mathfrak{E}}^{s,p}(\Gamma_j)$ and ${}_\gamma\|f\|_{s,p,\Gamma_j} := {}_{\mathfrak{E}}\|f\|_{s,p,\Gamma_j}$. One can readily verify that the spaces $W_{\mathfrak{E}}^{s,p}(\Gamma_j)$ are Banach spaces and that the following relations hold:

$$W_{\mathfrak{E}}^{s,p}(\Gamma_j) = \bigcap_{\gamma \in \mathfrak{E}} W_\gamma^{s,p}(\Gamma_j) \quad \text{and} \quad {}_{\mathfrak{E}}\|f\|_{s,p,\Gamma_j} \approx_{s,p} \sum_{\gamma \in \mathfrak{E}} {}_\gamma\|f\|_{s,p,\Gamma_j}. \tag{3.10}$$

Given a subset of edges $\mathfrak{E}$ of the reference triangle T, we define the spaces $W_{\mathfrak{E}}^{s,p}(T)$ analogously.

The first result states the continuity properties of $\mathcal{M}_{k,r}^{[1]}$.

Lemma 3.3. *Let $b \in C_c^\infty(T)$, $k, r \in \mathbb{N}_0$, and $(s, p) \in \mathcal{A}_k$. Then, for all $f \in W^{s-k-\frac{1}{p},p}(\Gamma_1) \cap W_{\gamma_{12}}^{\min\{s-k-\frac{1}{p}, r\},p}(\Gamma_1)$, there holds*

$$\partial_{\mathbf{n}}^m \mathcal{M}_{k,r}^{[1]}(f)|_{\Gamma_1} = \delta_{km} \left(\int_T b(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right) f, \qquad 0 \le m \le k, \tag{3.11a}$$

$$\partial_{\mathbf{n}}^j \mathcal{M}_{k,r}^{[1]}(f)|_{\Gamma_2} = 0, \qquad 0 \le j < \min\left\{ r, s - \frac{1}{p} \right\}, \tag{3.11b}$$

and

$$\|\mathcal{M}^{[1]}_{k,r}(f)\|_{s,p,K} \lesssim_{b,k,r,s,p} \begin{cases} {}_{\gamma_{12}}\|f\|_{s-k-\frac{1}{p},p,\Gamma_1} & \text{if } s \leq k+r+\frac{1}{p}, \\ \|f\|_{s-k-\frac{1}{p},p,\Gamma_1} & \text{if } s > k+r+\frac{1}{p}. \end{cases} \tag{3.12}$$

Moreover, if $f \in \mathcal{P}_N(\Gamma_1)$, $N \in \mathbb{N}_0$, *satisfies* $D^l_\Gamma f|_{\gamma_{12}} = 0$ *for* $0 \leq l \leq r-1$, *then* $\mathcal{M}^{[1]}_{k,r}(f) \in \mathcal{P}_{N+k}(K)$.

The proof of Lemma 3.3 appears in subsection 6.3. By mapping the other faces of K to Γ_1 and mapping K onto itself in an appropriate fashion, we may define operators corresponding to these faces. In particular, we define the following operator corresponding to Γ_2:

$$\mathcal{M}^{[2]}_{k,r}(f)(\boldsymbol{x}, z) := \mathcal{M}^{[1]}_{k,r}(f \circ \mathfrak{I}_2) \circ \mathfrak{R}_{12}(\boldsymbol{x}, z) \qquad \forall (\boldsymbol{x}, z) \in K,$$

where $\mathfrak{I}_2(\boldsymbol{x}) := (x_1, 0, x_2)$ and $\mathfrak{R}_{12}(\boldsymbol{x}, z) := (x_1, z, x_2)$ for all $(\boldsymbol{x}, z) \in K$.

Thanks to the chain rule and the smoothness of the mappings $\mathfrak{I}_2$ and $\mathfrak{R}_{12}$, the continuity and interpolation properties of $\mathcal{M}^{[2]}_{k,r}$ follow immediately from Lemma 3.3.

Corollary 3.4. *Let* $b \in C^\infty_c(T)$, $k, r \in \mathbb{N}_0$, *and* $(s,p) \in \mathcal{A}_k$. *Then, for all* $f \in W^{s-k-\frac{1}{p},p}(\Gamma_2) \cap W^{\min\{s-k-\frac{1}{p},r\},p}_{\gamma_{12}}(\Gamma_2)$, *there holds*

$$\partial^m_{\mathbf{n}} \mathcal{M}^{[2]}_{k,r}(f)|_{\Gamma_2} = \delta_{km} \left(\int_T b(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \right) f, \qquad 0 \leq m \leq k, \tag{3.13a}$$

$$\partial^j_{\mathbf{n}} \mathcal{M}^{[2]}_{k,r}(f)|_{\Gamma_1} = 0, \qquad 0 \leq j < \min\left\{ r, s - \frac{1}{p} \right\}, \tag{3.13b}$$

and

$$\|\mathcal{M}^{[2]}_{k,r}(f)\|_{s,p,K} \lesssim_{b,k,r,s,p} \begin{cases} {}_{\gamma_{12}}\|f\|_{s-k-\frac{1}{p},p,\Gamma_2} & \text{if } s \leq k+r+\frac{1}{p}, \\ \|f\|_{s-k-\frac{1}{p},p,\Gamma_2} & \text{if } s > k+r+\frac{1}{p}. \end{cases} \tag{3.14}$$

Moreover, if $f \in \mathcal{P}_N(\Gamma_2)$, $N \in \mathbb{N}_0$, *satisfies* $D^l_\Gamma f|_{\gamma_{12}} = 0$ *for* $0 \leq l \leq r-1$, *then* $\mathcal{M}^{[2]}_{k,r}(f) \in \mathcal{P}_{N+k}(K)$.

3.2.1. Regularity of partially vanishing traces. The operator $\mathcal{M}^{[2]}_{k,r}$ lifts traces from Γ_2 to K and has zero trace on Γ_1, which are the properties we desired to correct the traces of $\mathcal{L}^{[1]}_k$ on Γ_2. However, $\mathcal{M}^{[2]}_{k,r}$ acts on functions belonging to $W^{s-k-\frac{1}{p},p}(\Gamma_2) \cap W^{\min\{s-k-\frac{1}{p},r\},p}_{\gamma_{12}}(\Gamma_2)$ rather than just functions in $W^{s-k-\frac{1}{p},p}(\Gamma_2)$. The main result of this section characterizes one scenario in which traces belong to the space $W^{s-k-\frac{1}{p},p}(\Gamma_2) \cap W^{\min\{s-k-\frac{1}{p},r\},p}_{\gamma_{12}}(\Gamma_2)$, and fortunately, we will encounter exactly this scenario in our construction.

We have the following result which characterizes the regularity of the restriction of a trace $F \in \mathrm{Tr}^{s,p}_k(\Gamma_i \cup \Gamma_j)$ to Γ_j when F vanishes on Γ_i and the first l components of F vanish on Γ_j.

Lemma 3.5. *Let* $k \in \mathbb{N}_0$, $(s,p) \in \mathcal{A}_k$, $1 \leq l \leq k$, *and* $1 \leq i, j \leq 4$ *with* $i \neq j$ *be given. Suppose that* $F = (f^0, f^1, \ldots, f^k) \in \mathrm{Tr}^{s,p}_k(\Gamma_i \cup \Gamma_j)$ *satisfies*

(i) $F = (0, 0, \ldots, 0)$ *on* Γ_i;

(ii) $f^m_j = 0$ *on* Γ_j *for* $0 \leq m \leq l-1$.

Then, there holds $f_j^l \in W^{s-l-\frac{1}{p},p}(\Gamma_j) \cap W_{\gamma_{ij}}^{\min\{s-l-\frac{1}{p},k+1\},p}(\Gamma_j)$ *and*

$$ {}_{\gamma_{ij}}\|f_j^l\|_{s-l-\frac{1}{p},p,\Gamma_j} \lesssim_{k,s,p} \|F\|_{\mathrm{Tr}_k^{s,p},\Gamma_i\cup\Gamma_j} \qquad \text{if } s-l \le k+1+\frac{1}{p}, \tag{3.15a} $$

$$ \|f_j^l\|_{s-l-\frac{1}{p},p,\Gamma_j} \lesssim_{k,s,p} \|F\|_{\mathrm{Tr}_k^{s,p},\Gamma_i\cup\Gamma_j} \qquad \text{if } s-l > k+1+\frac{1}{p}. \tag{3.15b} $$

Proof. Without loss of generality, assume that $i < j$. We first show that for $\alpha \in \mathbb{N}_0^2$ there holds

$$ \left.\frac{\partial^{|\alpha|} f_j^l}{\partial \mathbf{t}_{ij}^{\alpha_1} \partial \mathbf{b}_{ji}^{\alpha_2}}\right|_{\gamma_{ij}} \equiv 0 \qquad 0 \le |\alpha| < \min\left\{s-l-\frac{2}{p}, k+1\right\}, \tag{3.16} $$

where $\mathbf{b}_{ij}$ and $\mathbf{b}_{ji}$ are defined in (2.12).

Step 1: $0 \le \alpha_2 \le k-l$ **and** $|\alpha| < \min\{s-l-2/p, k+1\}$**.** Manipulating definitions shows that

$$ \frac{\partial^{\alpha_1+r}\mathfrak{D}_j^l(F)}{\partial \mathbf{t}_{ij}^{\alpha_1}\partial \mathbf{b}_{ij}^r} = \mathbf{b}_{ij}^{\otimes r} \cdot \frac{\partial^{\alpha_1}\mathfrak{D}_j^{l+r}(F)}{\partial \mathbf{t}_{ij}^{\alpha_1}} \qquad 0 \le r \le k-l, \tag{3.17} $$

and so there holds

$$ \frac{\partial^{|\alpha|} f_j^l}{\partial \mathbf{t}_{ij}^{\alpha_1} \partial \mathbf{b}_{ji}^{\alpha_2}} = \mathbf{n}_j^{\otimes l} \cdot \frac{\partial^{|\alpha|}\mathfrak{D}_j^l(F)}{\partial \mathbf{t}_{ij}^{\alpha_1} \partial \mathbf{b}_{ji}^{\alpha_2}} = \mathbf{n}_j^{\otimes l} \cdot \mathbf{b}_{ji}^{\otimes \alpha_2} \cdot \frac{\partial^{\alpha_1}\mathfrak{D}_j^{l+\alpha_2}(F)}{\partial \mathbf{t}_{ij}^{\alpha_1}}, $$

and using that $\mathfrak{D}_i^{l+\alpha_2}(F)|_{\Gamma_i} = 0$ by (i) gives

$$ \frac{\partial^{|\alpha|} f_j^l}{\partial \mathbf{t}_{ij}^{\alpha_1} \partial \mathbf{b}_{ji}^{\alpha_2}} \circ \boldsymbol{F}_{ji} = \mathbf{n}_j^{\otimes l} \cdot \mathbf{b}_{ji}^{\otimes \alpha_2} \cdot \left(\frac{\partial^{\alpha_1}\mathfrak{D}_j^{l+\alpha_2}(F)}{\partial \mathbf{t}_{ij}^{\alpha_1}} \circ \boldsymbol{F}_{ji} - \frac{\partial^{\alpha_1}\mathfrak{D}_i^{l+\alpha_2}(F)}{\partial \mathbf{t}_{ij}^{\alpha_1}} \circ \boldsymbol{F}_{ij} \right) \tag{3.18} $$

on T. Equality (3.16) now follows from (2.18a).

Step 2: $k-l+1 \le \alpha_2 \le k$ **and** $|\alpha| < \min\{s-l-2/p, k+1\}$**.** The same arguments as in Step 1 show that

$$ \frac{\partial^{|\alpha|} f_j^l}{\partial \mathbf{t}_{ij}^{\alpha_1} \partial \mathbf{b}_{ji}^{\alpha_2}} = \mathbf{n}_j^{\otimes l} \cdot \mathbf{b}_{ji}^{\otimes k-l} \cdot \frac{\partial^{|\alpha|-k+l}\mathfrak{D}_j^k(F)}{\partial \mathbf{t}_{ij}^{\alpha_1} \partial \mathbf{b}_{ji}^{\alpha_2-k+l}}. $$

By construction, there exist constants a_1 and a_2 such that $\mathbf{n}_j = a_1\mathbf{b}_{ij} + a_2\mathbf{b}_{ji}$, and so

$$ \frac{\partial^{|\alpha|} f_j^l}{\partial \mathbf{t}_{ij}^{\alpha_1} \partial \mathbf{b}_{ji}^{\alpha_2}} = \sum_{r=0}^{l} c_r \mathbf{b}_{ij}^{\otimes r} \cdot \mathbf{b}_{ji}^{\otimes k-r} \cdot \frac{\partial^{|\alpha|-k+l}\mathfrak{D}_j^k(F)}{\partial \mathbf{t}_{ij}^{\alpha_1} \partial \mathbf{b}_{ji}^{|\alpha|-k+l}} = \sum_{r=0}^{l} c_r \mathbf{b}_{ij}^{\otimes r} \cdot \frac{\partial^{|\alpha|+l-r}\mathfrak{D}_j^r(F)}{\partial \mathbf{t}_{ij}^{\alpha_1} \partial \mathbf{b}_{ji}^{\alpha_2+l-r}} $$

for some suitable constants $\{c_r\}_{r=0}^l$. For $0 \le r \le l-1$, $\mathfrak{D}_j^r(F) = 0$ by (ii), and so

$$ \begin{aligned} \frac{\partial^{|\alpha|} f_j^l}{\partial \mathbf{t}_{ij}^{\alpha_1} \partial \mathbf{b}_{ji}^{\alpha_2}} = c_l \mathbf{b}_{ij}^{\otimes l} \cdot \frac{\partial^{|\alpha|}\mathfrak{D}_j^l(F)}{\partial \mathbf{t}_{ij}^{\alpha_1} \partial \mathbf{b}_{ji}^{\alpha_2}} &= c_l \mathbf{b}_{ij}^{\otimes l} \cdot \mathbf{b}_{ji}^{\otimes k-l} \cdot \frac{\partial^{|\alpha|-k+l}\mathfrak{D}_j^k(F)}{\partial \mathbf{t}_{ij}^{\alpha_1} \partial \mathbf{b}_{ji}^{\alpha_2-k+l}} \\ &= c_l \mathbf{b}_{ij}^{\otimes k-\alpha_2} \cdot \mathbf{b}_{ji}^{\otimes k-l} \cdot \left(\mathbf{b}_{ij}^{\otimes \alpha_2-k+l} \cdot \frac{\partial^{|\alpha|-k+l}\mathfrak{D}_j^k(F)}{\partial \mathbf{t}_{ij}^{\alpha_1} \partial \mathbf{b}_{ji}^{\alpha_2-k+l}} \right). \end{aligned} $$

Applying (i) then gives the following identity on T:

$$\frac{\partial^{|\alpha|} f_j^l}{\partial \mathbf{t}_{ij}^{\alpha_1} \partial \mathbf{b}_{ji}^{\alpha_2}} \circ \boldsymbol{F}_{ji} = c_l \mathbf{b}_{ij}^{\otimes k-\alpha_2} \cdot \mathbf{b}_{ji}^{\otimes k-l} \cdot \left(\mathbf{b}_{ij}^{\otimes \alpha_2 - k+l} \cdot \frac{\partial^{|\alpha|-k+l} \mathfrak{D}_j^k(F)}{\partial \mathbf{t}_{ij}^{\alpha_1} \partial \mathbf{b}_{ji}^{\alpha_2-k+l}} \circ \boldsymbol{F}_{ji} - \mathbf{b}_{ji}^{\otimes \alpha_2-k+l} \cdot \frac{\partial^{|\alpha|-k+l} \mathfrak{D}_i^k(F)}{\partial \mathbf{t}_{ij}^{\alpha_1} \partial \mathbf{b}_{ij}^{\alpha_2-k+l}} \circ \boldsymbol{F}_{ij} \right). \tag{3.19}$$

Equality (3.16) then follows from (2.18a).

Step 3: $f_j^l \in W^{s-l-\frac{1}{p},p}(\Gamma_j) \cap W_{\gamma_{ij}}^{\min\{s-l-\frac{1}{p},k+1\}p}(\Gamma_j)$**.** For $s - 2/p \notin \mathbb{Z}$, the inclusion $f_j^l \in W^{s-l-\frac{1}{p},p}(\Gamma_j) \cap W_{\gamma_{ij}}^{\min\{s-l-\frac{1}{p},k+1\},p}(\Gamma_j)$ follows from (3.16), and (3.15a) and (3.15b) are an immediate consequence of the definition of the $\|\cdot\|_{\mathrm{Tr}_k^{s,p},\Gamma_i\cup\Gamma_j}$ norm. For $s-2/p \in \mathbb{Z}$, and $|\alpha| = \min\{s-l-2/p, k+1\}$, there holds

$$\int_{\Gamma_j} \left| \frac{\partial^{|\alpha|} f_j^l}{\partial \mathbf{t}_{ij}^{\alpha_1} \partial \mathbf{b}_{ji}^{\alpha_2}}(\boldsymbol{x}) \right|^p \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{dist}(\boldsymbol{x}, \gamma_{ij})} \approx_p \int_T \left| \frac{\partial^{|\alpha|} f_j^l}{\partial \mathbf{t}_{ij}^{\alpha_1} \partial \mathbf{b}_{ji}^{\alpha_2}} \right|^p \circ \boldsymbol{F}_{ji}(\boldsymbol{x}) \frac{\mathrm{d}\boldsymbol{x}}{x_2}, \tag{3.20}$$

and so the inclusion $f_j^l \in W_{\gamma_{ij}}^{\min\{s-l-\frac{1}{p},k+1\},p}(\Gamma_j)$ follows from (3.18), (3.19), and (2.18b), while (3.15a) follows from the definition of the norm. ∎

3.2.2. Construction of lifting. In the following lemma, we construct the lifting operator $\mathcal{L}_k^{[2]}$ in the same fashion as $\mathcal{L}_k^{[1]}$ (3.5), replacing the use of $\mathcal{E}_m^{[1]}$ with $\mathcal{M}_{m,k+1}^{[2]}$.

LEMMA 3.6. *Let $b \in C_c^\infty(T)$ with $\int_T b(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 1$, $k \in \mathbb{N}_0$, and $\mathcal{S} = \{1,2\}$. For $F = (f^0, f^1, \dots, f^k) \in L^p(\Gamma_1 \cup \Gamma_2)^{k+1}$, we formally define the following operators:*

$$\mathcal{L}_{k,0}^{[2]}(F) := \mathcal{L}_k^{[1]}(F) + \mathcal{M}_{0,k+1}^{[2]}(f_2^0 - \mathcal{L}_k^{[1]}(F)|_{\Gamma_2}), \tag{3.21a}$$

$$\mathcal{L}_{k,m}^{[2]}(F) := \mathcal{L}_{k,m-1}^{[2]}(F) + \mathcal{M}_{m,k+1}^{[2]}(f_2^m - \partial_{\mathbf{n}}^m \mathcal{L}_{k,m-1}^{[2]}(F)|_{\Gamma_2}), \qquad 1 \le m \le k, \tag{3.21b}$$

$$\mathcal{L}_k^{[2]}(F) := \mathcal{L}_{k,k}^{[2]}(F). \tag{3.21c}$$

Then, for all $(s,p) \in \mathcal{A}_k$ and $F \in \mathrm{Tr}_k^{s,p}(\Gamma_{\mathcal{S}})$, $\mathcal{L}_k^{[2]}(F)$ is well-defined and there holds

$$\partial_{\mathbf{n}}^m \mathcal{L}_k^{[2]}(F)|_{\Gamma_j} = f_j^m, \quad 0 \le m \le k, \ j \in \mathcal{S}, \quad \text{and} \quad \|\mathcal{L}_k^{[2]}(F)\|_{s,p,K} \lesssim_{b,k,s,p} \|F\|_{\mathrm{Tr}_k^{s,p},\Gamma_{\mathcal{S}}}. \tag{3.22}$$

Moreover, if for some $N \in \mathbb{N}_0$, F satisfies (2.20), then $\mathcal{L}_k^{[2]}(F) \in \mathcal{P}_N(K)$.

Proof. Let $k \in \mathbb{N}_0$, $(s,p) \in \mathcal{A}_k$, and $f \in \mathrm{Tr}_k^{s,p}(\mathcal{S})$ be given.

Step 1: $m = 0$**.** Thanks to Lemma 3.2, the traces $G = (g^0, g^1, \dots, g^k)$ given by

$$g_i^l := f_i^l - \partial_{\mathbf{n}} \mathcal{L}_k^{[1]}(F)|_{\Gamma_i}, \qquad 0 \le l \le k, \ 1 \le i \le 2,$$

satisfy the hypotheses of Lemma 3.5 with $(i,j) = (1,2)$ and $l = 1$. Thanks to Lemma 3.5 and Corollary 3.4, $\mathcal{M}_{0,k+1}^{[2]}(g_2^0)$, and hence $\mathcal{L}_{k,0}^{[2]}(F)$, is well-defined with

$$\|\mathcal{L}_{k,0}^{[2]}(F)\|_{s,p,K} \lesssim_{b,k,s,p} \|\mathcal{L}_k^{[1]}(F)\|_{s,p,K} + \begin{cases} {}_{\gamma_{12}}\left\| f_2^0 - \mathcal{L}_k^{[1]}(F) \right\|_{s-\frac{1}{p},p,\Gamma_2} & \text{if } s \le k+1+\frac{1}{p}, \\ \|f_2^0 - \mathcal{L}_k^{[1]}(F)\|_{s-\frac{1}{p},p,\Gamma_2} & \text{if } s > k+1+\frac{1}{p}. \end{cases}$$

Applying (3.6) and (3.13a) gives

$$\partial_{\mathbf{n}}^l \mathcal{L}_{k,0}^{[2]}(F)|_{\Gamma_1} = f_1^l, \qquad 0 \le l \le k, \quad \text{and} \quad \mathcal{L}_{k,0}^{[2]}(F)|_{\Gamma_2} = f_2^0,$$

and applying (3.6) and (3.15) gives

$$\|\mathcal{L}_{k,0}^{[2]}(F)\|_{s,p,K} \lesssim_{b,k,s,p} \|F\|_{\mathrm{Tr}_k^{s,p},\Gamma_{\mathcal{S}}}.$$

Moreover, if F satisfies (2.20) and for some $N \in \mathbb{N}_0$, then $F \in \mathrm{Tr}_k^{s,p}(\Gamma_{\mathcal{S}})$ by Lemma 2.5 and $\mathcal{L}_k^{[1]}(F) \in \mathcal{P}_N(K)$ by Lemma 3.2. Thus, the trace G satisfies (2.20) for $\{i,j\} \subseteq \{1,2\}$ and $G \in \mathrm{Tr}_k^{s,p}(\Gamma_{\mathcal{S}})$ for all $(s,p) \in \mathcal{A}_k$. By Lemma 3.5, $g_2^0 \in W_{\gamma_{12}}^{k+1,p}(\Gamma_2)$ for all $p \in (1,\infty)$, and so $D_\Gamma^l g_2^0|_{\gamma_{12}} = 0$ for $0 \le l \le k$. Thanks to Corollary 3.4, $\mathcal{L}_{k,0}^{[2]}(F) \in \mathcal{P}_N(K)$.

Step 2: Induction on m. Assume that for some m such that $0 \le m \le k-1$, $\mathcal{L}_{k,m}^{[2]}(F)$ is well-defined and satisfies

$$\partial_{\mathbf{n}}^l \mathcal{L}_{k,m}^{[2]}(F)|_{\Gamma_1} = f_1^l, \qquad 0 \le l \le k, \qquad \partial_{\mathbf{n}}^l \mathcal{L}_{k,m}^{[2]}(F)|_{\Gamma_2} = f_2^l, \qquad 0 \le l \le m, \tag{3.23}$$

and

$$\|\mathcal{L}_{k,m}^{[2]}(F)\|_{s,p,K} \lesssim_{b,k,s,p} \|F\|_{\mathrm{Tr}_k^{s,p},\Gamma_{\mathcal{S}}}. \tag{3.24}$$

Additionally assume that if F satisfies (2.20) for $\{i,j\} \subseteq \{1,2\}$ and for some $N \in \mathbb{N}_0$, then $\mathcal{L}_{k,m}^{[2]}(F) \in \mathcal{P}_N(K)$.

Thanks to (3.23), the traces $G = (g^0, g^1, \ldots, g^k)$ given by

$$g_i^l := f_i^l - \partial_{\mathbf{n}}^l \mathcal{L}_{k,m}^{[2]}(F)|_{\Gamma_i}, \qquad 0 \le l \le k, \ 1 \le i \le 2,$$

satisfy the hypotheses of Lemma 3.5 with $(i,j) = (1,2)$ and $l = m+1$. Thanks to Lemma 3.5 and Corollary 3.4, $\mathcal{M}_{m+1,k+1}^{[2]}(g_2^{m+1})$, and hence $\mathcal{L}_{k,m+1}^{[2]}(F)$ is well-defined with

$$\begin{aligned}\|\mathcal{L}_{k,m+1}^{[2]}(F)\|_{s,p,K} &\lesssim_{b,k,s,p} \|\mathcal{L}_{k,m}^{[2]}(F)\|_{s,p,K} \\ &\quad + \begin{cases} {}_{\gamma_{12}}\left\| f_2^{m+1} - \partial_{\mathbf{n}}^{m+1}\mathcal{L}_{k,m}^{[2]}(F)\right\|_{s-m-1-\frac{1}{p},p,\Gamma_2} & \text{if } s-m-1 \le k+1+\frac{1}{p}, \\ \|f_2^{m+1} - \partial_{\mathbf{n}}^{m+1}\mathcal{L}_{k,m}^{[2]}(F)\|_{s-m-1-\frac{1}{p},p,\Gamma_2} & \text{if } s-m-1 > k+1+\frac{1}{p}. \end{cases}\end{aligned}$$

Applying (3.23) and (3.13a) gives (3.23) for $m+1$, while applying (3.24) and (3.15) gives (3.24) for $m+1$.

Moreover, if F satisfies (2.20) for some $N \in \mathbb{N}_0$, then $\mathcal{L}_{k,m}^{[2]}(F) \in \mathcal{P}_N(K)$ by assumption and so the trace G satisfies (2.20) and $G \in \mathrm{Tr}_k^{s,p}(\Gamma_{\mathcal{S}})$ for all $(s,p) \in \mathcal{A}_k$. By Lemma 3.5, $g_2^{m+1} \in W_{\gamma_{12}}^{k+1,p}(\Gamma_2)$ for all $p \in (1,\infty)$, and so $D_\Gamma^l g_2^{m+1}|_{\gamma_{12}} = 0$ for $0 \le l \le k$. Thanks to Corollary 3.4, $\mathcal{L}_{k,m+1}^{[2]}(F) \in \mathcal{P}_N(K)$. ∎

3.3. Lifting from three faces. We continue in the spirit of the previous two sections and define another lifting operator from Γ_1 with vanishing traces on Γ_2 and Γ_3. Given nonnegative integers $k, r \in \mathbb{N}_0$, a smooth compactly supported function $b \in C_c^\infty(T)$, and a function $f : T \to \mathbb{R}$, we define the operator $\mathcal{S}_{k,r}^{[1]}$ formally by the rule

$$\begin{aligned}\mathcal{S}_{k,r}^{[1]}(f)(\boldsymbol{x}, z) &:= (x_1x_2)^r \mathcal{E}_k^{[1]}((\omega_1\omega_2)^{-r} f)(\boldsymbol{x}, z) \\ &= (x_1x_2)^r \frac{(-z)^k}{k!} \int_T \frac{b(\boldsymbol{y}) f(\boldsymbol{x} + z\boldsymbol{y})}{((x_1+zy_1)(x_2+zy_2))^r} \,\mathrm{d}\boldsymbol{y} \qquad \forall (\boldsymbol{x}, z) \in K.\end{aligned} \tag{3.25}$$

Note that when $r = 0$, we have $\mathcal{S}^{[1]}_{k,0} = \mathcal{E}^{[1]}_k$. For functions $f : \Gamma_1 \to \mathbb{R}$, we again abuse the notation and set $\mathcal{S}^{[1]}_{k,r}(f) := \mathcal{S}^{[1]}_{k,r}(f \circ \mathfrak{I}_1)$.

We require one additional family of spaces with vanishing traces. Let $s = m + \sigma$ with $m \in \mathbb{N}_0$ and $\sigma \in [0, 1)$ and $1 < p < \infty$. Given $r \in \mathbb{N}$, a face Γ_j, $1 \leq j \leq 4$, and $\mathfrak{E}$ a subset of the edges of Γ_j, we define the following subspaces of $W^{s,p}_{\mathfrak{E}}(\Gamma_j)$:

$$W^{s,p}_{\mathfrak{E},r}(\Gamma_j) := \left\{ f \in W^{s,p}(\Gamma_j) \cap W^{\min\{s,r\},p}_{\mathfrak{E}}(\Gamma_j) : \ {}_{\mathfrak{E},r}\|f\|_{s,p,T} < \infty \right\}, \tag{3.26}$$

where the norm on $W^{s,p}_{\mathfrak{E},r}(\Gamma_j)$ is given by

$$\begin{aligned} {}_{\mathfrak{E},r}\|f\|^p_{s,p,\Gamma_j} := &\begin{cases} {}_{\mathfrak{E}}\|f\|^p_{s,p,T} & \text{if } s \leq r, \\ \|f\|^p_{s,p,T} & \text{if } s > r, \end{cases} \\ &+ \begin{cases} \left\| \operatorname{dist}(\cdot, \bigcup_{\gamma \in \mathfrak{E}} \gamma)^{-\sigma} \frac{\partial^{m-r+1} D^{r-1}_{\Gamma} f}{\partial \mathbf{t}^{m-r+1}_{\gamma}} \right\|^p_{p,\Gamma_j} & \text{if } s > r,\ \sigma p = 1, \mathfrak{E} \neq \emptyset, \\ 0 & \text{otherwise}, \end{cases} \end{aligned}$$

where $\mathbf{t}_\gamma$ is a unit-tangent vector on the edge $\gamma \in \mathfrak{E}$. For $r = 0$, we set $W^{s,p}_{\mathfrak{E},0}(\Gamma_j) := W^{s,p}(\Gamma_j)$. When $\mathfrak{E}$ consists of only one element γ, we set $W^{s,p}_{\gamma,r}(\Gamma_j) := W^{s,p}_{\mathfrak{E},r}(\Gamma_j)$ and ${}_{\gamma,r}\|f\|_{s,p,\Gamma_j} := {}_{\mathfrak{E},r}\|f\|_{s,p,\Gamma_j}$. One can again verify that $W^{s,p}_{\mathfrak{E},r}(\Gamma_j)$ are Banach spaces and that the following analogue of (3.10) holds:

$$W^{s,p}_{\mathfrak{E},r}(\Gamma_j) = \bigcap_{\gamma \in \mathfrak{E}} W^{s,p}_{\gamma,r}(\Gamma_j) \quad \text{and} \quad {}_{\mathfrak{E},r}\|f\|_{s,p,\Gamma_j} \approx_{s,p} \sum_{\gamma \in \mathfrak{E}} {}_{\gamma,r}\|f\|_{s,p,\Gamma_j}. \tag{3.27}$$

The following result shows that the continuity of $\mathcal{S}^{[1]}_{k,r}$ can be characterized with these spaces.

Lemma 3.7. *Let $b \in C^\infty_c(T)$, $k, r \in \mathbb{N}_0$, $(s,p) \in \mathcal{A}_k$, and $\mathfrak{E} = \{\gamma_{12}, \gamma_{13}\}$. Then, for all $f \in W^{s-k-\frac{1}{p},p}_{\mathfrak{E},r}(\Gamma_1)$, there holds*

(3.28a)

$$\partial^m_{\mathbf{n}} \mathcal{S}^{[1]}_{k,r}(f)|_{\Gamma_1} = \delta_{km} \left(\int_T b(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right) f, \qquad 0 \leq m \leq k,$$

(3.28b)

$$\partial^j_{\mathbf{n}} \mathcal{S}^{[1]}_{k,r}(f)|_{\Gamma_i} = 0, \qquad 0 \leq j < \min\left\{ r, s - \frac{1}{p} \right\},\ 2 \leq i \leq 3,$$

and

$$\|\mathcal{S}^{[1]}_{k,r}(f)\|_{s,p,K} \lesssim_{b,k,r,s,p} {}_{\mathfrak{E},r}\|f\|_{s-k-\frac{1}{p},p,\Gamma_1}. \tag{3.29}$$

Moreover, if $f \in \mathcal{P}_N(\Gamma_1)$, $N \in \mathbb{N}_0$, satisfies $D^l_\Gamma f|_{\gamma_{12}} = D^l_\Gamma f|_{\gamma_{13}} = 0$ for $0 \leq l \leq r-1$, then $\mathcal{S}^{[1]}_{k,r}(f) \in \mathcal{P}_{N+k}(K)$.

The proof of Lemma 3.7 appears in subsection 6.4. We define the analogous operator associated to Γ_3 as follows.

$$\mathcal{S}^{[3]}_{k,r}(f)(\boldsymbol{x}, z) := \mathcal{S}^{[1]}_{k,r}(f \circ \mathfrak{I}_3) \circ \mathfrak{R}_{13}(\boldsymbol{x}, z) \qquad \forall (\boldsymbol{x}, z) \in K,$$

where $\mathfrak{I}_3(\boldsymbol{x}) := (0, x_2, x_1)$ and $\mathfrak{R}_{13}(\boldsymbol{x}, z) := (z, x_2, x_1)$ for all $(\boldsymbol{x}, z) \in K$. Thanks to the chain rule and the smoothness of the mappings $\mathfrak{I}_3$ and $\mathfrak{R}_{13}$, the continuity and interpolation properties of $\mathcal{S}^{[3]}_{k,r}$ follow immediately from Lemma 3.7.

COROLLARY 3.8. *Let $b \in C_c^\infty(T)$, $k, r \in \mathbb{N}_0$, $(s,p) \in \mathcal{A}_k$, and $\mathfrak{E} = \{\gamma_{13}, \gamma_{23}\}$. Then, for all $f \in W_{\mathfrak{E},r}^{s-k-\frac{1}{p},p}(\Gamma_3)$, there holds*

$$\partial_{\mathbf{n}}^m \mathcal{S}_{k,r}^{[3]}(f)|_{\Gamma_3} = \delta_{km} \left(\int_T b(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right) f, \qquad 0 \le m \le k, \tag{3.30a}$$

$$\partial_{\mathbf{n}}^j \mathcal{S}_{k,r}^{[3]}(f)|_{\Gamma_i} = 0, \qquad 0 \le j < \min\left\{ r, s - \frac{1}{p} \right\}, \ 1 \le i \le 2 \tag{3.30b}$$

and

$$\|\mathcal{S}_{k,r}^{[3]}(f)\|_{s,p,K} \lesssim_{b,k,r,s,p} \ {}_{\mathfrak{E},r}\|f\|_{s-k-\frac{1}{p},p,\Gamma_3}. \tag{3.31}$$

Moreover, if $f \in \mathcal{P}_N(\Gamma_3)$, $N \in \mathbb{N}_0$, satisfies $D_\Gamma^l f|_{\gamma_{13}} = D_\Gamma^l f|_{\gamma_{23}} = 0$ for $0 \le l \le r-1$, then $\mathcal{S}_{k,r}^{[3]}(f) \in \mathcal{P}_{N+k}(K)$.

We also have the following analogue of Lemma 3.5.

LEMMA 3.9. *Let $k \in \mathbb{N}_0$, $(s,p) \in \mathcal{A}_k$, $1 \le l \le k$, and $1 \le i, j \le 4$ with $i \ne j$ be given. Suppose that $F = (f^0, f^1, \dots, f^k) \in \mathrm{Tr}_k^{s,p}(\Gamma_i \cup \Gamma_j)$ satisfies*

(i) $F = (0, 0, \dots, 0)$ on Γ_i;

(ii) $f_j^m = 0$ on Γ_j for $0 \le m \le l-1$.

Then, there holds $f_j^l \in W_{\gamma_{ij},k+1}^{s-l-\frac{1}{p},p}(\Gamma_j)$ and

$$ {}_{\gamma_{ij},k+1}\left\| f_j^l \right\|_{s-l-\frac{1}{p},p,\Gamma_j} \lesssim_{k,s,p} \|F\|_{\mathrm{Tr}_k^{s,p},\Gamma_i \cup \Gamma_j}. \tag{3.32}$$

Proof. The result follows from applying inequality (3.20) and identity (3.19). ∎

We now construct the lifting operator $\mathcal{L}_k^{[3]}$ in the same fashion as $\mathcal{L}_k^{[2]}$ (3.21), replacing the use of $\mathcal{M}_{m,k}^{[2]}$ with $\mathcal{S}_{m,k}^{[3]}$.

LEMMA 3.10. *Let $b \in C_c^\infty(T)$ with $\int_T b(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = 1$, $k \in \mathbb{N}_0$, and $\mathcal{S} = \{1, 2, 3\}$. For $F = (f^0, f^1, \dots, f^k) \in L^p(\Gamma_{\mathcal{S}})^{k+1}$, we formally define the following operators:*

$$\mathcal{L}_{k,0}^{[3]}(F) := \mathcal{L}_k^{[2]}(F) + \mathcal{S}_{0,k+1}^{[3]}(f_3^0 - \mathcal{L}_k^{[2]}(F)|_{\Gamma_3}), \tag{3.33a}$$

$$\mathcal{L}_{k,m}^{[3]}(F) := \mathcal{L}_{k,m-1}^{[3]}(F) + \mathcal{S}_{m,k+1}^{[3]}(f_3^m - \partial_{\mathbf{n}}^m \mathcal{L}_{k,m-1}^{[3]}(F)|_{\Gamma_3}), \qquad 1 \le m \le k, \tag{3.33b}$$

$$\mathcal{L}_k^{[3]}(F) := \mathcal{L}_{k,k}^{[3]}(F). \tag{3.33c}$$

Then, for all $(s,p) \in \mathcal{A}_k$ and $F \in \mathrm{Tr}_k^{s,p}(\Gamma_{\mathcal{S}})$, there holds

$$\partial_{\mathbf{n}}^m \mathcal{L}_k^{[3]}(F)|_{\Gamma_j} = f_j^m, \quad 0 \le m \le k, \ j \in \mathcal{S}, \quad \text{and} \quad \|\mathcal{L}_k^{[3]}(F)\|_{s,p,K} \lesssim_{b,k,s,p} \|F\|_{\mathrm{Tr}_k^{s,p},\Gamma_{\mathcal{S}}}. \tag{3.34}$$

Moreover, if F satisfies (2.20) and for some $N \in \mathbb{N}_0$, then $\mathcal{L}_k^{[3]}(F) \in \mathcal{P}_N(K)$.

Proof. Let $k \in \mathbb{N}_0$, $(s,p) \in \mathcal{A}_k$, and $f \in \mathrm{Tr}_k^{s,p}(\Gamma_{\mathcal{S}})$ be given. Let $\mathfrak{E} = \{\gamma_{13}, \gamma_{23}\}$.
Step 1: $m = 0$. Thanks to Lemma 3.6, the traces $G = (g^0, g^1, \dots, g^k)$ given by

$$g_i^l := f_i^l - \partial_{\mathbf{n}} \mathcal{L}_k^{[2]}(F)|_{\Gamma_i}, \qquad 0 \le l \le k, \ 1 \le i \le 3,$$

satisfy the hypotheses of Lemma 3.9 with $(i,j) \in \{(1,3),(2,3)\}$ and $l=1$. Thanks to (3.27) and Lemma 3.9, $g_3^0 \in W_{\mathfrak{E},k+1}^{s-\frac{1}{p},p}(\Gamma_3)$. Consequently, $\mathcal{S}_{0,k+1}^{[3]}(g_3^0)$ is well-defined by Corollary 3.8, and hence $\mathcal{L}_{k,0}^{[3]}(F)$ is well-defined with

$$\|\mathcal{L}_{k,0}^{[3]}(F)\|_{s,p,K} \lesssim_{b,k,s,p} \|\mathcal{L}_{k}^{[2]}(F)\|_{s,p,K} + {}_{\mathfrak{E},k+1}\left\| f_3^0 - \mathcal{L}_k^{[2]}(F) \right\|_{s-\frac{1}{p},p,\Gamma_3}.$$

Applying (3.22) and (3.30a) gives

$$\partial_{\mathbf{n}}^l \mathcal{L}_{k,0}^{[3]}(F)|_{\Gamma_i} = f_i^l, \qquad 0 \le l \le k,\ 1 \le i \le 2, \qquad \mathcal{L}_{k,0}^{[3]}(F)|_{\Gamma_3} = f_3^0,$$

and applying (3.10), (3.22), and (3.15) gives

$$\|\mathcal{L}_{k,0}^{[3]}(F)\|_{s,p,K} \lesssim_{b,k,s,p} \|F\|_{\mathrm{Tr}_k^{s,p},\Gamma_{\mathcal{S}}}.$$

Moreover, if F satisfies (2.20) for some $N \in \mathbb{N}_0$, then $\mathcal{L}_k^{[2]}(F) \in \mathcal{P}_N(K)$ by Lemma 3.6, and so the trace G satisfies (2.20) and $G \in \mathrm{Tr}_k^{s,p}(\Gamma_{\mathcal{S}})$ for all $(s,p) \in \mathcal{A}_k$ thanks to Lemma 2.5. By (3.27) and Lemma 3.9, $g_3^0 \in W_{\mathfrak{E}}^{k+1,p}(\Gamma_3)$ for all $p \in (1,\infty)$, and so $D_\Gamma^l g_3^0|_{\gamma_{13}} = D_\Gamma^l g_3^0|_{\gamma_{23}} = 0$ for $0 \le l \le k$. Thanks to Corollary 3.8, $\mathcal{L}_{k,0}^{[3]}(F) \in \mathcal{P}_N(K)$.

Step 2: Induction on m. Assume that for some m such that $0 \le m \le k-1$, $\mathcal{L}_{k,m}^{[3]}(F)$ is well-defined and satisfies

$$\partial_{\mathbf{n}}^l \mathcal{L}_{k,m}^{[3]}(F)|_{\Gamma_i} = f_i^l, \qquad 0 \le l \le k,\ 1 \le i \le 2, \tag{3.35a}$$

$$\partial_{\mathbf{n}}^l \mathcal{L}_{k,m}^{[3]}(F)|_{\Gamma_3} = f_3^l, \qquad 0 \le l \le m, \tag{3.35b}$$

and

$$\|\mathcal{L}_{k,m}^{[3]}(F)\|_{s,p,K} \lesssim_{b,k,s,p} \|F\|_{\mathrm{Tr}_k^{s,p},\Gamma_{\mathcal{S}}}. \tag{3.36}$$

Additionally assume that if F satisfies (2.20) for some $N \in \mathbb{N}_0$, then $\mathcal{L}_{k,m}^{[3]}(F) \in \mathcal{P}_N(K)$.

Thanks to (3.35), the traces $G = (g^0, g^1, \dots, g^k)$ given by

$$g_i^l := f_i^l - \partial_{\mathbf{n}}^l \mathcal{L}_{k,m}^{[3]}(F)|_{\Gamma_i}, \qquad 0 \le l \le k,\ 1 \le i \le 3,$$

satisfy the hypotheses of Lemma 3.9 with $(i,j) \in \{(1,3),(2,3)\}$ and $l = m+1$. Thanks to (3.27) and Lemma 3.9, there holds $g_3^{m+1} \in W_{\mathfrak{E},k+1}^{s-m-1-\frac{1}{p},p}(\Gamma_3)$. Consequently, $\mathcal{S}_{m+1,k+1}^{[3]}(g_3^{m+1})$ is well-defined by Corollary 3.8, and hence $\mathcal{L}_{k,m+1}^{[3]}(F)$ is well-defined with

$$\begin{aligned}\|\mathcal{L}_{k,m+1}^{[3]}(F)\|_{s,p,K} &\lesssim_{b,k,s,p} \|\mathcal{L}_{k,m}^{[3]}(F)\|_{s,p,K} \\ &\quad + {}_{\mathfrak{E},k+1}\left\| f_3^{m+1} - \partial_{\mathbf{n}}^{m+1} \mathcal{L}_{k,m}^{[3]}(F) \right\|_{s-m-1-\frac{1}{p},p,\Gamma_3}.\end{aligned}$$

Applying (3.35) and (3.30a) gives (3.35) for $m+1$, while applying (3.27), (3.36), and (3.32) gives (3.36) for $m+1$.

Moreover, if F satisfies (2.20) for some $N \in \mathbb{N}_0$, then $\mathcal{L}_{k,m}^{[3]}(F) \in \mathcal{P}_N(K)$ by assumption and so the trace G satisfies (2.20) and $G \in \mathrm{Tr}_k^{s,p}(\Gamma_{\mathcal{S}})$ for all $(s,p) \in \mathcal{A}_k$ thanks to Lemma 2.5. By (3.10) and Lemma 3.5, $g_3^{m+1} \in W_{\mathfrak{E}}^{k+1,p}(\Gamma_3)$ for all $p \in (1,\infty)$, and so $D_\Gamma^l g_3^{m+1}|_{\gamma_{13}} = D_\Gamma^l g_3^{m+1}|_{\gamma_{23}} = 0$ for $0 \le l \le k$. Thanks to Corollary 3.8, $\mathcal{L}_{k,m+1}^{[3]}(F) \in \mathcal{P}_N(K)$. ∎

3.4. Lifting from four faces. To complete the construction of the lifting operator from the entire boundary, we define one final single face lifting operator from Γ_1 that vanishes on the remaining faces. Given nonnegative integers $k, r \in \mathbb{N}_0$, a smooth compactly supported function $b \in C_c^\infty(T)$, and a function $f : T \to \mathbb{R}$, we define the operator $\mathcal{R}_{k,r}^{[1]}$ formally by the rule

$$\begin{aligned} &\mathcal{R}_{k,r}^{[1]}(f)(\boldsymbol{x}, z) \\ &:= (x_1 x_2 (1 - x_1 - x_2 - z))^r \mathcal{E}_k^{[1]}((\omega_1\omega_2\omega_3)^{-r} f)(\boldsymbol{x}, z) \\ &= (x_1 x_2 (1 - x_1 - x_2 - z))^r \frac{(-z)^k}{k!} \int_T \frac{b(\boldsymbol{y}) f(\boldsymbol{w})\, \mathrm{d}\boldsymbol{y}}{(\omega_1\omega_2\omega_3)^r(\boldsymbol{w})} \bigg|_{\boldsymbol{w}=\boldsymbol{x}+z\boldsymbol{y}} \quad \forall (\boldsymbol{x}, z) \in K. \end{aligned} \tag{3.37}$$

Note that when $r = 0$, we have $\mathcal{R}_{k,r}^{[1]} = \mathcal{E}_k^{[1]}$. For functions $f : \Gamma_1 \to \mathbb{R}$, we again abuse notation and set $\mathcal{R}_{k,r}^{[1]}(f) := \mathcal{R}_{k,r}^{[1]}(f \circ \mathfrak{I}_1)$. The weighted spaces $W_{\mathfrak{E},r}^{s,p}(\Gamma_1)$ again play a role in the continuity of $\mathcal{R}_{k,r}^{[1]}$ as the following result shows.

Lemma 3.11. *Let $b \in C_c^\infty(T)$, $k, r \in \mathbb{N}_0$, $(s, p) \in \mathcal{A}_k$, and $\mathfrak{E} = \{\gamma_{12}, \gamma_{13}, \gamma_{14}\}$. Then, for all $f \in W_{\mathfrak{E},r}^{s-k-\frac{1}{p},p}(\Gamma_1)$, there holds*

$$\partial_{\mathbf{n}}^m \mathcal{R}_{k,r}^{[1]}(f)|_{\Gamma_1} = \delta_{km} \left(\int_T b(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \right) f, \qquad 0 \le m \le k, \tag{3.38a}$$

$$\partial_{\mathbf{n}}^j \mathcal{R}_{k,r}^{[1]}(f)|_{\Gamma_i} = 0, \qquad 0 \le j < \min\left\{ r, s - \frac{1}{p} \right\},\ 2 \le i \le 4, \tag{3.38b}$$

and

$$\|\mathcal{R}_{k,r}^{[1]}(f)\|_{s,p,K} \lesssim_{b,k,r,s,p} \ _{\mathfrak{E},r}\|f\|_{s-k-\frac{1}{p},p,\Gamma_1}. \tag{3.39}$$

Moreover, if $f \in \mathcal{P}_N(\Gamma_1)$, $N \in \mathbb{N}_0$, satisfies $D_\Gamma^l f|_{\partial\Gamma_1} = 0$ for $0 \le l \le r - 1$, then $\mathcal{R}_{k,r}^{[1]}(f) \in \mathcal{P}_{N+k}(K)$.

The proof of Lemma 3.11 appears in subsection 6.5. The analogous operator associated to Γ_4 is given by

$$\mathcal{R}_{k,r}^{[4]}(f)(\boldsymbol{x}, z) := 3^{-\frac{k}{2}} \mathcal{R}_{k,r}^{[1]}(f \circ \mathfrak{I}_4) \circ \mathfrak{R}_{14}(\boldsymbol{x}, z) \qquad \forall (\boldsymbol{x}, z) \in K,$$

where $\mathfrak{I}_4(\boldsymbol{x}) := (x_1, x_2, 1 - x_1 - x_2)$ and $\mathfrak{R}_{14}(\boldsymbol{x}, z) := (x_1, x_2, 1 - x_1 - x_2 - z)$ for all $(\boldsymbol{x}, z) \in K$. Thanks to the chain rule and the smoothness of the mappings $\mathfrak{I}_4$ and $\mathfrak{R}_{14}$, the continuity and interpolation properties of $\mathcal{R}_{k,r}^{[4]}$ follow immediately from Lemma 3.11.

Corollary 3.12. *Let $b \in C_c^\infty(T)$, $k, r \in \mathbb{N}_0$, $(s, p) \in \mathcal{A}_k$, and $\mathfrak{E} = \{\gamma_{14}, \gamma_{24}, \gamma_{34}\}$. Then, for all $f \in W_{\mathfrak{E},r}^{s-k-\frac{1}{p},p}(\Gamma_4)$, there holds*

$$\partial_{\mathbf{n}}^m \mathcal{R}_{k,r}^{[4]}(f)|_{\Gamma_4} = \delta_{km} \left(\int_T b(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \right) f, \qquad 0 \le m \le k, \tag{3.40a}$$

$$\partial_{\mathbf{n}}^j \mathcal{R}_{k,r}^{[4]}(f)|_{\Gamma_i} = 0, \qquad 0 \le j < \min\left\{ r, s - \frac{1}{p} \right\},\ 1 \le i \le 3 \tag{3.40b}$$

and

$$\|\mathcal{R}^{[4]}_{k,r}(f)\|_{s,p,K} \lesssim_{b,k,r,s,p} \ {}_{\mathfrak{E},r}\|f\|_{s-k-\frac{1}{p},p,\Gamma_4}. \tag{3.41}$$

Moreover, if $f \in \mathcal{P}_N(\Gamma_4)$*,* $N \in \mathbb{N}_0$*, satisfies* $D^l_\Gamma f|_{\partial\Gamma_4} = 0$ *for* $0 \leq l \leq r-1$*, then* $\mathcal{R}^{[4]}_{k,r}(f) \in \mathcal{P}_{N+k}(K)$*.*

Finally, we construct the lifting operator $\mathcal{L}^{[4]}_k$ in the same fashion as $\mathcal{L}^{[3]}_k$ (3.33), replacing the use of $\mathcal{S}^{[3]}_{m,k+1}$ with $\mathcal{R}^{[4]}_{m,k+1}$.

Lemma 3.13. *Let* $b \in C^\infty_c(T)$ *with* $\int_T b(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 1$ *and* $k \in \mathbb{N}_0$*. For* $F = (f^0, f^1, \ldots, f^k) \in L^p(\partial K)^{k+1}$*, we formally define the following operators:*

$$\mathcal{L}^{[4]}_{k,0}(F) := \mathcal{L}^{[3]}_k(F) + \mathcal{R}^{[4]}_{0,k+1}(f^0_4 - \mathcal{L}^{[3]}_k(F)|_{\Gamma_4}), \tag{3.42a}$$

$$\mathcal{L}^{[4]}_{k,m}(F) := \mathcal{L}^{[4]}_{k,m-1}(F) + \mathcal{R}^{[4]}_{m,k+1}(f^m_4 - \partial^m_{\mathbf{n}}\mathcal{L}^{[4]}_{k,m-1}(F)|_{\Gamma_4}), \qquad 1 \leq m \leq k, \tag{3.42b}$$

$$\mathcal{L}^{[4]}_k(F) := \mathcal{L}^{[4]}_{k,k}(F). \tag{3.42c}$$

Then, for all $(s,p) \in \mathcal{A}_k$ *and* $F \in \mathrm{Tr}^{s,p}_k(\partial K)$*, there holds*

$$\partial^m_{\mathbf{n}}\mathcal{L}^{[4]}_k(F)|_{\partial K} = f^m, \quad 0 \leq m \leq k, \quad \text{and} \quad \|\mathcal{L}^{[4]}_k(F)\|_{s,p,K} \lesssim_{b,k,s,p} \|F\|_{\mathrm{Tr}^{s,p}_k,\partial K}. \tag{3.43}$$

Moreover, if F *satisfies* (2.20) *for some* $N \in \mathbb{N}_0$*, then* $\mathcal{L}^{[4]}_k(F) \in \mathcal{P}_N(K)$*.*

Proof. Let $k \in \mathbb{N}_0$, $(s,p) \in \mathcal{A}_k$, and $f \in \mathrm{Tr}^{s,p}_k(\partial K)$ be given and set $\mathfrak{E} := \{\gamma_{14}, \gamma_{24}, \gamma_{34}\}$.

Step 1: $m = 0$**.** Thanks to Lemma 3.10, the traces $G = (g^0, g^1, \ldots, g^k)$ given by

$$g^l_i := f^l_i - \partial_{\mathbf{n}}\mathcal{L}^{[3]}_k(F)|_{\Gamma_i}, \qquad 0 \leq l \leq k,\ 1 \leq i \leq 4,$$

satisfies the hypotheses of Lemma 3.9 with $(i,j) \in \{(1,4),(2,4),(3,4)\}$ and $l = 1$. Thanks to (3.27) and Lemma 3.9, $g^0_4 \in W^{s-\frac{1}{p},p}_{\mathfrak{E},k+1}(\Gamma_4)$. Consequently, $\mathcal{R}^{[4]}_{0,k+1}(g^0_4)$ is well-defined by Corollary 3.12, and hence $\mathcal{L}^{[4]}_{k,0}(F)$ is well-defined with

$$\|\mathcal{L}^{[4]}_{k,0}(F)\|_{s,p,K} \lesssim_{b,k,s,p} \|\mathcal{L}^{[3]}_k(F)\|_{s,p,K} + \ {}_{\mathfrak{E},k+1}\left\|f^0_4 - \mathcal{L}^{[3]}_k(F)\right\|_{s-\frac{1}{p},p,\Gamma_4}.$$

Applying (3.34) and (3.40a) gives

$$\partial^l_{\mathbf{n}}\mathcal{L}^{[4]}_{k,0}(F)|_{\Gamma_i} = f^l_i, \qquad 0 \leq l \leq k,\ 1 \leq i \leq 3, \qquad \mathcal{L}^{[4]}_{k,0}(F)|_{\Gamma_3} = f^0_4,$$

and applying (3.27), (3.34), and (3.32) gives

$$\|\mathcal{L}^{[4]}_{k,0}(F)\|_{s,p,K} \lesssim_{b,k,s,p} \|F\|_{\mathrm{Tr}^{s,p}_k,\partial K}.$$

Moreover, if F satisfies (2.20) for some $N \in \mathbb{N}_0$, then $\mathcal{L}^{[3]}_k(F) \in \mathcal{P}_N(K)$ by Lemma 3.6, and so the trace G satisfies (2.20) and $G \in \mathrm{Tr}^{s,p}_k(\partial K)$ for all $(s,p) \in \mathcal{A}_k$ thanks to Lemma 2.5. By (3.27) and Lemma 3.9, $g^0_4 \in W^{k+1,p}_{\mathfrak{E}}(\Gamma_4)$ for all $p \in (1,\infty)$, and so $D^l_\Gamma g^0_4|_{\partial\Gamma_4} = 0$ for $0 \leq l \leq k$. Thanks to Corollary 3.12, $\mathcal{L}^{[4]}_{k,0}(F) \in \mathcal{P}_N(K)$.

Step 2: Induction on m. Assume that for some $0 \leq m \leq k-1$, $\mathcal{L}^{[4]}_{k,m}(F)$ is well-defined and satisfies

$$\partial^l_{\mathbf{n}} \mathcal{L}^{[4]}_{k,m}(F)|_{\Gamma_i} = f^l_i, \qquad 0 \leq l \leq k,\ 1 \leq i \leq 3, \tag{3.44a}$$

$$\partial^l_{\mathbf{n}} \mathcal{L}^{[4]}_{k,m}(F)|_{\Gamma_4} = f^l_4, \qquad 0 \leq l \leq m, \tag{3.44b}$$

and

$$\|\mathcal{L}^{[4]}_{k,m}(F)\|_{s,p,K} \lesssim_{b,k,s,p} \|F\|_{\mathrm{Tr}^{s,p}_k, \partial K}. \tag{3.45}$$

Additionally assume that if F satisfies (2.20) for some $N \in \mathbb{N}_0$, then $\mathcal{L}^{[4]}_{k,m}(F) \in \mathcal{P}_N(K)$.

Thanks to (3.44), the traces $G = (g^0, g^1, \ldots, g^k)$ given by

$$g^l_i := f^l_i - \partial^l_{\mathbf{n}} \mathcal{L}^{[4]}_{k,m}(F)|_{\Gamma_i}, \qquad 0 \leq l \leq k,\ 1 \leq i \leq 4,$$

satisfies the hypotheses of Lemma 3.9 with $(i,j) \in \{(1,4),(2,4),(3,4)\}$ and $l = m+1$. Thanks to (3.27) and Lemma 3.9, $g^{m+1}_4 \in W^{s-m-1-\frac{1}{p},p}_{\mathfrak{E},k+1}(\Gamma_4)$. Consequently, $\mathcal{R}^{[4]}_{m+1,k+1}(g^{m+1}_4)$ is well-defined by Corollary 3.12, and hence $\mathcal{L}^{[4]}_{k,m+1}(F)$ is well-defined with

$$\begin{aligned}\|\mathcal{L}^{[4]}_{k,m+1}(F)\|_{s,p,K} \lesssim_{b,k,s,p} \|\mathcal{L}^{[4]}_{k,m}(F)\|_{s,p,K}& \\ + {}_{\mathfrak{E},k+1}\left\| f^{m+1}_4 - \partial^{m+1}_{\mathbf{n}} \mathcal{L}^{[4]}_{k,m}(F) \right\|_{s-m-1-\frac{1}{p},p,\Gamma_4}&.\end{aligned}$$

Applying (3.44) and (3.40a) gives (3.44) for $m+1$, while applying (3.27), (3.45), and (3.32) gives (3.45) for $m+1$.

Moreover, if F satisfies (2.20) for some $N \in \mathbb{N}_0$, then $\mathcal{L}^{[4]}_{k,m}(F) \in \mathcal{P}_N(K)$ by assumption and so the trace G satisfies (2.20) and $G \in \mathrm{Tr}^{s,p}_k(\partial K)$ for all $(s,p) \in \mathcal{A}_k$ thanks to Lemma 2.5. By (3.27) and Lemma 3.9, $g^{m+1}_4 \in W^{k+1,p}_{\mathfrak{E}}(\Gamma_4)$ for all $p \in (1,\infty)$, and so $D^l_{\Gamma} g^{m+1}_4|_{\partial \Gamma_4} = 0$ for $0 \leq l \leq k$. Thanks to Corollary 3.12, $\mathcal{L}^{[4]}_{k,m+1}(F) \in \mathcal{P}_N(K)$. ∎

3.5. Proof of Theorem 2.6. Let $b \in C^\infty_c(T)$ be any smooth function satisfying $\int_T b(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 1$. Then, $\mathcal{L}_k := \mathcal{L}^{[4]}_k$, where $\mathcal{L}^{[4]}_k$ is defined in (3.42) satisfies the desired properties thanks to Lemma 3.13. ∎

4. Whole space operators.

In this section, we examine the continuity properties of the following operators, which are the whole space extensions of the lifting operators $\mathcal{E}^{[1]}_k$ (3.1): Given $k \in \mathbb{N}_0$, $\chi \in C^\infty_c(\mathbb{R})$, and $b \in C^\infty_c(\mathbb{R}^2)$ and a function $f : \mathbb{R}^2 \to \mathbb{R}$, we define the lifting operator $\tilde{\mathcal{E}}_k$ by the rule

$$\tilde{\mathcal{E}}_k(f)(\boldsymbol{x}, z) := \chi(z) z^k \int_{\mathbb{R}^2} b(\boldsymbol{y}) f(\boldsymbol{x} + z\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \qquad \forall (\boldsymbol{x}, z) \in \mathbb{R}^2 \times \mathbb{R}. \tag{4.1}$$

We use the notation $\tilde{\mathcal{E}}_k[\chi, b]$ when we want to make the dependence on χ and b explicit. The advantage of working with the operator $\tilde{\mathcal{E}}_k$ is that we shall capitalize on the abundance of equivalent $W^{s,p}(U)$-norms when U is all of $\mathbb{R}^d$ or the half-space $\mathbb{R}^d_+ = \mathbb{R}^{d-1} \times (0,\infty)$, $d > 1$. In particular, we recall the following norm-equivalence on $W^{s,p}(U)$, $0 < s < 1$, $1 < p < \infty$, with $U = \mathbb{R}^d$ or $\mathbb{R}^d_+$ (see e.g. [44, Theorems 6.38 & 6.61]):

$$|f|^p_{s,p,U} \approx_{s,p,d} \sum_{i=1}^{d} \int_0^\infty \int_U \frac{|f(\boldsymbol{x} + t\mathbf{e}_i) - f(\boldsymbol{x})|^p}{t^{1+sp}} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \qquad \forall f \in W^{s,p}(U). \tag{4.2}$$

The main result of this section is the following analogue of Lemma 3.1.

THEOREM 4.1. *Let $\chi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} \chi \in (-2, 2)$, $b \in C_c^\infty(\mathbb{R}^2)$ with $\operatorname{supp} b \subset T$, and $k \in \mathbb{N}_0$ be given. Then, for $(s, p) \in \mathcal{A}_k \cup (k + \frac{1}{2}, 2)$, there holds*

$$\|\tilde{\mathcal{E}}_k(f)\|_{s,p,\mathbb{R}^3_+} \lesssim_{\chi,b,k,s,p} \|f\|_{s-k-\frac{1}{p},p,\mathbb{R}^2} \qquad \forall f \in W^{s-k-\frac{1}{p},p}(\mathbb{R}^2). \tag{4.3}$$

The proof of Theorem 4.1 appears in subsection 4.3.

4.1. Continuity of $\tilde{\mathcal{E}}_0$. We begin by recording the particular case of Theorem 4.1 with $k = 0$, which follows from the same arguments as in the proof of [44, Theorem 9.21].

LEMMA 4.2. *Let $\chi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} \chi \in (-2, 2)$ and $b \in C_c^\infty(\mathbb{R}^2)$ with $\operatorname{supp} b \subset T$. Then, for $1 < p < \infty$ and $1/p < s < 1$, there holds*

$$\|\tilde{\mathcal{E}}_0(f)\|_{s,p,\mathbb{R}^3_+} \lesssim_{\chi,b,s,p} \|f\|_{s-\frac{1}{p},p,\mathbb{R}^2} \qquad \forall f \in W^{s-\frac{1}{p},p}(\mathbb{R}^2). \tag{4.4}$$

When $p = 2$, the above result is also true for $s = 1/2$ as the following lemma shows.

LEMMA 4.3. *Let $\chi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} \chi \in (-2, 2)$ and $b \in C_c^\infty(\mathbb{R}^2)$ with $\operatorname{supp} b \subset T$. Then, there holds*

$$\|\tilde{\mathcal{E}}_0(f)\|_{\frac{1}{2},2,\mathbb{R}^3_+} \lesssim_{\chi,b} \|f\|_{2,\mathbb{R}^2} \qquad \forall f \in L^2(\mathbb{R}^2). \tag{4.5}$$

Proof. By density, it suffices to consider $f \in C_c^\infty(\mathbb{R}^2)$. For $k \in \mathbb{N}_0$ define

$$g_k(\boldsymbol{x}, z) := z^k \int_{\mathbb{R}^2} b(\boldsymbol{y}) f(\boldsymbol{x} + z\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}, \qquad (\boldsymbol{x}, z) \in \mathbb{R}^3. \tag{4.6}$$

Step 1: $H^{1/2}(\mathbb{R}^3_+)$ bound for g_0. Thanks to (4.2), there holds

$$|g_0|^2_{\frac{1}{2},2,\mathbb{R}^3_+} \approx \int_0^\infty |g_0(\cdot, z)|^2_{\frac{1}{2},2,\mathbb{R}^2} \, \mathrm{d}z + \int_{\mathbb{R}^2} |g_0(\boldsymbol{x}, \cdot)|^2_{\frac{1}{2},2,\mathbb{R}_+} \, \mathrm{d}\boldsymbol{x} =: I_1 + I_2.$$

We now follow the steps in the proof of [18, Theorem 2.2]. Let $\hat{\cdot}$ denote the Fourier transform with respect to the $\boldsymbol{x}$-variable. Then,

$$\begin{aligned} I_1 \approx \int_0^\infty \int_{\mathbb{R}^2} |\boldsymbol{\xi}| \cdot |\hat{g}_0(\boldsymbol{\xi}, z)|^2 \, \mathrm{d}\boldsymbol{\xi} \, \mathrm{d}z &= \int_0^\infty \int_{\mathbb{R}^2} |\boldsymbol{\xi}| \cdot |\hat{b}(\boldsymbol{\xi} z) \hat{f}(\boldsymbol{\xi})|^2 \, \mathrm{d}\boldsymbol{\xi} \, \mathrm{d}z \\ &= \int_{\mathbb{R}^2} \left(|\boldsymbol{\xi}| \cdot \|\hat{b}(\boldsymbol{\xi} \cdot)\|^2_{2,\mathbb{R}_+} \right) |\hat{f}(\boldsymbol{\xi})|^2 \, \mathrm{d}\boldsymbol{\xi}, \end{aligned}$$

where we used the following convolution identity for $z > 0$:

$$g_0(\boldsymbol{x}, z) = \int_{\mathbb{R}^2} z^{-2} b\left(\frac{\boldsymbol{y} - \boldsymbol{x}}{z} \right) f(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} \implies \hat{g}_0(\boldsymbol{\xi}, z) = \hat{b}(\boldsymbol{\xi} z) \hat{f}(\boldsymbol{\xi}). \tag{4.7}$$

Similarly, there holds

$$I_2 \approx \int_{\mathbb{R}^2} |\hat{g}(\boldsymbol{\xi},\cdot)|^2_{\frac{1}{2},2,\mathbb{R}_+} \,\mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^2} |\hat{b}(\boldsymbol{\xi}\cdot)|^2_{\frac{1}{2},2,\mathbb{R}_+} |\hat{f}(\boldsymbol{\xi})|^2 \,\mathrm{d}\boldsymbol{\xi}.$$

Thanks to a change of variables, we obtain

$$\begin{aligned} |\boldsymbol{\xi}| \cdot \|\hat{b}(\boldsymbol{\xi}\cdot)\|^2_{2,\mathbb{R}_+} + |\hat{b}(\boldsymbol{\xi}\cdot)|^2_{\frac{1}{2},2,\mathbb{R}_+} &\leq \sup_{\boldsymbol{\omega}\in\mathbb{S}^2} \left(|\boldsymbol{\xi}| \cdot \|\hat{b}(|\boldsymbol{\xi}|\boldsymbol{\omega}\cdot)\|^2_{2,\mathbb{R}_+} + |\hat{b}(|\boldsymbol{\xi}|\boldsymbol{\omega}\cdot)|^2_{\frac{1}{2},2,\mathbb{R}_+} \right) \\ &= \sup_{\boldsymbol{\omega}\in\mathbb{S}^2} \|\hat{b}(\boldsymbol{\omega}\cdot)\|^2_{\frac{1}{2},2,\mathbb{R}_+}, \end{aligned}$$

which is finite since $\hat{b}$ is a Schwartz function, and so $|g_0|_{\frac{1}{2},2,\mathbb{R}^3_+} \lesssim_b \|f\|_{2,\mathbb{R}^2}$.

Step 2: $H^{1/2}(\mathbb{R}^3_+)$ bound on $\tilde{\mathcal{E}}_0(f)$. For $i = 1, 2$, there holds

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^3_+} \frac{|\tilde{\mathcal{E}}_0(f)(\boldsymbol{x}+t\mathbf{e}_i, z) - \tilde{\mathcal{E}}_0(f)(\boldsymbol{x}, z)|^2}{t^2} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z\,\mathrm{d}t \\ &\qquad = \int_0^\infty \int_{\mathbb{R}^3_+} |\chi(z)|^2 \frac{|g_0(\boldsymbol{x}+t\mathbf{e}_i, z) - g_0(\boldsymbol{x}, z)|^2}{t^2} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z\,\mathrm{d}t \lesssim_{\chi,b} \|f\|_{2,\mathbb{R}^2}. \end{aligned}$$

where we used (4.2) and step 1. Thanks to the relation

$$\begin{aligned} |\tilde{\mathcal{E}}_0(f)(\boldsymbol{x}, z+t) - \tilde{\mathcal{E}}_0(f)(\boldsymbol{x}, z)|^2 \lesssim |\chi(z+t)|^2 |g_0(\boldsymbol{x}, z+t) - g_0(\boldsymbol{x}, t)|^2 \\ + |\chi(z+t) - \chi(z)|^2 |g_0(\boldsymbol{x}, z)|^2, \end{aligned}$$

we obtain

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^3_+} \frac{|\tilde{\mathcal{E}}_0(f)(\boldsymbol{x}, z+t) - \tilde{\mathcal{E}}_0(f)(\boldsymbol{x}, z)|^2}{t^2} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z\,\mathrm{d}t \\ &\qquad \lesssim \|\chi\|^2_{\infty,\mathbb{R}_+} |g_0|^2_{\frac{1}{2},2,\mathbb{R}^3_+} + \int_0^\infty \int_{\mathbb{R}^3_+} \frac{|\chi(z+t) - \chi(z)|^2}{t^2} |g_0(\boldsymbol{x}, z)|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z\,\mathrm{d}t. \end{aligned}$$

Now, applying Hardy's inequality [40, Theorem 327] gives

$$\begin{aligned} \int_0^\infty \frac{|\chi(z+t) - \chi(z)|^2}{t^2} \,\mathrm{d}t &= \int_0^\infty \left(\frac{1}{t} \int_z^{z+t} \chi'(r) \,\mathrm{d}r \right)^2 \,\mathrm{d}t \\ &= \int_0^\infty \left(\frac{1}{t} \int_0^t \chi'(r+z) \,\mathrm{d}r \right)^2 \,\mathrm{d}t \lesssim \|\chi'(\cdot+z)\|^2_{2,\mathbb{R}_+}, \end{aligned}$$

and so

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^3_+} \frac{|\chi(z+t) - \chi(z)|^2}{t^2} |g_0(\boldsymbol{x}, z)|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z\,\mathrm{d}t &\lesssim \int_{\mathbb{R}^3_+} \|\chi'(\cdot+z)\|^2_{2,\mathbb{R}_+} |g_0(\boldsymbol{x}, z)|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z \\ &\leq \|\chi'\|^2_{2,\mathbb{R}_+} \int_0^2 \int_{\mathbb{R}^2} |g_0(\boldsymbol{x}, z)|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z. \end{aligned}$$

Applying Young's inequality to the convolution form of g_0, (4.7) then gives

$$\int_0^2 \int_{\mathbb{R}^2} |g_0(\boldsymbol{x}, z)|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z \leq 2\|b\|^2_{1,\mathbb{R}^2} \|f\|^2_{2,\mathbb{R}^2}.$$

Inequality (4.5) now follows on collecting results and applying (4.2). ∎

We shall also need the stability of the lifting of the derivative of a smooth function.

LEMMA 4.4. *Let $\chi \in C_c^\infty(\mathbb{R})$ and $b \in C_c^\infty(\mathbb{R}^2)$. For $1 < p < \infty$, there holds*

$$\sum_{i=1}^{2} \|\tilde{\mathcal{E}}_0(\partial_i f)\|_{p,\mathbb{R}^3_+} \lesssim_{\chi,b,p} \|f\|_{1-\frac{1}{p},p,\mathbb{R}^2} \qquad \forall f \in C_c^\infty(\mathbb{R}^2). \tag{4.8}$$

Proof. Let $1 < p < \infty$, $f \in C_c^\infty(\mathbb{R}^2)$, and $i \in \{1, 2\}$. Integrating by parts gives

$$\begin{aligned}\tilde{\mathcal{E}}_0(\partial_i f)(\boldsymbol{x}, z) = \chi(z) \int_{\mathbb{R}^2} b(\boldsymbol{y})(\partial_i f)(\boldsymbol{x} + z\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} &= \frac{\chi(z)}{z} \int_{\mathbb{R}^2} b(\boldsymbol{y})\partial_{y_i}\{f(\boldsymbol{x} + z\boldsymbol{y})\}\, \mathrm{d}\boldsymbol{y} \\ &= -\frac{\chi(z)}{z} \int_{\mathbb{R}^2} (\partial_i b)(\boldsymbol{y}) f(\boldsymbol{x} + z\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}.\end{aligned}$$

Since $b \in C_c^\infty(\mathbb{R}^2)$, there holds $\int_{\mathbb{R}} (\partial_i b)(\boldsymbol{y})\, \mathrm{d}y_i = 0$, and so

$$\tilde{\mathcal{E}}_0(\partial_i f)(\boldsymbol{x}, z) = \chi(z) \int_{\mathbb{R}^2} (\partial_i b)(\boldsymbol{y}) \frac{f(\boldsymbol{x} + z(\boldsymbol{y} - y_i \boldsymbol{e}_i)) - f(\boldsymbol{x} + z\boldsymbol{y})}{z}\, \mathrm{d}\boldsymbol{y}.$$

Applying Hölder's inequality, we obtain

$$|\tilde{\mathcal{E}}_0(\partial_i f)(\boldsymbol{x}, z)|^p \lesssim_{\chi,b,p} \int_{\mathbb{R}^2} |y_i(\partial_i b)(\boldsymbol{y})| \left| \frac{f(\boldsymbol{x} + z(\boldsymbol{y} - y_i \boldsymbol{e}_i)) - f(\boldsymbol{x} + z\boldsymbol{y})}{y_i z} \right|^p \mathrm{d}\boldsymbol{y}.$$

Integrating over $\mathbb{R}^3_+$ then gives

$$\begin{aligned}\|\tilde{\mathcal{E}}_0(\partial_i f)\|_{p,\mathbb{R}^3_+}^p &\lesssim_{\chi,b,p} \int_{\mathbb{R}^5_+} |y_i(\partial_i b)(\boldsymbol{y})| \left| \frac{f(\boldsymbol{x} + z(\boldsymbol{y} - y_i \boldsymbol{e}_i)) - f(\boldsymbol{x} + z\boldsymbol{y})}{y_i z} \right|^p \mathrm{d}\boldsymbol{y}\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}z \\ &\overset{\tilde{x}_j = x_j + (1-\delta_{ij}) y_j z}{=} \int_{\mathbb{R}^5_+} |y_i(\partial_i b)(\boldsymbol{y})| \left| \frac{f(\tilde{\boldsymbol{x}}) - f(\tilde{\boldsymbol{x}} + y_i z \boldsymbol{e}_i)}{y_i z} \right|^p \mathrm{d}\boldsymbol{y}\, \mathrm{d}\tilde{\boldsymbol{x}}\, \mathrm{d}z \\ &\overset{t = y_i z}{\lesssim} \|\partial_i b\|_{1,\mathbb{R}^2} \int_{\mathbb{R}^3_+} \frac{|f(\tilde{\boldsymbol{x}}) - f(\tilde{\boldsymbol{x}} + t\boldsymbol{e}_i)|^p}{t^p}\, \mathrm{d}\tilde{\boldsymbol{x}}\, \mathrm{d}t.\end{aligned}$$

Inequality (4.8) now follows from summing over i and applying (4.2). ∎

4.2. Continuity of $\tilde{\mathcal{E}}_k$. We now show how the continuity of the operator $\tilde{\mathcal{E}}_0$ can be used to deduce the continuity of $\tilde{\mathcal{E}}_k$ for $k \in \mathbb{N}_0$. We begin with a partial result.

LEMMA 4.5. *Let $\chi \in C_c^\infty(\mathbb{R})$ and $b \in C_c^\infty(\mathbb{R}^2)$ be as in Theorem 4.1 and $k \in \mathbb{N}_0$ be given. Then, for $1 < p < \infty$, there holds*

$$\|\tilde{\mathcal{E}}_k(f)\|_{p,\mathbb{R}^3_+} \lesssim_{\chi,b,k,p} \|f\|_{p,\mathbb{R}^2} \qquad \forall f \in C_c^\infty(\mathbb{R}^2), \tag{4.9}$$

and for $1/p < s < 1$ or $(s, p) = (\frac{1}{2}, 2)$, there holds

$$\|\tilde{\mathcal{E}}_k(f)\|_{s,p,\mathbb{R}^3_+} \lesssim_{\chi,b,k,s,p} \|f\|_{s-\frac{1}{p},p,\mathbb{R}^2} \qquad \forall f \in C_c^\infty(\mathbb{R}^2). \tag{4.10}$$

Proof. Let $k \in \mathbb{N}_0$, $1 < p < \infty$, and $f \in C_c^\infty(\mathbb{R}^2)$. Since the function $\tilde{\chi} := z^k \chi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} \tilde{\chi} = \operatorname{supp} \chi$, we have $\tilde{\mathcal{E}}_k[\chi, b](f) = \tilde{\mathcal{E}}_0[\tilde{\chi}, b](f)$. Consequently, it suffices to prove (4.9) and (4.10) in the case $k = 0$. To this end, we apply Jensen's inequality to the identity (4.7) to obtain

$$\begin{aligned}\|\tilde{\mathcal{E}}_0(f)\|_{p,\mathbb{R}^3_+}^p &\le \|f\|_{p,\mathbb{R}^2}^p \int_{\mathbb{R}} |\chi(z)|^p \left(\int_{\mathbb{R}^2} \left| z^{-2} b\left(\frac{\boldsymbol{x}}{z}\right) \right| \mathrm{d}\boldsymbol{x} \right)^p \mathrm{d}z \\ &= \|f\|_{p,\mathbb{R}^2}^p \|b\|_{1,\mathbb{R}^2}^p \|\chi(z)\|_{1,\mathbb{R}}^p,\end{aligned}$$

and (4.9) follows. Inequality (4.10) for $1/p < s < 1$ is an immediate consequence of (4.4), while the case $(s,p) = (\frac{1}{2}, 2)$ follows from Lemma 4.3. ◻

For more precise results, we shall show the effect of taking partial derivatives of $\tilde{\mathcal{E}}_k(f)$ on the index k and on the function f. To this end, we recall an integration-by-parts formula for tensors. Given two d-dimensional tensors S and T, let $S : T$ denote the usual tensor contraction

$$S : T := S_{i_1 i_2 \cdots i_d} T_{i_1 i_2 \cdots i_d},$$

where we are using Einstein summation notation. Given a d-dimensional tensor S with $d \geq 0$ and $k \geq 0$, let $D^k S$ denote the k-th derivative tensor of S:

$$(D^k S)_{i_1 i_2 \cdots i_{d+k}} := \partial_{i_{d+1}} \partial_{i_{d+2}} \cdots \partial_{i_{d+k}} S_{i_1 i_2 \cdots i_d},$$

and let $\operatorname{div} S$ denote the $(d-1)$-dimensional tensor given by

$$(\operatorname{div} S)_{i_1 i_2 \cdots i_{d-1}} := \partial_j S_{i_1 i_2 \cdots i_{d-1} j},$$

while $\operatorname{div}^k S$, $0 \leq k \leq d$, denotes k applications of div to S. With this notation, we have the following integration by parts formula for symmetric, smooth, compactly supported tensors S and T of dimension d and $0 \leq k \leq d$, respectively:

$$\int_{\mathbb{R}^2} S : D^{d-k} T \, \mathrm{d}\boldsymbol{x} = (-1)^{d-k} \int_{\mathbb{R}^2} \operatorname{div}^{d-k} S : T \, \mathrm{d}\boldsymbol{x}.$$

With this notation in hand, we have the following identity that shows that the derivatives of $\tilde{\mathcal{E}}_k(f)$ are linear combinations of liftings of derivatives of f.

Lemma 4.6. *Let $\chi \in C_c^\infty(\mathbb{R})$, $b \in C_c^\infty(\mathbb{R}^2)$, and $k \in \mathbb{N}_0$ be given. For all $\alpha \in \mathbb{N}_0^3$ and $f \in C_c^\infty(\mathbb{R}^2)$, there holds*

$$\begin{aligned} D^\alpha \tilde{\mathcal{E}}_k(f)(\boldsymbol{x}, z) \\ = \sum_{i=0}^{\alpha_3} \chi_i(z) z^{\max\{k+i-|\alpha|, 0\}} \int_{\mathbb{R}^2} B_{ki\alpha}(\boldsymbol{y}) : (D^{\max\{|\alpha|-k-i, 0\}} f)(\boldsymbol{x} + z\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} \end{aligned} \tag{4.11}$$

for suitable $\chi_i \in C_c^\infty(\mathbb{R})$ and $\max\{|\alpha|-i-k, 0\}$-dimensional tensors $B_{ki\alpha}$ with entries in $C_c^\infty(\mathbb{R}^2)$.

Proof. Let $f \in C_c^\infty(\mathbb{R}^2)$ and let g_k be defined as in (4.6). For integers $m \geq k$, there holds

$$\begin{aligned} \partial_z^m g_k(\boldsymbol{x}, z) &= \sum_{j=0}^{k} c_{kmj} z^{k-j} \int_{\mathbb{R}^2} b(\boldsymbol{y}) (D^{m-j} f)(\boldsymbol{x} + z\boldsymbol{y}) : \boldsymbol{y}^{\otimes m-j} \, \mathrm{d}\boldsymbol{y} \\ &= \sum_{j=0}^{k} c_{kmj} \int_{\mathbb{R}^2} (b(\boldsymbol{y}) \boldsymbol{y}^{\otimes m-j}) : D_{\boldsymbol{y}}^{k-j} \{ (D^{m-k} f)(\boldsymbol{x} + z\boldsymbol{y}) \} \, \mathrm{d}\boldsymbol{y} \\ &= \int_{\mathbb{R}^2} \left\{ \sum_{j=0}^{k} (-1)^{k-j} c_{kmj} \operatorname{div}^{k-j} (b(\boldsymbol{y}) \boldsymbol{y}^{\otimes m-j}) \right\} : (D^{m-k} f)(\boldsymbol{x} + z\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} \\ &=: \int_{\mathbb{R}^2} B_{km}(\boldsymbol{y}) : (D^{m-k} f)(\boldsymbol{x} + z\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}, \end{aligned}$$

where c_{kmj} are suitable constants, $\boldsymbol{y}^{\otimes n}$ is the tensor product of n copies of $\boldsymbol{y}$, and $D_{\boldsymbol{y}}$ denotes the derivative operator with respect to $\boldsymbol{y}$. For $0 \leq m < k$, there holds

$$\begin{aligned}
\partial_z^m g_k(\boldsymbol{x}, z) &= \sum_{j=0}^{m} c_{kmj} z^{k-j} \int_{\mathbb{R}^2} b(\boldsymbol{y})(D^{m-j} f)(\boldsymbol{x} + z\boldsymbol{y}) : \boldsymbol{y}^{\otimes m-j} \,\mathrm{d}\boldsymbol{y} \\
&= \sum_{j=0}^{m} c_{kmj} z^{k-m} \int_{\mathbb{R}^2} (b(\boldsymbol{y})\boldsymbol{y}^{\otimes m-j}) : D_{\boldsymbol{y}}^{m-j} \{f(\boldsymbol{x} + z\boldsymbol{y})\} \,\mathrm{d}\boldsymbol{y} \\
&= z^{k-m} \int_{\mathbb{R}^2} \left\{ -\sum_{j=0}^{m} (-1)^{m-j} c_{kmj} \operatorname{div}^{m-j}(b(\boldsymbol{y})\boldsymbol{y}^{\otimes m-j}) \right\} : f(\boldsymbol{x} + z\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y} \\
&=: z^{k-m} \int_{\mathbb{R}^2} B_{km}(\boldsymbol{y}) : f(\boldsymbol{x} + z\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y}.
\end{aligned}$$

Consequently, there holds

$$\partial_z^m g_k(\boldsymbol{x}, z) = z^{\max\{k-m,0\}} \int_{\mathbb{R}^2} B_{km}(\boldsymbol{y}) : (D^{\max\{m-k,0\}} f)(\boldsymbol{x} + z\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y} \qquad \forall m \in \mathbb{N}_0.$$

Now let $\beta \in \mathbb{N}_0^2$ with $|\beta| \geq k$. Then, there holds

$$\begin{aligned}
D_{\boldsymbol{x}}^{\beta} g_k(\boldsymbol{x}, z) &= z^k \int_{\mathbb{R}^2} b(\boldsymbol{y})(D^{\beta} f)(\boldsymbol{x} + z\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y} = \int_{\mathbb{R}^2} b(\boldsymbol{y}) D_{\boldsymbol{y}}^{\tilde{\beta}} \{(D^{\beta-\tilde{\beta}} f)(\boldsymbol{x} + z\boldsymbol{y})\} \,\mathrm{d}\boldsymbol{y} \\
&= (-1)^{|\tilde{\beta}|} \int_{\mathbb{R}^2} (D^{\tilde{\beta}} b)(\boldsymbol{y})(D^{\beta-\tilde{\beta}} f)(\boldsymbol{x} + z\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y} =: \int_{\mathbb{R}^2} B_{k\beta}(\boldsymbol{y})(D^{\beta-\tilde{\beta}} f)(\boldsymbol{x} + z\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y},
\end{aligned}$$

where $D_{\boldsymbol{x}}^{\beta} := \partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2}$ and $\tilde{\beta} \in \mathbb{N}_0^2$ is any fixed multi-index such that $|\tilde{\beta}| = k$ and $\beta - \tilde{\beta} \in \mathbb{N}_0^2$. Similar arguments show that for $\beta \in \mathbb{N}_0^2$ with $|\beta| < k$, there holds

$$\begin{aligned}
D_{\boldsymbol{x}}^{\beta} g_k(\boldsymbol{x}, z) &= (-1)^{|\beta|} z^{k-|\beta|} \int_{\mathbb{R}^2} (D^{\beta} b)(\boldsymbol{y}) f(\boldsymbol{x} + z\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y} \\
&=: z^{k-|\beta|} \int_{\mathbb{R}^2} B_{k\beta}(\boldsymbol{y}) f(\boldsymbol{x} + z\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y}.
\end{aligned}$$

Collecting results, for any $\alpha \in \mathbb{N}_0^3$, there holds

$$D^{\alpha} g_k(\boldsymbol{x}, z) = z^{\max\{k-|\alpha|,0\}} \int_{\mathbb{R}^2} B_{k\alpha}(\boldsymbol{y}) : (D^{\max\{|\alpha|-k,0\}} f)(\boldsymbol{x} + z\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y}$$

for suitable $\max\{|\alpha|-k, 0\}$-dimensional tensors $B_{k\alpha}$ with entries in $C_c^{\infty}(\mathbb{R}^2)$. Equality (4.11) now follows from the product rule. ∎

4.3. Proof of Theorem 4.1. Let $k \in \mathbb{N}_0$, $1 < p < \infty$, and $f \in C_c^{\infty}(\mathbb{R}^2)$. For $\alpha \in \mathbb{N}_0^3$, (4.11) gives

$$\|D^{\alpha} \tilde{\mathcal{E}}_k(f)\|_{\sigma,p,\mathbb{R}^3_+} \leq \sum_{\substack{0 \leq i \leq \alpha_3 \\ \beta \in \mathbb{N}_0^2 \\ |\beta| = \max\{|\alpha|-k-i,0\}}} \|\tilde{\mathcal{E}}_{\max\{k+i-|\alpha|,0\}}[\chi_i, b_{ki\beta}](D^{\beta} f)\|_{\sigma,p,\mathbb{R}^3_+}, \tag{4.12}$$

where $\chi_i \in C_c^{\infty}(\mathbb{R})$ and $b_{ki\beta} \in C_c^{\infty}(\mathbb{R}^2)$ are suitable functions depending on χ and b respectively and $0 \leq \sigma < 1$.

Step 1: L^p bounds on derivatives. For $k + i - |\alpha| \geq 0$ (so that $|\beta| = 0$), (4.9) gives

$$\|\tilde{\mathcal{E}}_{\max\{k+i-|\alpha|,0\}}[\chi_i, b_{ki\beta}](D^\beta f)\|_{p,\mathbb{R}^3_+} = \|\tilde{\mathcal{E}}_{k+i-|\alpha|}[\chi_i, b_{ki\beta}](f)\|_{p,\mathbb{R}^3_+} \lesssim_{\chi,b,k,p} \|f\|_{p,\mathbb{R}^2}.$$

For $k + i - |\alpha| < 0$ (so that $|\alpha| \geq k$ and $|\beta| \geq 1$), there exists $j \in \{1, 2\}$ such that $\beta_j \geq 1$, and so we apply (4.8) to obtain

$$\begin{aligned}\|\tilde{\mathcal{E}}_0[\chi_i, b_{ki\beta}](D^\beta f)\|_{p,\mathbb{R}^3_+} = \|\tilde{\mathcal{E}}_0[\chi_i, b_{ki\beta}](\partial_j D^{\beta-\mathbf{e}_j} f)\|_{p,\mathbb{R}^3_+} &\lesssim_{\chi,b,k,p} \|D^{\beta-\mathbf{e}_j} f\|_{1-\frac{1}{p},p,\mathbb{R}^2} \\ &\leq \|f\|_{|\alpha|-k-i-\frac{1}{p},p,\mathbb{R}^2}.\end{aligned}$$

Consequently, for all $f \in C_c^\infty(\mathbb{R}^2)$, there holds

$$\|\tilde{\mathcal{E}}_k(f)\|_{m,p,\mathbb{R}^3_+} \lesssim_{\chi,b,k,m,p} \|f\|_{m-k-\frac{1}{p},p,\mathbb{R}^2}, \qquad m \in \{k+1, k+2, \ldots\}. \tag{4.13}$$

By density, (4.13) holds for all $f \in W^{m-k-1/p,p}(\mathbb{R}^2)$.
Step 2: The case $s \geq k + 1$. Inequality (4.3) for real $s \geq k+1$ with $(s, p) \in \mathcal{A}_k$ follows from (4.13) using a standard interpolation argument.
Step 3: The case $k + 1/p \leq s < k + 1$. For $s = k + \sigma$, where $1/p < \sigma < 1$ or $(\sigma, p) = (1/2, 2)$, we take $|\alpha| = k$ in (4.12) and apply (4.10) to obtain

$$|\tilde{\mathcal{E}}_k(f)|_{s,p,\mathbb{R}^3_+} \leq \sum_{0\leq i\leq\alpha_3} \|\tilde{\mathcal{E}}_i[\chi_i, b_{ki\beta}](f)\|_{\sigma,p,\mathbb{R}^3_+} \lesssim_{\chi,b,k,p,s} \|f\|_{\sigma-\frac{1}{p},p,\mathbb{R}^2},$$

which completes the proof. □

5. Weighted L^p continuity of whole-space operators. In the previous section in Theorem 4.1, we established that the lifting operators $\tilde{\mathcal{E}}_k$ are continuous from $W^{s-k-1/p,p}(\mathbb{R}^2)$ to $W^{s,p}(\mathbb{R}^3_+)$ provided that $s > k+1/p$. We now turn to the stability of the operator $\tilde{\mathcal{E}}_k$ with respect to lower-order Sobolev spaces. In particular, we seek to obtain bounds on $\|\tilde{\mathcal{E}}_k(f)\|_{s,p,\mathcal{O}_1}$ for $0 \leq s < k+1/p$, where $\mathcal{O}_1 := (0,\infty)^3 \supset K$ is the first octant. It turns out that one suitable space for the lifted function f is a weighted L^p space. Let $\mathcal{Q}_1 = (0, \infty)^2 \supset T$ denote the first quadrant and let $\rho \in L^\infty(\mathcal{Q}_1)$ be a weight function that satisfying $\rho > 0$ almost everywhere. Then, for $1 < p < \infty$, define

$$L^p(\mathcal{Q}_1; \rho\,\mathrm{d}\boldsymbol{x}) := \left\{ f \text{ measurable} : \int_{\mathcal{Q}_1} |f(\boldsymbol{x})|^p \rho(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} < \infty \right\}. \tag{5.1}$$

The weight that will appear in our estimates are powers of ω_1 (3.7) extended to all of $\mathbb{R}^2$ by

$$\omega_1(\boldsymbol{x}) = \min\{x_1, 1\} \qquad \forall \boldsymbol{x} \in \mathbb{R}^2. \tag{5.2}$$

In particular, the main result of this section is as follows.

THEOREM 5.1. *Let $\chi \in C_c^\infty(\mathbb{R})$ and $b \in C_c^\infty(\mathbb{R}^2)$ be as in Theorem 4.1 and $k \in \mathbb{N}_0$ be given. For $1 < p < \infty$ and $0 \leq s < k + 1/p$, there holds*

$$\|\tilde{\mathcal{E}}_k(f)\|_{s,p,\mathcal{O}_1} \lesssim_{\chi,b,k,s,p} \|\omega_1^{\frac{1}{p}+k-s} f\|_{p,\mathcal{Q}_1} \qquad \forall f \in L^p(\mathcal{Q}_1, \omega_1^{1+(k-s)p}\,\mathrm{d}\boldsymbol{x}). \tag{5.3}$$

The proof proceeds in several steps and appears in subsection 5.3.

5.1. Auxiliary results. We begin by recording a number of technical lemmas. Throughout the rest of the section we use the notation $\fint_{\mathcal{O}} f d\boldsymbol{x} := |\mathcal{O}|^{-1} \int_{\mathcal{O}} f d\boldsymbol{x}$.

LEMMA 5.2. *For $1 \le p < \infty$ and $0 < h < \infty$, there holds*

$$\int_0^\infty \left| \fint_x^{x+h} f(y)\,\mathrm{d}y \right|^p \mathrm{d}x \le h^{p-1} \int_0^\infty |f(x)|^p\,\mathrm{d}x \qquad \forall f \text{ measurable.} \tag{5.4}$$

Proof. The result follows on applying Hölder's inequality and changing the order of integration:

$$\begin{aligned}\int_0^\infty \left| \frac{1}{h} \int_x^{x+h} f(y)\,\mathrm{d}y \right|^p \mathrm{d}x &\le h^{p-2} \int_0^\infty \int_x^{x+h} |f(y)|^p\,\mathrm{d}y \\ &= h^{p-2} \left(\int_0^h \int_0^y + \int_h^\infty \int_{y-h}^y \right) |f(y)|^p\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

□

LEMMA 5.3. *Let $1 < p < \infty$, $0 < s < 1$, and $0 \le a \le \infty$. Then, there holds*

$$\int_{(0,a)^2} \frac{|f(x)-f(y)|^p}{|x-y|^{1+sp}}\,\mathrm{d}x\,\mathrm{d}y \lesssim_{s,p} \int_{(0,a)} x^{(1-s)p} |f'(x)|^p\,\mathrm{d}x \qquad \forall f \in W^{1,p}_{\mathrm{loc}}((0,a)). \tag{5.5}$$

Proof. The proof follows the same arguments as those used in the proof of [44, Theorem 1.28], which considers the case $a = \infty$. The full details are given below.

By symmetry, there holds

$$\begin{aligned}\int_{(0,a)^2} \frac{|f(x)-f(y)|^p}{|x-y|^{1+sp}}\,\mathrm{d}x\,\mathrm{d}y &= 2 \int_0^a \int_y^a \frac{|f(x)-f(y)|^p}{(x-y)^{1+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &= 2 \int_0^a \int_y^a \frac{1}{(x-y)^{1+sp}} \left| \int_y^x f'(t)\,\mathrm{d}t \right|^p \mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Performing a change of variable and applying Hardy's inequality [44, Theorem 1.3], we obtain

$$\begin{aligned}\int_y^a \frac{1}{(x-y)^{1+sp}} \left| \int_y^x f'(t)\,\mathrm{d}t \right|^p \mathrm{d}x &\overset{\substack{\tilde{x}=x-y \\ \tau=t-y}}{\le} \int_0^{a-y} \frac{1}{\tilde{x}^{1+sp}} \left(\int_0^{\tilde{x}} |\tau f'(y+\tau)| \frac{\mathrm{d}\tau}{\tau} \right)^p \mathrm{d}\tilde{x} \\ &\le \frac{1}{s^p} \int_0^{a-y} \frac{|f'(y+\tilde{x})|^p}{\tilde{x}^{1+(s-1)p}}\,\mathrm{d}\tilde{x} \\ &\overset{x=\tilde{x}+y}{=} \frac{1}{s^p} \int_y^a \frac{|f'(x)|^p}{(x-y)^{1+(s-1)p}}\,\mathrm{d}x.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{(0,a)^2} \frac{|f(x)-f(y)|^p}{|x-y|^{1+sp}}\,\mathrm{d}x\,\mathrm{d}y &\le \frac{2}{s^p} \int_0^a \int_y^a \frac{|f'(x)|^p}{(x-y)^{1+(s-1)p}}\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{2}{s^p} \int_0^a |f'(x)|^p \int_0^x \frac{1}{(x-y)^{1+(s-1)p}}\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{2}{s^p(1-s)p} \int_0^a x^{(1-s)p} |f'(x)|^p\,\mathrm{d}x,\end{aligned}$$

which completes the proof. □

LEMMA 5.4. *Let $1 < p < \infty$ and $0 < s < 1/p$. For all $f \in L^p(\mathcal{Q}_1; \omega_1^{1-sp}\,\mathrm{d}\boldsymbol{x})$, there holds*

$$\int_{\mathcal{Q}_1}\int_0^2\int_{x_1}^{x_1+z}\int_{x_2}^{x_2+z}\frac{1}{z^{2+sp}}|f(\boldsymbol{y})|^p\,\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}z\,\mathrm{d}\boldsymbol{x} \lesssim_{s,p} \|\omega_1^{\frac{1}{p}-s}f\|_{p,\mathcal{Q}_1}^p. \tag{5.6}$$

Proof. Applying (5.4) and using that $0 < z < 2$ gives

$$\int_0^\infty \fint_{x_2}^{x_2+z}\left(\int_{x_1}^{x_1+z}|f(\boldsymbol{y})|^p\,\mathrm{d}y_1\right)\mathrm{d}y_2\,\mathrm{d}x_2 \lesssim_p \int_0^\infty\int_{x_1}^{x_1+z}|f(y_1,x_2)|^p\,\mathrm{d}y_1\,\mathrm{d}x_2.$$

Moreover, there holds

$$\begin{aligned}\int_0^\infty\int_0^2\int_{x_1}^{x_1+z}&\frac{1}{z^{1+sp}}|f(y_1,x_2)|^p\,\mathrm{d}y_1\,\mathrm{d}z\,\mathrm{d}x_1\\ &= \int_0^\infty\int_{x_1}^{x_1+2}|f(y_1,x_2)|^p\int_{y_1-x_1}^2\frac{1}{z^{1+sp}}\,\mathrm{d}z\,\mathrm{d}y_1\,\mathrm{d}x_1\\ &\lesssim_{s,p}\int_0^\infty\int_{x_1}^{x_1+2}(y_1-x_1)^{-sp}|f(y_1,x_2)|^p\,\mathrm{d}y_1\,\mathrm{d}x_1\\ &= \left(\int_0^2\int_0^{y_1}+\int_2^\infty\int_{y_1-2}^{y_1}\right)(y_1-x_1)^{-sp}|f(y_1,x_2)|^p\,\mathrm{d}x_1\,\mathrm{d}y_1\\ &\lesssim_{s,p}\int_0^\infty\min\{y_1,2\}^{1-sp}|f(y_1,x_2)|^p\,\mathrm{d}y_1.\end{aligned}$$

The result now follows on integrating over $0 < x_2 < \infty$. ∎

LEMMA 5.5. *Let $\chi \in C_c^\infty(\mathbb{R})$ and $b \in C_c^\infty(\mathbb{R}^2)$ be as in Theorem 4.1 and $1 < p < \infty$. Let $k \in \mathbb{N}_0$ and $f \in L^p(\mathcal{Q}_1; \omega_1^{1+kp}\,\mathrm{d}\boldsymbol{x})$. For $0 < t < 2$, there holds*

$$\int_0^t\int_{\mathcal{Q}_1}|g_k(\boldsymbol{x},z)|^p\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}z \lesssim_{b,k,p}\int_{\mathcal{Q}_1}\min\{x_1,t\}^{1+kp}|f(\boldsymbol{x})|^p\,\mathrm{d}\boldsymbol{x}, \tag{5.7}$$

where g_k is defined in (4.6)

Proof. Let $k \in \mathbb{N}_0$, $1 < p < \infty$, and $f \in L^p(\mathcal{Q}_1; \omega_1^{1+kp}\,\mathrm{d}\boldsymbol{x})$ be given. Let $z \in (0,t)$. Then, for $\boldsymbol{x} \in \mathcal{Q}_1$ and $\boldsymbol{y} \in (0,1)^2$, there holds $z \le \min\{x_1 + zy_1, t\}/y_1$, and so

$$\begin{aligned}|g_k(\boldsymbol{x},z)| &\le \int_{(0,1)^2}\min\{x_1+zy_1,t\}^k|y_1^{-k}b(\boldsymbol{y})||f(\boldsymbol{x}+z\boldsymbol{y})|\,\mathrm{d}\boldsymbol{y}\\ &\overset{\boldsymbol{u}=\boldsymbol{x}+z\boldsymbol{y}}{\lesssim_{b,k}}\fint_{x_2}^{x_2+z}\fint_{x_1}^{x_1+z}\min\{u_1,t\}^k|f(\boldsymbol{u})|\,\mathrm{d}u_1\,\mathrm{d}u_2.\end{aligned}$$

Integrating over $x_2 \in (0,\infty)$ and applying (5.4) to the function

$$\tilde{f}(u_2;x_1,z) = \fint_{x_1}^{x_1+z}|\tilde{\omega}_1^k f(\boldsymbol{u})|\,\mathrm{d}u_1, \qquad \text{where } \tilde{\omega}_1(\boldsymbol{u}) := \min\{u_1,t\}^k$$

and using that $0 < z < t < 2$ gives

$$\begin{aligned}\int_0^\infty|g_k(\boldsymbol{x},z)|^p\,\mathrm{d}x_2 &\le \int_0^\infty\left(\fint_{x_2}^{x_2+z}\tilde{f}(u_2;x_1,z)\,\mathrm{d}u_2\right)^p\mathrm{d}x_2\\ &\le 2^{p-1}\int_0^\infty\left(\fint_{x_1}^{x_1+z}|(\tilde{\omega}_1^k f)(u_1,x_2)|\,\mathrm{d}u_1\right)^p\mathrm{d}x_2.\end{aligned}$$

Hardy's inequality [40, Theorem 327] then shows that, for $0 < x_2 < \infty$, there holds

$$\int_0^t \left(\fint_{x_1}^{x_1+z} |(\tilde{\omega}_1^k f)(u_1, x_2)| \, \mathrm{d}u_1 \right)^p \mathrm{d}z \overset{v=x_1+z}{=} \int_{x_1}^{x_1+t} \left(\fint_{x_1}^{v} |(\tilde{\omega}_1^k f)(u_1, x_2)| \, \mathrm{d}u_1 \right)^p \mathrm{d}v$$
$$\lesssim_p \int_{x_1}^{x_1+t} |(\tilde{\omega}_1^k f)(v, x_2)|^p \, \mathrm{d}v,$$

and so

$$\int_0^t \int_0^\infty |g_k(\boldsymbol{x}, z)|^p \, \mathrm{d}x_2 \, \mathrm{d}z \lesssim_{b,k,p} \int_0^\infty \int_{x_1}^{x_1+t} |(\tilde{\omega}_1^k f)(v, x_2)|^p \, \mathrm{d}v \, \mathrm{d}x_2.$$

Integrating over x_1 and changing the order of integration gives

$$\int_0^t \int_{\mathcal{Q}_1} |g_k(\boldsymbol{x}, z)|^p \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}z \lesssim_{b,k,p} \int_{\mathcal{Q}_1} \int_{x_1}^{x_1+t} |(\tilde{\omega}_1^k f)(v, x_2)|^p \, \mathrm{d}v \, \mathrm{d}\boldsymbol{x}$$
$$= \int_0^\infty \left(\int_0^t \int_0^v + \int_t^\infty \int_{v-t}^v \right) |(\tilde{\omega}_1^k f)(v, x_2)|^p \, \mathrm{d}x_1 \, \mathrm{d}v \, \mathrm{d}x_2$$
$$\leq \int_{\mathcal{Q}_1} \tilde{\omega}_1(v, x_2)^{1+kp} |f(v, x_2)|^p \, \mathrm{d}v \, \mathrm{d}x_2,$$

which completes the proof. ∎

5.2. Continuity of $\tilde{\mathcal{E}}_0$. In this section, we prove Theorem 5.1 in the case $k = 0$. We will utilize the following equivalent norm on $W^{s,p}(\mathcal{O}_1)$.

LEMMA 5.6. *For all $p \in (1, \infty)$, $s \in (0, 1)$, and $f \in W^{s,p}(\mathcal{O}_1)$, there holds*

$$\|f\|_{s,p,\mathcal{O}_1}^p \approx_{s,p} \|f\|_{p,\mathcal{O}_1}^p + \sum_{i=1}^3 \int_0^1 \int_{\mathcal{O}_1} \frac{|f(\boldsymbol{x} + t\mathbf{e}_i) - f(\boldsymbol{x})|^p}{t^{1+sp}} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t. \tag{5.8}$$

Proof. Let $f \in W^{s,p}(\mathcal{O}_1)$. Thanks to [44, Theorem 6.38], there holds

$$|f|_{s,p,\mathcal{O}_1}^p \approx_{s,p} \sum_{i=1}^3 \int_0^\infty \int_{\mathcal{O}_1} \frac{|f(\boldsymbol{x} + t\mathbf{e}_i) - f(\boldsymbol{x})|^p}{t^{1+sp}} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t,$$

and (5.8) now follows on noting that

$$\sum_{i=1}^3 \int_1^\infty \int_{\mathcal{O}_1} \frac{|f(\boldsymbol{x} + t\mathbf{e}_i) - f(\boldsymbol{x})|^p}{t^{1+sp}} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \lesssim_{s,p} \|f\|_{p,\mathcal{O}_1}^p.$$ ∎

We now estimate each term in (5.8). The first result deals with terms involving translations in the first two coordinate directions.

LEMMA 5.7. *Let $\chi \in C_c^\infty(\mathbb{R})$ and $b \in C_c^\infty(\mathbb{R}^2)$ be as in Theorem 4.1. For $1 < p < \infty$, $0 < s < 1/p$, and $1 \leq i \leq 2$, there holds*

$$\int_0^1 \int_{\mathcal{O}_1} \frac{|\tilde{\mathcal{E}}_0(f)(\boldsymbol{x} + t\boldsymbol{e}_i, z) - \tilde{\mathcal{E}}_0(f)(\boldsymbol{x}, z)|^p}{t^{1+sp}} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}z \, \mathrm{d}t \lesssim_{\chi,b,s,p} \|\omega_1^{\frac{1}{p}-s} f\|_{p,\mathcal{Q}_1}^p \tag{5.9}$$

for all $f \in C_c^\infty(\mathcal{Q}_1)$.

Proof. Let $1 < p < \infty$, $0 < s < 1/p$, and $f \in C_c^\infty(\mathcal{Q}_1)$ be given. Let $g_0(\cdot,\cdot)$ be as in (4.6) with $k = 0$ and let $\tilde{g}(\boldsymbol{x}, z, t) := g_0(\boldsymbol{x} + t\mathbf{e}_i, z) - g_0(\boldsymbol{x}, z)$.

Step 1. Let $1 \le i \le 2$. We will show that

$$\int_0^1 \int_0^2 \int_{\mathcal{Q}_1} \frac{|\tilde{g}(\boldsymbol{x}, z, t)|^p}{t^{1+sp}} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}z \, \mathrm{d}t \lesssim_{b,s,p} \|\omega_1^{1/p-s} f\|_{p,\mathcal{Q}_1}^p. \tag{5.10}$$

We begin by decomposing the above integral into two terms:

$$\begin{aligned}
&\int_0^1 \int_0^2 \int_{\mathcal{Q}_1} \frac{|\tilde{g}(\boldsymbol{x}, z, t)|^p}{t^{1+sp}} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}z \, \mathrm{d}t \\
&\qquad = \left(\int_0^1 \int_0^t \int_{\mathcal{Q}_1} + \int_0^1 \int_t^2 \int_{\mathcal{Q}_1} \right) \frac{|\tilde{g}(\boldsymbol{x}, z, t)|^p}{t^{1+sp}} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}z \, \mathrm{d}t =: A_i + B_i.
\end{aligned}$$

Part (a): A_i**.** Let $0 < t < 1$. Then, $f(\cdot + t\mathbf{e}_i) - f(\cdot) \in L^p(\mathcal{Q}_1, \omega_1 \, \mathrm{d}\boldsymbol{x})$ and

$$\tilde{g}(\boldsymbol{x}, z, t) = \int_{\mathcal{Q}_1} b(\boldsymbol{y}) \left[f(\boldsymbol{x} + z\boldsymbol{y} + t\mathbf{e}_i) - f(\boldsymbol{x} + z\boldsymbol{y}) \right] \mathrm{d}\boldsymbol{y}.$$

Integrating (5.7) over $0 < t < 1$ then gives

$$A_i \lesssim_{b,p} \int_{\mathcal{Q}_1} \int_0^1 \min\{x_1, t\} \frac{|f(\boldsymbol{x} + t\mathbf{e}_i)|^p + |f(\boldsymbol{x})|^p}{t^{1+sp}} \, \mathrm{d}t \, \mathrm{d}\boldsymbol{x}.$$

For $i = 1$, there holds

$$\begin{aligned}
\int_0^\infty \int_0^1 t^{-sp} |f(\boldsymbol{x} + t\boldsymbol{e}_1)|^p \, \mathrm{d}t \, \mathrm{d}x_1 &\overset{\tilde{x}_1 = x_1 + t}{=} \int_0^1 \int_t^\infty t^{-sp} |f(\tilde{x}_1, x_2)|^p \, \mathrm{d}\tilde{x}_1 \, \mathrm{d}t \\
&= \left(\int_0^1 \int_0^{\tilde{x}_1} + \int_1^\infty \int_0^1 \right) t^{-sp} |f(\tilde{x}_1, x_2)|^p \, \mathrm{d}t \, \mathrm{d}\tilde{x}_1 \\
&\lesssim_{s,p} \int_0^\infty \min\{\tilde{x}_1, 1\}^{1-sp} |f(\tilde{x}_1, x_2)|^p \, \mathrm{d}\tilde{x}_1.
\end{aligned}$$

On the other hand, note that for any $0 \le u < \infty$ and $\tilde{f}(\boldsymbol{x}) = f(\boldsymbol{x} + u\mathbf{e}_2)$, there holds

$$\begin{aligned}
&\int_0^\infty \int_0^1 \min\{x_1, t\} t^{-(1+sp)} |\tilde{f}(\boldsymbol{x})|^p \, \mathrm{d}t \, \mathrm{d}x_1 \\
&\quad = \left(\int_0^1 \int_0^{x_1} + \int_1^\infty \int_0^1 \right) t^{-sp} |\tilde{f}(\boldsymbol{x})|^p \, \mathrm{d}t \, \mathrm{d}x_1 + \int_0^1 \int_{x_1}^1 t^{-(1+sp)} x_1 |\tilde{f}(\boldsymbol{x})|^p \, \mathrm{d}t \, \mathrm{d}x_1 \\
&\quad \lesssim_{s,p} \int_0^\infty \min\{x_1, 1\}^{1-sp} |\tilde{f}(\boldsymbol{x})|^p \, \mathrm{d}x_1 + \int_0^1 (x_1^{-sp} - 1) x_1 |\tilde{f}(\boldsymbol{x})|^p \, \mathrm{d}x_1 \\
&\quad \lesssim \int_0^\infty \min\{x_1, 1\}^{1-sp} |f(x_1, x_2 + u)|^p \, \mathrm{d}x_1.
\end{aligned}$$

The bound $A_i \lesssim_{b,s,p} \|\omega_1^{\frac{1}{p}-s} f\|_{p,\mathcal{Q}_1}^p$ now follows on performing a change of variables and collecting results.

Part (b): B_i**.** Using identity (4.7), we obtain

$$\begin{aligned}
\tilde{g}(\boldsymbol{x}, z, t) &= z^{-2} \int_{\mathbb{R}^2} \left[b\left(\frac{\boldsymbol{y} - \boldsymbol{x} - t\mathbf{e}_i}{z} \right) - b\left(\frac{\boldsymbol{y} - \boldsymbol{x}}{z} \right) \right] f(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} \\
&= -z^{-3} \int_{\mathbb{R}^2} \int_0^t (\partial_i b) \left(\frac{\boldsymbol{y} - \boldsymbol{x} - r\mathbf{e}_i}{z} \right) f(\boldsymbol{y}) \, \mathrm{d}r \, \mathrm{d}\boldsymbol{y}.
\end{aligned}$$

Writing $z^{-2}|\partial_i b| = (z^{-2}|\partial_i b|)^{1-1/p}(z^{-2}|\partial_i b|)^{1/p}$ and applying Hölder's inequality gives

$$\begin{aligned}|\tilde{g}(\boldsymbol{x}, z, t)|^p &\leq \left(\int_0^t \int_{\mathbb{R}^2} \frac{1}{z^2}|\partial_i b|\left(\frac{\boldsymbol{y}-\boldsymbol{x}-r\mathbf{e}_i}{z}\right) \mathrm{d}\boldsymbol{y}\, \mathrm{d}r\right)^{p-1} \\ &\quad \times \frac{1}{z^{p+2}} \int_0^t \int_{\mathbb{R}^2} |\partial_i b|\left(\frac{\boldsymbol{y}-\boldsymbol{x}-r\mathbf{e}_i}{z}\right)|f(\boldsymbol{y})|^p \,\mathrm{d}\boldsymbol{y}\, \mathrm{d}r \\ &\lesssim_{b,p} \frac{t^{p-1}}{z^{p+2}} \int_0^t \int_{\mathbb{R}^2} |\partial_i b|\left(\frac{\boldsymbol{y}-\boldsymbol{x}-r\mathbf{e}_i}{z}\right)|f(\boldsymbol{y})|^p \,\mathrm{d}\boldsymbol{y}\, \mathrm{d}r.\end{aligned}$$

Integrating over $\boldsymbol{x}$ gives

$$\begin{aligned}\int_{\mathcal{Q}_1} |\tilde{g}(\boldsymbol{x}, z, t)|^p \,\mathrm{d}\boldsymbol{x} &\overset{\tilde{\boldsymbol{x}}=\boldsymbol{x}+r\mathbf{e}_i}{\leq} \frac{t^{p-1}}{z^{p+2}} \int_0^t \int_{\mathcal{Q}_1}\int_{\mathbb{R}^2} |\partial_i b|\left(\frac{\boldsymbol{y}-\tilde{\boldsymbol{x}}}{z}\right)|f(\boldsymbol{y})|^p \,\mathrm{d}\boldsymbol{y}\, \mathrm{d}\tilde{\boldsymbol{x}}\, \mathrm{d}r \\ &\leq \frac{t^p}{z^{p+2}} \int_{\mathcal{Q}_1}\int_{\mathbb{R}^2} |\partial_i b|\left(\frac{\boldsymbol{y}-\tilde{\boldsymbol{x}}}{z}\right)|f(\boldsymbol{y})|^p \,\mathrm{d}\boldsymbol{y}\, \mathrm{d}\tilde{\boldsymbol{x}} \\ &\lesssim_b \frac{t^p}{z^{p+2}} \int_{\mathcal{Q}_1}\int_{x_1}^{x_1+z}\int_{x_2}^{x_2+z} |f(\boldsymbol{y})|^p \,\mathrm{d}y_2\, \mathrm{d}y_1\, \mathrm{d}\boldsymbol{x}.\end{aligned}$$

Integrating over z and t, we obtain

$$\begin{aligned}B_i &\lesssim_{b,p} \int_0^1\int_t^2 \frac{t^{(1-s)p-1}}{z^{p+2}} \int_{\mathcal{Q}_1}\int_{x_1}^{x_1+z}\int_{x_2}^{x_2+z} |f(\boldsymbol{y})|^p \,\mathrm{d}y_2\, \mathrm{d}y_1\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}z\, \mathrm{d}t \\ &= \int_0^2\int_0^z \frac{t^{(1-s)p-1}}{z^{p+2}} \int_{\mathcal{Q}_1}\int_{x_1}^{x_1+z}\int_{x_2}^{x_2+z} |f(\boldsymbol{y})|^p \,\mathrm{d}y_2\, \mathrm{d}y_1\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t\, \mathrm{d}z \\ &= \frac{1}{p(1-s)} \int_{\mathcal{Q}_1}\int_0^2\int_{x_1}^{x_1+z}\int_{x_2}^{x_2+z} \frac{1}{z^{2+sp}}|f(\boldsymbol{y})|^p \,\mathrm{d}y_2\, \mathrm{d}y_1\, \mathrm{d}z\, \mathrm{d}\boldsymbol{x}.\end{aligned}$$

Applying (5.6), we obtain $B_i \lesssim_{b,s,p} \|\omega_1^{\frac{1}{p}-s} f\|_{p,\mathcal{Q}_1}^p$, which completes the proof of (5.10).
Step 2. Since $\operatorname{supp}\chi \subset B(0,2)$, there holds

$$\begin{aligned}&\int_0^1\int_{\mathcal{O}_1} \frac{|\tilde{\mathcal{E}}_0(f)(\boldsymbol{x}+t\mathbf{e}_i, z) - \tilde{\mathcal{E}}_0(f)(\boldsymbol{x}, z)|^p}{t^{1+sp}} \,\mathrm{d}\boldsymbol{x}\, \mathrm{d}z\, \mathrm{d}t \\ &\quad = \int_0^1\int_0^2\int_{\mathcal{Q}_1} |\chi(z)|^p \frac{|\tilde{g}(\boldsymbol{x}, z, t)|^p}{t^{1+sp}} \,\mathrm{d}\boldsymbol{x}\, \mathrm{d}z\, \mathrm{d}t \lesssim_{\chi,b,s,p} \int_{\mathcal{Q}_1} \min\{x_1, 1\}^{1-sp} f(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x},\end{aligned}$$

which completes the proof. ∎

The next result deals with the term involving a translation in the z-direction.

Lemma 5.8. *Let $\chi \in C_c^\infty(\mathbb{R})$ and $b \in C_c^\infty(\mathbb{R}^2)$ be as in Theorem 4.1. For $1 < p < \infty$ and $0 < s < 1/p$, there holds*

$$\int_0^1\int_{\mathcal{O}_1} \frac{|\tilde{\mathcal{E}}_0(f)(\boldsymbol{x}, z+t) - \tilde{\mathcal{E}}_0(f)(\boldsymbol{x}, z)|^p}{t^{1+sp}} \,\mathrm{d}\boldsymbol{x}\, \mathrm{d}z\, \mathrm{d}t \lesssim_{\chi,b,s,p} \|\omega_1^{\frac{1}{p}-s} f\|_{p,\mathcal{Q}_1}^p \tag{5.11}$$

for all $f \in C_c^\infty(\mathcal{Q}_1)$.

Proof. Let $1 < p < \infty$, $0 < s < 1/p$, and $f \in C_c^\infty(\mathcal{Q}_1)$ be given. Let $g_0(\cdot,\cdot)$ be defined as in (4.6).

Step 1. We will first show that

$$\int_0^1 \int_0^2 \int_{\mathcal{Q}} \frac{|g_0(\boldsymbol{x}, z+t) - g_0(\boldsymbol{x}, z)|^p}{t^{1+sp}} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}z \, \mathrm{d}t \lesssim_{b,s,p} \|\omega_1^{\frac{1}{p}-s} f\|_{p,\mathcal{Q}_1}^p. \tag{5.12}$$

Applying (5.5) gives

$$\int_0^1 \int_0^2 \frac{|g_0(\boldsymbol{x}, z+t) - g_0(\boldsymbol{x}, z)|^p}{t^{1+sp}} \, \mathrm{d}z \, \mathrm{d}t \leq \int_0^2 z^{(1-s)p} |\partial_z g_0(\boldsymbol{x}, z)|^p \, \mathrm{d}z. \tag{5.13}$$

Applying identity (4.7), we obtain

$$\begin{aligned} \partial_z g_0(\boldsymbol{x}, z) &= \int_{\mathbb{R}^2} \partial_z \left\{ z^{-2} b \left(\frac{\boldsymbol{y} - \boldsymbol{x}}{z} \right) \right\} f(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} \\ &= -\int_{\mathbb{R}^2} \left\{ 2z^{-3} b \left(\frac{\boldsymbol{y} - \boldsymbol{x}}{z} \right) + z^{-4} Db \left(\frac{\boldsymbol{y} - \boldsymbol{x}}{z} \right) \cdot (\boldsymbol{y} - \boldsymbol{x}) \right\} f(\boldsymbol{y}) \, \mathrm{d}y_1 \, \mathrm{d}y_2. \end{aligned}$$

For $\boldsymbol{y} \in (x_1, x_1 + z) \times (x_2, x_2 + z)$, there holds

$$\left| 2z^{-3} b \left(\frac{\boldsymbol{y} - \boldsymbol{x}}{z} \right) + z^{-4} Db \left(\frac{\boldsymbol{y} - \boldsymbol{x}}{z} \right) \cdot (\boldsymbol{y} - \boldsymbol{x}) \right| \lesssim_b z^{-3},$$

where we used that b and Db are uniformly bounded. Since $\operatorname{supp} b \subset (0,1)^2$, we obtain

$$\int_0^2 z^{(1-s)p} |\partial_z g_0(\boldsymbol{x}, z)| \, \mathrm{d}z \lesssim_b \int_0^2 \frac{1}{z^{2+sp}} \int_{x_2}^{x_2+z} \int_{x_1}^{x_1+z} |f(\boldsymbol{y})| \, \mathrm{d}y_1 \, \mathrm{d}y_2 \, \mathrm{d}z.$$

Inequality (5.12) now follows on integrating (5.13) over $\boldsymbol{x} \in \mathcal{Q}_1$ and applying (5.6).

Step 2. For $0 < t < 2$ and $\boldsymbol{x} \in \mathcal{Q}_1$, we add and subtract $\chi(z+t) g_0(\boldsymbol{x}, t)$ to obtain

$$\begin{aligned} |\tilde{\mathcal{E}}_0(f)(\boldsymbol{x}, z+t) - \tilde{\mathcal{E}}_0(f)(\boldsymbol{x}, z)|^p \lesssim_p |\chi(z+t)|^p |g_0(\boldsymbol{x}, z+t) - g_0(\boldsymbol{x}, t)|^p& \\ + |\chi(z+t) - \chi(z)|^p |g_0(\boldsymbol{x}, z)|^p&. \end{aligned}$$

For the first term, we use that $\operatorname{supp} \chi \in (-2, 2)$ and apply (5.12) to obtain

$$\begin{aligned} &\int_0^1 \int_{\mathcal{O}_1} |\chi(z+t)|^p \frac{|g_0(\boldsymbol{x}, z+t) - g_0(\boldsymbol{x}, t)|^p}{t^{1+sp}} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}z \, \mathrm{d}t \\ &\qquad \lesssim_{\chi,p} \int_0^1 \int_0^2 \int_{\mathcal{Q}_1} \frac{|g_0(\boldsymbol{x}, z+t) - g_0(\boldsymbol{x}, t)|^p}{t^{1+sp}} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}z \, \mathrm{d}t \lesssim_{b,s,p} \|\omega_1^{\frac{1}{p}-s} f\|_{p,\mathcal{Q}_1}^p. \end{aligned}$$

For the second term, we again use that $\operatorname{supp} \chi \in (-2, 2)$ as well as the assumption $0 < s < 1/p$:

$$\begin{aligned} &\int_0^1 \int_{\mathcal{O}_1} \frac{|\chi(z+t) - \chi(z)|^p}{t^{1+sp}} |g_0(\boldsymbol{x}, z)|^p \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}z \, \mathrm{d}t \\ &\quad = \int_0^1 \int_0^2 \int_{\mathcal{Q}_1} \frac{1}{t^{1+sp}} \left| \int_z^{z+t} \chi'(r) \, \mathrm{d}r \right|^p |g_0(\boldsymbol{x}, z)|^p \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}z \, \mathrm{d}t \\ &\quad \lesssim_{\chi,p} \int_0^1 \int_0^2 \int_{\mathcal{Q}_1} t^{(1-s)p-1} |g_0(\boldsymbol{x}, z)|^p \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}z \, \mathrm{d}t \\ &\quad \lesssim_{s,p} \int_0^2 \int_{\mathcal{Q}_1} |g_0(\boldsymbol{x}, z)|^p \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}z. \end{aligned}$$

Inequality (5.11) now follows from (5.7). ∎

We now obtain Theorem 5.1 in the case $k = 0$.

LEMMA 5.9. *Let $\chi \in C_c^\infty(\mathbb{R})$ and $b \in C_c^\infty(\mathbb{R}^2)$ be as in Theorem 4.1. For $1 < p < \infty$ and $0 \le s < 1/p$, there holds*

$$\|\tilde{\mathcal{E}}_0(f)\|_{s,p,\mathcal{O}_1} \lesssim_{\chi,b,k,s,p} \|\omega_1^{\frac{1}{p}-s} f\|_{p,\mathcal{Q}_1} \qquad \forall f \in L^p(\mathcal{Q}_1, \omega_1^{1-sp}\,\mathrm{d}\boldsymbol{x}). \tag{5.14}$$

Proof. The case $s = 0$ follows on taking $t = 2$ in (5.7) and using the fact that $\|\tilde{\mathcal{E}}_0(f)\|_{p,\mathcal{O}_1} \lesssim_{\chi,p} \|g\|_{p,\mathcal{Q}_1\times(0,2)}$, where g is defined in (4.6). The case $0 < s < 1/p$ follows from the norm equivalence (5.8), the bounds (5.9) and (5.11), and the density of $C_c^\infty(\mathcal{O}_1)$ in $L^p(\mathcal{O}_1, \omega_1^{1-sp}\,\mathrm{d}\boldsymbol{x})$. ∎

5.3. Proof of Theorem 5.1. Let $k \in \mathbb{N}_0$, $1 < p < \infty$.
Step 1: $s = 0$. Taking $t = 2$ in (5.7) and using the fact that $\|\tilde{\mathcal{E}}_k(f)\|_{p,\mathcal{O}_1} \lesssim_{\chi,p} \|g_k\|_{p,\mathcal{Q}_1\times(0,2)}$, where g_k is defined in (4.6), we obtain (5.3) in the case $s = 0$.
Step 2: $s \in \{1, 2, \ldots, k\}$. Let $f \in C_c^\infty(\mathcal{Q}_1)$. Applying (4.11) with $|\alpha| \le k$, we obtain

$$\|D^\alpha \tilde{\mathcal{E}}_k(f)\|_{\sigma,p,\mathcal{O}_1} \le \sum_{0\le i\le \alpha_3} \|\tilde{\mathcal{E}}_{k+i-|\alpha|}[\chi_i, b_{ki}](f)\|_{\sigma,p,\mathcal{O}_1}, \tag{5.15}$$

where $\chi_i \in C_c^\infty(\mathbb{R})$ and $b_{ki} \in C_c^\infty(\mathbb{R}^2)$ are suitable functions depending on χ and b respectively and $0 \le \sigma < 1$. Applying (5.3) with $s = 0$ then gives

$$|\tilde{\mathcal{E}}_k(f)|_{s,p,\mathcal{O}_1} \lesssim_{\chi,b,k,s,p} \|\omega_1^{\frac{1}{p}+k-s} f\|_{p,\mathcal{Q}_1},$$

where we used that $k + i - |\alpha| \ge k - m$ for $0 \le i \le \alpha_3$. By density, (5.3) holds for $s \in \{0, 1, \ldots, k\}$.
Step 3: $0 \le s \le k$. This case follows from interpolating Step 2 (see e.g. [23, Theorem 14.2.3] and [17, Theorem 5.4.1]).
Step 4: $k < s < k + 1/p$. Let $\sigma = s - k$ so that $0 < \sigma < 1/p$. Setting $\tilde{\chi}_{ik\alpha}(z) := z^{k+i-|\alpha|}\chi_i \in C^{\infty_c}(\mathcal{Q}_1)$ so that $\operatorname{supp}\tilde{\chi}_{ik\alpha} \in (-2, 2)$ and applying (5.14) and (5.15) then gives

$$\|D^\alpha \tilde{\mathcal{E}}_k(f)\|_{\sigma,p,\mathcal{O}_1} \le \sum_{0\le i\le \alpha_3} \|\tilde{\mathcal{E}}_0[\tilde{\chi}_{ik\alpha}, b_{ki}](f)\|_{\sigma,p,\mathcal{O}_1} \lesssim_{\chi,b,k,\sigma,p} \|\omega_1^{\frac{1}{p}-\sigma} f\|_{p,\mathcal{Q}_1}.$$

Inequality (5.3) now follows. ∎

6. Continuity of fundamental operators. In this section, we prove the continuity and interpolation properties of the four fundamental operators $\mathcal{E}_k^{[1]}$ defined in (3.1), $\mathcal{M}_{k,r}^{[1]}$ defined in (3.8), $\mathcal{S}_{k,r}^{[1]}$ defined in (3.25), and $\mathcal{R}_{k,r}^{[1]}$ defined in (3.37). We begin with the properties of $\mathcal{E}_k^{[1]}$, which rely on the results of section 4. Then, in subsection 6.2, we show that the four fundamental operators are continuous from weighted L^p spaces (5.1) to $W^{s,p}(K)$ for small s, which will be useful for the analysis of $\mathcal{M}_{k,r}^{[1]}$, $\mathcal{S}_{k,r}^{[1]}$, and $\mathcal{R}_{k,r}^{[1]}$. This section concludes with the proofs of Lemmas 3.3, 3.7, and 3.11.

6.1. Proof of Lemma 3.1. Step 1: Continuity (3.4). Let $\tilde{b}$ denote the extension by zero of b to $\mathbb{R}^2$ and let $\chi \in C_c^\infty(\mathbb{R})$ with $\chi \equiv 1$ on $(-1, 1)$ and $\operatorname{supp}\chi \in (-2, 2)$. Let $f \in W^{s-k-\frac{1}{p},p}(T)$ be given and let $\tilde{f}$ denote a bounded extension f to

$\mathbb{R}^2$ satisfying $\|\tilde{f}\|_{s,p,\mathbb{R}^2} \lesssim_{s,p} \|f\|_{s,p,T}$; see e.g. [34]. Thanks to the identity

$$\mathcal{E}_k(f) = \frac{(-1)^k}{k!}\tilde{\mathcal{E}}_k(f)[\chi, \tilde{b}](f) \qquad \text{on } K, \tag{6.1}$$

where $\tilde{E}_k$ is defined in (4.1), inequality (3.4) immediately follows from (4.3) and the smoothness of the mapping $\mathfrak{I}_1$ defined in (3.2).

Step 2: Trace property (3.3). Direct computation shows that (3.3) holds.

Step 3: Polynomial preservation. If $f \in \mathcal{P}_N(\Gamma_1)$, $N \in \mathbb{N}_0$, then direct inspection reveals that $\mathcal{E}_k^{[1]}(f) \in \mathcal{P}_{N+k}(K)$. □

6.2. Weighted continuity. We begin with the continuity of $\mathcal{E}_k^{[1]}$.

LEMMA 6.1. *Let $b \in C_c^\infty(T)$, $k \in \mathbb{N}_0$, $1 < p < \infty$, and $0 \le s < k + 1/p$ or $(s,p) = (k+\frac{1}{2}, 2)$. Then, for all $t_1, t_2, t_3 \in [0,\infty)$ such that $t_1 + t_2 + t_3 = k - s + 1/p$, there holds*

$$\|\mathcal{E}_k^{[1]}(f)\|_{s,p,K} \lesssim_{b,k,s,p} \|\omega_1^{t_1}\omega_2^{t_2}\omega_3^{t_3} f\|_{p,T} \qquad \forall f \in L^p(T; (\omega_1^{t_1}\omega_2^{t_2}\omega_3^{t_3})^p \,\mathrm{d}\boldsymbol{x}), \tag{6.2}$$

where ω_i are defined in (3.7).

Proof. Let $t = k - s + 1/p$.

Step 1: $t_2 = t_3 = 0$. Let $\tilde{b}$ denote the extension by zero of b to $\mathbb{R}^2$ and let $\chi \in C_c^\infty(\mathbb{R})$ with $\chi \equiv 1$ on $(-1,1)$ and $\operatorname{supp}\chi \in (-2,2)$. Let $f \in C_c^\infty(T)$ be given and let $\tilde{f}$ denote the extension by zero of f to $\mathbb{R}^2$. Thanks to the identity (6.1), (6.2) with $t_2 = t_3 = 0$ follows from (5.3), where we recall that ω_1 is extended to $\mathbb{R}^2$ by (5.2), and a standard density argument. The case $(s,p) = (k+\frac{1}{2}, 2)$ follows from a similar argument using (4.3).

Step 2: $t_1 = t_3 = 0$. We define transformations $\mathfrak{F}_1 : T \to T$ and $\mathfrak{G}_1 : K \to K$ as follows:

$$\mathfrak{F}_1(\boldsymbol{x}) := (x_2, x_1) \quad \text{and} \quad \mathfrak{G}_1(\boldsymbol{x}, z) := (x_2, x_1, z) \qquad (\boldsymbol{x}, z) \in K. \tag{6.3}$$

Then, a change of variable shows that $\mathcal{E}_k^{[1]}(f) \circ \mathfrak{G}_1 = \mathcal{E}_k^{[1]}[b \circ \mathfrak{F}_1](f \circ \mathfrak{F}_1)$, and so

$$\|\mathcal{E}_k^{[1]}(f)\|_{s,p,K} = \|\mathcal{E}_k^{[1]}(f) \circ \mathfrak{G}_1\|_{s,p,K} \lesssim_{b,k,s,p} \|\omega_1^t (f \circ \mathfrak{F}_1)\|_{p,T} = \|\omega_2^t f\|_{p,T},$$

where we applied Step 1 in the middle inequality.

Step 3: $t_3 = 0$. Applying Steps 1 and 2 and interpolating between $L^p(T; \omega_1^{tp} d\boldsymbol{x})$ and $L^p(T; \omega_2^{tp} d\boldsymbol{x})$ (see e.g. [17, Theorem 5.4.1]) then gives (6.2).

Step 4: $t_1 = t_2 = 0$. We define transformations $\mathfrak{F}_2 : T \to T$ and $\mathfrak{G}_2 : K \to K$ as follows:

$$\mathfrak{F}_2(\boldsymbol{x}) := (x_2, 1 - x_1 - x_2) \quad \text{and} \quad \mathfrak{G}_2(\boldsymbol{x}, z) := (1 - x_1 - x_2 - z, x_1, z) \qquad (\boldsymbol{x}, z) \in K. \tag{6.4}$$

A change of variables then gives $\mathcal{E}_k^{[1]}(f) \circ \mathfrak{G}_2 = \mathcal{E}_k^{[1]}[b \circ \mathfrak{F}_2](f \circ \mathfrak{F}_2)$, and so

$$\|\mathcal{E}_k^{[1]}(f)\|_{t,p,K} \lesssim \|\mathcal{E}_k^{[1]}(f) \circ \mathfrak{G}_2\|_{t,p,K} \lesssim_{b,k,t,p} \|\omega_1^t (f \circ \mathfrak{F}_2)\|_{p,T} = \|\omega_3^t f\|_{p,T},$$

where we applied Step 1 in the middle inequality.

Step 5: General case. Applying Steps 3 and 4 and interpolating between $L^p(T; (\omega_1^{r_1}\omega_2^{r_2})^p d\boldsymbol{x})$ with $r_1, r_2 \in \mathbb{R}_+$ with $r_1 + r_2 = t$ and $L^p(T; \omega_3^{tp} d\boldsymbol{x})$ (see e.g. [17, Theorem 5.4.1]) gives (6.2). □

We now turn to the continuity of $\mathcal{M}^{[1]}_{k,r}$.

LEMMA 6.2. *Let $b \in C_c^\infty(T)$, $k, r \in \mathbb{N}_0$, $1 < p < \infty$, and $0 \le s < k + 1/p$ or $(s, p) = (k + \frac{1}{2}, 2)$. Then, for all $t_1, t_2, t_3 \in [0, \infty)$ such that $t_1 + t_2 + t_3 = k - s + 1/p$, there holds*

$$\|\mathcal{M}^{[1]}_{k,r}(f)\|_{s,p,K} \lesssim_{b,k,r,s,p} \|\omega_1^{t_1}\omega_2^{t_2}\omega_3^{t_3} f\|_{p,T} \qquad \forall f \in L^p(T; (\omega_1^{t_1}\omega_2^{t_2}\omega_3^{t_3})^p \, \mathrm{d}\boldsymbol{x}), \tag{6.5}$$

where ω_i are defined in (3.7).

Proof. Let $0 \le s < k + 1/p$ or $(s, p) = (k + \frac{1}{2}, 2)$. We proceed by induction on r. The case $r = 0$ follows from (6.2), so assume that (6.5) holds for some $r \in \mathbb{N}_0$. Direction computation gives

$$\begin{aligned}\mathcal{M}^{[1]}_{k,r+1}(f)(\boldsymbol{x}, z) - \mathcal{M}^{[1]}_{k,r}(f)(\boldsymbol{x}, z) &= x_2^r \frac{(-z)^k}{k!} \int_T b(\boldsymbol{y}) \frac{f(\boldsymbol{x} + z\boldsymbol{y})}{(x_2 + zy_2)^r} \left(\frac{x_2}{x_2 + zy_2} - 1 \right) \mathrm{d}\boldsymbol{y} \\ &= x_2^r \frac{(-z)^{k+1}}{k!} \int_T y_2 b(\boldsymbol{y}) \frac{f(\boldsymbol{x} + z\boldsymbol{y})}{(x_2 + zy_2)^{r+1}} \, \mathrm{d}\boldsymbol{y} \\ &= (k+1)\mathcal{M}^{[1]}_{k+1,r}[\omega_2 b](\omega_2^{-1} f)(\boldsymbol{x}, z),\end{aligned}$$

which leads to the following identity

$$\mathcal{M}^{[1]}_{k,r+1}(f) = (k+1)\mathcal{M}^{[1]}_{k+1,r}[\omega_2 b](\omega_2^{-1} f) + \mathcal{M}^{[1]}_{k,r}(f). \tag{6.6}$$

Consequently, there holds

$$\|\mathcal{M}^{[1]}_{k,r+1}(f)\|_{s,p,K} \le (k+1)\|\mathcal{M}^{[1]}_{k+1,r}[\omega_2 b](\omega_2^{-1} f)\|_{s,p,K} + \|\mathcal{M}^{[1]}_{k,r}(f)\|_{s,p,K}.$$

Applying (6.5) with $\tau_1 = t_1$, $\tau_2 = t_2 + 1$ and $\tau_3 = t_3$ gives

$$\|\mathcal{M}^{[1]}_{k+1,r}[\omega_2 b](\omega_2^{-1} f)\|_{s,p,K} \lesssim_{b,k,r,s,p} \|\omega_1^{\tau_1}\omega_2^{\tau_2 - 1}\omega_3^{\tau_3} f\|_{p,T} = \|\omega_1^{t_1}\omega_2^{t_2}\omega_3^{t_3} f\|_{p,T},$$

and so $\|\mathcal{M}^{[1]}_{k,r+1}(f)\|_{s,p,K} \lesssim_{b,k,r,s,p} \|\omega_1^{t_1}\omega_2^{t_2}\omega_3^{t_3} f\|_{p,T}$, which completes the proof. ∎

It will be convenient to define a three-parameter version of $\mathcal{S}^{[1]}_{k,r}$ as follows:

$$\mathcal{S}^{[1]}_{k,r,q}(f)(\boldsymbol{x}, z) := x_1^q x_2^r \mathcal{E}^{[1]}_k(\omega_1^{-q}\omega_2^{-r} f)(\boldsymbol{x}, z) \tag{6.7}$$

for $k, r, q \in \mathbb{N}_0$. This three-parameter version satisfies the same continuity properties as $\mathcal{E}^{[1]}_k$ and $\mathcal{M}^{[1]}_{k,r}$.

LEMMA 6.3. *Let $b \in C_c^\infty(T)$, $k, r, q \in \mathbb{N}_0$, $1 < p < \infty$, and $0 \le s < k + 1/p$ or $(s, p) = (k + \frac{1}{2}, 2)$. Then, for all $t_1, t_2, t_3 \in [0, \infty)$ such that $t_1 + t_2 + t_3 = k - s + 1/p$, there holds*

$$\|\mathcal{S}^{[1]}_{k,r,q}(f)\|_{s,p,K} \lesssim_{b,k,r,q,s,p} \|\omega_1^{t_1}\omega_2^{t_2}\omega_3^{t_3} f\|_{p,T} \qquad \forall f \in L^p(T; (\omega_1^{t_1}\omega_2^{t_2}\omega_3^{t_3})^p \, \mathrm{d}\boldsymbol{x}), \tag{6.8}$$

where ω_i are defined in (3.7).

Proof. Let $0 \le s < k + 1/p$ or $(s, p) = (k + \frac{1}{2}, 2)$. We proceed by induction on q. The case $q = 0$ follows from (6.5), so assume that (6.8) holds for some $q \in \mathbb{N}_0$. Direct computation gives

$$\begin{aligned}
&\mathcal{S}^{[1]}_{k,r,q+1}(f)(\boldsymbol{x},z) - \mathcal{S}^{[1]}_{k,r,q}(f)(\boldsymbol{x},z) \\
&\quad = x_1^q x_2^r \frac{(-z)^k}{k!} \int_T b(\boldsymbol{y}) \frac{f(\boldsymbol{x}+z\boldsymbol{y})}{(x_1+zy_1)^q (x_2+zy_2)^r} \left(\frac{x_1}{x_1+zy_1} - 1 \right) \mathrm{d}\boldsymbol{y} \\
&\quad = x_1^q x_2^r \frac{(-z)^{k+1}}{k!} \int_T y_1 b(\boldsymbol{y}) \frac{f(\boldsymbol{x}+z\boldsymbol{y})}{(x_1+zy_1)^{q+1} (x_2+zy_2)^r} \mathrm{d}\boldsymbol{y} \\
&\quad = (k+1) \mathcal{S}^{[1]}_{k+1,r,q}[\omega_1 b](\omega_1^{-1} f)(\boldsymbol{x},z),
\end{aligned}$$

which leads to the following identity

$$\mathcal{S}^{[1]}_{k,r,q+1}(f) = (k+1)\mathcal{S}^{[1]}_{k+1,r,q}[\omega_1 b](\omega_1^{-1} f) + \mathcal{S}^{[1]}_{k,r,q}(f). \tag{6.9}$$

Consequently, there holds

$$\|\mathcal{S}^{[1]}_{k,r,q+1}(f)\|_{s,p,K} \leq (k+1)\|\mathcal{S}^{[1]}_{k+1,r,q}[\omega_1 b](\omega_1^{-1} f)\|_{s,p,K} + \|\mathcal{M}^{[1]}_{k,r,q}(f)\|_{s,p,K}.$$

Applying (6.8) with $\tau_1 = t_1 + 1$, $\tau_2 = t_2$ and $\tau_3 = t_3$ gives

$$\|\mathcal{S}^{[1]}_{k+1,r,q}[\omega_1 b](\omega_1^{-1} f)\|_{s,p,K} \lesssim_{b,k,r,q,s,p} \|\omega_1^{\tau_1 - 1}\omega_2^{\tau_2}\omega_3^{\tau_3} f\|_{p,T} = \|\omega_1^{t_1}\omega_2^{t_2}\omega_3^{t_3} f\|_{p,T},$$

and so $\|\mathcal{S}^{[1]}_{k,r,q+1}(f)\|_{s,p,K} \lesssim_{b,k,r,q,s,p} \|\omega_1^{t_1}\omega_2^{t_2}\omega_3^{t_3} f\|_{p,T}$. ∎

6.3. Proof of Lemma 3.3. Step 1: Continuity (3.12). We first show that (3.12) holds with Γ_1 replaced by T and γ_{12} replaced by γ_2, where we recall that the edges of T are labeled as in Figure 1b: For all $k, r \in \mathbb{N}_0$, $(s,p) \in \mathcal{A}_k \cup \{(k+\frac{1}{2}, 2)\}$, and $f \in W^{s-k-\frac{1}{p},p}(T) \cap W^{\min\{s-k-\frac{1}{p}, r\},p}_{\gamma_2}(T)$, there holds

$$\|\mathcal{M}^{[1]}_{k,r}(f)\|_{s,p,K} \lesssim_{b,k,r,s,p} \begin{cases} \|f\|_{2,T} & \text{if } (s,p) = (k+\frac{1}{2}, 2), \\ {}_{\gamma_2}\|f\|_{s-k-\frac{1}{p},p,T} & \text{if } k+\frac{1}{p} < s \leq k+r+\frac{1}{p}, \\ \|f\|_{s-k-\frac{1}{p},p,T} & \text{if } s > k+r+\frac{1}{p}. \end{cases} \tag{6.10}$$

We proceed by induction on r. The case $r = 0$ follows from (3.4) and (6.2), so assume that (6.10) holds for some $r \in \mathbb{N}_0$ and all $k \in \mathbb{N}_0$ and $(s,p) \in \mathcal{A}_k$. Let $k \in \mathbb{N}_0$, $(s,p) \in \mathcal{A}_k \cup \{(k+\frac{1}{2}, 2)\}$, and $f \in W^{s-k-\frac{1}{p},p}(T) \cap W^{\min\{s-k-\frac{1}{p}, r+1\},p}_{\gamma_2}(T)$ be given. Thanks to (6.6), there holds

$$\|\mathcal{M}^{[1]}_{k,r+1}(f)\|_{s,p,K} \leq (k+1)\|\mathcal{M}^{[1]}_{k+1,r}[\omega_2 b](\omega_2^{-1} f)\|_{s,p,K} + \|\mathcal{M}^{[1]}_{k,r}(f)\|_{s,p,K}.$$

Part (a): $k + 1/p \leq s \leq k + 1 + 1/p$. Thanks to Theorem A.3, there holds $\omega_2^{-1} f \in L^p(T; \omega_2^{(k-s+1)p+1}\,\mathrm{d}\boldsymbol{x})$ and (A.7) and (A.9) give

$$\|\omega_2^{k-s+1+\frac{1}{p}}\omega_2^{-1} f\|_{p,T} = \|\omega_2^{k-s+\frac{1}{p}} f\|_{p,T} \lesssim_{k,s,p} {}_{\gamma_2}\|f\|_{s-k-\frac{1}{p},T}.$$

Consequently, we apply (6.5) to obtain

$$\|\mathcal{M}^{[1]}_{k+1,r}[\omega_2 b](\omega_2^{-1} f)\|_{s,p,K} \lesssim_{b,k,r,s,p} \|\omega_2^{k-s+\frac{1}{p}} f\|_{p,T} \lesssim_{k,s,p} {}_{\gamma_2}\|f\|_{s-k-\frac{1}{p},p,T}.$$

Part (b): $k+1+1/p < s \le k+r+1+1/p$**.** Theorem A.3 shows that $\omega_2^{-1} f \in W^{s-k-1-\frac{1}{p},p}_{\gamma_2}(T)$ and (6.10) and (A.8) then give

$$\|\mathcal{M}^{[1]}_{k+1,r}[\omega_2 b](\omega_2^{-1} f)\|_{s,p,K} \lesssim_{b,k,r,s,p} {}_{\gamma_2}\|\omega_2^{-1} f\|_{s-k-1-\frac{1}{p},p,T} \lesssim_{k,s,p} {}_{\gamma_2}\|f\|_{s-k-\frac{1}{p},p,T}\,.$$

Part (c): $s > k+r+1+1/p$**.** Thanks to Theorem A.3, there holds $\omega_2^{-1} f \in W^{s-k-1-\frac{1}{p},p}(T) \cap W^{r}_{\gamma_2}(T)$, and so we apply (6.10) and (A.7) to obtain

$$\|\mathcal{M}^{[1]}_{k+1,r}[\omega_2 b](\omega_2^{-1} f)\|_{s,p,K} \lesssim_{b,k,r,s,p} \|\omega_2^{-1} f\|_{s-k-1-\frac{1}{p},p,T} \lesssim_{k,s,p} \|f\|_{s-k-\frac{1}{p},p,T}.$$

Inequality (6.10) for $r+1$ now follows from the triangle inequality. The smoothness of the mapping $\mathfrak{I}_1$ defined in (3.2) then gives (3.12).
Step 2: Trace properties (3.11a) and (3.11b). Direct computation shows that (3.11a) and (3.11b) hold.
Step 3: Polynomial preservation. Suppose that $f \in \mathcal{P}_N(\Gamma_1)$, $N \in \mathbb{N}_0$, satisfies $D^l_\Gamma f|_{\gamma_{12}} = 0$ for $0 \le l \le r-1$. Then, $f \circ \mathfrak{I}_1 = \omega_2^r g$ for some $g \in \mathcal{P}_{N-r}(T)$, and so $\mathcal{M}^{[1]}_{k,r}(f) = x_2^r \mathcal{E}^{[1]}_k(g) \in \mathcal{P}_{N+k}(K)$ thanks to Lemma 3.1. ☐

6.4. Proof of Lemma 3.7. Step 1: Continuity (3.29). We first show that the following analogue of (3.29) holds: Let $b \in C^\infty_c(T)$, $k, r \in \mathbb{N}_0$, $(s,p) \in \mathcal{A}_k \cup \{(k+1/2, 2)\}$, and $\mathfrak{E} = \{\gamma_1, \gamma_2\}$. For all $f \in W^{s-k-\frac{1}{p},p}_{\mathfrak{E},r}(T)$, there holds

$$\|\mathcal{S}^{[1]}_{k,r}(f)\|_{s,p,K} \lesssim_{b,k,r,s,p} \begin{cases} \|f\|_{2,T} & \text{if } (s,p) = (k+\frac{1}{2}, 2), \\ {}_{\mathfrak{E},r}\|f\|_{s-k-\frac{1}{p},p,T} & \text{otherwise.} \end{cases} \tag{6.11}$$

We proceed by induction on r. The case $r = 0$ follows from (3.4) and (6.2). Now let $r \in \mathbb{N}_0$ be given, and assume that (6.11) holds for all $k \in \mathbb{N}_0$ and $(s,p) \in \mathcal{A}_k \cup \{(k+1/2, 2)\}$.

Let $k \in \mathbb{N}_0$, $(s,p) \in \mathcal{A}_k \cup \{(k+1/2, 2)\}$, and $f \in W^{s-k-\frac{1}{p},p}_{\mathfrak{E},r+1}(T)$ be given. Then, applying (6.6) and (6.9) gives

$$\begin{aligned}
\mathcal{S}^{[1]}_{k,r+1}(f) &= (k+1)\mathcal{S}^{[1]}_{k+1,r+1,r}[\omega_1 b](\omega_1^{-1} f) + \mathcal{S}^{[1]}_{k,r+1,r}(f) \\
&= x^r \left((k+1)\mathcal{M}^{[1]}_{k+1,r+1}[\omega_1 b](\omega_1^{-(r+1)} f) + \mathcal{M}^{[1]}_{k,r+1}(\omega_1^{-r} f) \right) \\
&= x^r \big[(k+1)(k+2)\mathcal{M}^{[1]}_{k+2,r}[\omega_1\omega_2 b](\omega_1^{-(r+1)}\omega_2^{-1} f) \\
&\quad + \mathcal{M}^{[1]}_{k+1,r}[\omega_1 b](\omega_1^{-(r+1)} f) + \mathcal{M}^{[1]}_{k+1,r}[\omega_2 b](\omega_1^{-r}\omega_2^{-1} f) + \mathcal{M}^{[1]}_{k+1,r}(\omega_1^{-r} f) \big],
\end{aligned}$$

where $S^{[1]}_{k,r,q}$ is defined in (6.7), and so

$$\begin{aligned}
\mathcal{S}^{[1]}_{k,r+1}(f) &= (k+1)(k+2)\mathcal{S}^{[1]}_{k+2,r}[\omega_1\omega_2 b]((\omega_1\omega_2)^{-1} f) \\
&\quad + (k+1)\left(\mathcal{S}^{[1]}_{k+1,r}[\omega_1 b](\omega_1^{-1} f) + \mathcal{S}^{[1]}_{k+1,r}[\omega_2 b](\omega_2^{-1} f) \right) + \mathcal{S}^{[1]}_{k,r}(f).
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
\|\mathcal{S}^{[1]}_{k,r+1}(f)\|_{s,p,K} &\lesssim_k \|\mathcal{S}^{[1]}_{k,r}(f)\|_{s,p,K} + \sum_{i=1}^{2} \|\mathcal{S}^{[1]}_{k+1,r}[\omega_i b](\omega_i^{-1} f)\|_{s,p,K} \\
&\quad + \|\mathcal{S}^{[1]}_{k+2,r}[\omega_1\omega_2 b]((\omega_1\omega_2)^{-1} f)\|_{s,p,K}
\end{aligned} \tag{6.12}$$

Part (a). We first consider the terms $\|\mathcal{S}^{[1]}_{k+1,r}[\omega_i b](\omega_i^{-1} f)\|_{s,p,K}$, $1 \leq i \leq 2$. For $k+1/p \leq s \leq k+1+1/p$, Theorem A.3 shows that $\omega_i^{-1} f \in L^p(T; \omega_i^{(k-s+1)p+1}\, \mathrm{d}\boldsymbol{x})$ and (A.8) and (A.9) gives

$$\|\omega_i^{k-s+1+\frac{1}{p}} \omega_i^{-1} f\|_{p,T} = \|\omega_i^{k-s+\frac{1}{p}} f\|_{p,T} \lesssim_{k,s,p} {}_{\mathfrak{E}}\|f\|_{s-k-\frac{1}{p},T} = {}_{\mathfrak{E},r+1}\|f\|_{s-k-\frac{1}{p},T}$$

for $1 \leq i \leq 2$. Applying (6.8) then gives

$$\|\mathcal{S}^{[1]}_{k+1,r}[\omega_i b](\omega_i^{-1} f)\|_{s,p,K} \lesssim_{b,k,r,s,p} {}_{\mathfrak{E},r+1}\|f\|_{s-k-\frac{1}{p},T}, \qquad 1 \leq i \leq 2. \tag{6.13}$$

Now let $s > k+1+1/p$. Corollary A.4 shows that $\omega_i^{-1} f \in W^{s-k-1-\frac{1}{p},p}_{\mathfrak{E},r}(T)$ and (A.11) then gives

$${}_{\mathfrak{E},r}\left\|\omega_i^{-1} f\right\|_{s-k-1-\frac{1}{p},p,T} \lesssim_{k,s,p,r} {}_{\mathfrak{E},r+1}\|f\|_{s-k-\frac{1}{p},p,T}.$$

Inequality (6.13) then follows from (6.11).

Part (b). We now turn to the term $\|\mathcal{S}^{[1]}_{k+2,r}[\omega_1\omega_2 b]((\omega_1\omega_2)^{-1} f)\|_{s,p,K}$. Assume first that $k+1/p \leq s \leq k+1+1/p$. Theorem A.3 shows that $(\omega_1\omega_2)^{-1} f \in L^p(T; \omega_1^p \omega_2^{(k-s+1)p+1}\, \mathrm{d}\boldsymbol{x})$, and (A.8) and (A.9) give

$$\|\omega_1\omega_2^{k-s+1+\frac{1}{p}} \omega_1^{-1}\omega_2^{-1} f\|_{p,T} \lesssim_{k,s,p} {}_{\mathfrak{E}}\|f\|_{s-k-\frac{1}{p},T} = {}_{\mathfrak{E},r+1}\|f\|_{s-k-\frac{1}{p},T}.$$

Applying (6.8) then gives

$$\|\mathcal{S}^{[1]}_{k+2,r}[\omega_1\omega_2 b]((\omega_1\omega_2)^{-1} f)\|_{s,p,K} \lesssim_{b,k,r,s,p} {}_{\mathfrak{E},r+1}\|f\|_{s-k-\frac{1}{p},T}. \tag{6.14}$$

Now assume that $k+1+1/p < s \leq k+2+1/p$. Thanks to Corollary A.4, $\omega_2^{-1} f \in W^{s-k-1-\frac{1}{p},p}_{\gamma_1,r+1}(T)$, and so Theorem A.3 gives $(\omega_1\omega_2)^{-1} f \in L^p(T; \omega_1^{(k-s+2)p+1}\, \mathrm{d}\boldsymbol{x})$. Inequalities (A.8) and (A.11b) then give

$$\|\omega_1^{k-s+2+\frac{1}{p}} \omega_1^{-1}\omega_2^{-1} f\|_{p,T} \lesssim_{k,s,p} {}_{\mathfrak{E},r}\left\|\omega_2^{-1} f\right\|_{s-k-1-\frac{1}{p},T} \lesssim_{k,s,p} {}_{\mathfrak{E},r+1}\|f\|_{s-k-\frac{1}{p},T}.$$

Applying (6.8) then gives (6.14).

Now assume that $s > k+2+1/p$. Two applications of Corollary A.4 show that $(\omega_1\omega_2)^{-1} f \in W^{s-k-2-\frac{1}{p},p}_{\mathfrak{E},r}(T)$ and (3.27) and (A.11b) give

$$\begin{aligned}
{}_{\mathfrak{E},r}\left\|(\omega_1\omega_2)^{-1} f\right\|_{s-k-2-\frac{1}{p},T} &\lesssim_{k,s,p,r} {}_{\gamma_1,r}\left\|\omega_1^{-1} f\right\|_{s-k-1-\frac{1}{p},T} + {}_{\gamma_2,r+1}\left\|\omega_1^{-1} f\right\|_{s-k-1-\frac{1}{p},T} \\
&\lesssim_{k,s,p,r} {}_{\gamma_1,r+1}\|f\|_{s-k-\frac{1}{p},T} + {}_{\gamma_2,r+1}\|f\|_{s-k-\frac{1}{p},T} \\
&\lesssim_{k,s,p} {}_{\mathfrak{E},r+1}\|f\|_{s-k-\frac{1}{p},T}.
\end{aligned}$$

Applying (6.11) then gives (6.14). Inequality (6.11) for $r+1$ now follows from the triangle inequality, (6.13), and (6.14). The smoothness of the mapping $\mathfrak{I}_1$ defined in (3.2) then gives (3.29).

Step 2: Trace properties (3.28a) and (3.28b). Direct computation shows that (3.28a) and (3.28b) hold.

Step 3: Polynomial preservation. Suppose that $f \in \mathcal{P}_N(\Gamma_1)$, $N \in \mathbb{N}_0$, satisfies $D^l_\Gamma f|_{\gamma_{12}} = D^l_\Gamma f|_{\gamma_{13}} = 0$ for $0 \leq l \leq r-1$. Then, $f \circ \mathfrak{I}_1 = (\omega_1\omega_2)^r g$ for some $g \in \mathcal{P}_{N-2r}(T)$, and so $\mathcal{S}^{[1]}_{k,r}(f) = (x_1 x_2)^r \mathcal{E}^{[1]}_k(g) \in \mathcal{P}_{N+k}(K)$ thanks to Lemma 3.1. ∎

6.5. Proof of Lemma 3.11. Step 1: Continuity (3.39). We first show that the following analogue of (3.39) holds: For $b \in C_c^\infty(T)$, $k, r \in \mathbb{N}_0$, $(s,p) \in \mathcal{A}_k$, and $\mathfrak{E} = \{\gamma_1, \gamma_2, \gamma_3\}$, there holds

$$\|\mathcal{R}_{k,r}^{[1]}(f)\|_{s,p,K} \lesssim_{b,k,r,s,p} {}_{\mathfrak{E},r}\|f\|_{s-k-\frac{1}{p},p,T} \qquad \forall f \in W_{\mathfrak{E},r}^{s-k-\frac{1}{p},p}(T). \tag{6.15}$$

Part (a): Variants of $\mathcal{S}_{k,r}^{[1]}$. We begin with a brief aside. Let $\mathfrak{E}_{ij} = \{\gamma_i, \gamma_j\}$ for $1 \le i < j \le 3$. Formally define the following analogue of $\mathcal{S}_{k,r}^{[1]}$ (3.25):

$$\begin{aligned}\mathcal{S}_{k,r}^{[1],(13)}(f)(\boldsymbol{x}, z) &:= (x_1(1 - x_1 - x_2 - z))^r \mathcal{E}_k^{[1]}((\omega_1\omega_3)^{-r} f)(\boldsymbol{x}, z) \\ &= \mathcal{S}_{k,r}^{[1]}[b \circ \mathfrak{F}_2](f \circ \mathfrak{F}_2) \circ \mathfrak{G}_2(\boldsymbol{x}, z), \qquad (\boldsymbol{x}, z) \in K,\end{aligned}$$

where $\mathfrak{F}_2$ and $\mathfrak{G}_2$ are defined in (6.4). Note that for any $s \ge 0$ and $r \in \mathbb{N}_0$, there holds $f \in W_{\mathfrak{E}_{13},r}^{s,p}(T)$ if and only if $f \circ \mathfrak{F}_2 \in W_{\mathfrak{E}_{12},r}^{s,p}(T)$. Thanks to Lemma 3.7, for $b \in C_c^\infty(T)$, $k, r \in \mathbb{N}_0$, $(s,p) \in \mathcal{A}_k$, there holds

$$\|\mathcal{S}_{k,r}^{[1],(13)}(f)\|_{s,p,K} \lesssim_{b,k,r,s,p} {}_{\mathfrak{E}_{13},r}\|f\|_{s-k-\frac{1}{p},p,T} \qquad \forall f \in W_{\mathfrak{E}_{13},r}^{s-k-\frac{1}{p},p}(T), \tag{6.16}$$

where we used that $\|f\|_{t,p,T} \approx_{t,p} \|f \circ \mathfrak{F}_2\|_{t,p,T}$ and ${}_{\mathfrak{E}_{13},r}\|f\|_{t,p,T} \approx_{t,p} {}_{\mathfrak{E}_{12},r}\|f \circ \mathfrak{F}_2\|_{t,p,T}$. Analogous arguments show that the operator

$$\begin{aligned}\mathcal{S}_{k,r}^{[1],(23)}(f)(\boldsymbol{x}, z) &:= (x_2(1 - x_1 - x_2 - z))^r \mathcal{E}_k^{[1]}((\omega_2\omega_3)^{-r} f)(\boldsymbol{x}, z) \\ &= \mathcal{S}_{k,r}^{[1]}[b \circ \mathfrak{F}_2^{-1}](f \circ \mathfrak{F}_2^{-1}) \circ \mathfrak{G}_2^{-1}(\boldsymbol{x}, z) \qquad (\boldsymbol{x}, z) \in K\end{aligned}$$

satisfies the following for $b \in C_c^\infty(T)$, $k, r \in \mathbb{N}_0$, $(s,p) \in \mathcal{A}_k$:

$$\|\mathcal{S}_{k,r}^{[1],(23)}(f)\|_{s,p,K} \lesssim_{b,k,r,s,p} {}_{\mathfrak{E}_{23},r}\|f\|_{s-k-\frac{1}{p},p,T} \qquad \forall f \in W_{\mathfrak{E}_{23},r}^{s-k-\frac{1}{p},p}(T). \tag{6.17}$$

Part (b): Key identity for $\mathcal{R}_{k,r}^{[1]}$. Thanks to Lemma C.1, there holds

$$\begin{aligned}\mathcal{R}_{k,r}^{[1]}(f) &= (x_1x_2(1 - x_1 - x_2 - z))^r \sum_{\substack{\alpha \in \mathbb{N}_0^3 \\ \alpha_j \le k \\ |\alpha| \ge 2}} \mathcal{E}_k^{[1]}\left(\frac{c_{\alpha,1} f}{\omega_1^{\alpha_1}\omega_2^{\alpha_2}} + \frac{c_{\alpha,2} f}{\omega_1^{\alpha_1}\omega_3^{\alpha_3}} + \frac{c_{\alpha,3} f}{\omega_2^{\alpha_2}\omega_3^{\alpha_3}}\right) \\ &= \sum_{1 \le i < j \le 3} \lambda_{m(i,j)}^r \sum_{l=1}^{r} (\lambda_i\lambda_j)^{r-l} \sum_{n=0}^{l} \left(d_{ln}^{(ij)} S_{k,l}^{[1],(ij)}(\omega_i^n f) + d_{ln}^{(ji)} S_{k,l}^{[1],(ij)}(\omega_j^n f)\right),\end{aligned}$$

where $\lambda_1 := x_1$, $\lambda_2 := x_2$, $\lambda_3 := 1 - x_1 - x_2 - z$, $m(i,j)$ is the lone element of $\{1,2,3\} \setminus \{i,j\}$, $d_{ln}^{(ij)}$ and $d_{ln}^{(ji)}$ are suitable constants, and $S_{k,r}^{[1],(12)} := \mathcal{S}_{k,r}^{[1]}$.

Let $b \in C_c^\infty(T)$, $k, r \in \mathbb{N}_0$, $(s,p) \in \mathcal{A}_k$, and $f \in W_{\mathfrak{E},r}^{s-k-\frac{1}{p},p}(T)$ be given. For any $n \in \mathbb{N}_0$ and real $t \ge 0$, the mapping $g \mapsto \omega_i^n g$ is continuous from $W_{\mathfrak{E},r}^{t,p}(T)$ to $W_{\mathfrak{E},r}^{t,p}(T)$. Similarly, for any $\alpha \in \mathbb{N}_0^3$, the mapping $g \mapsto \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3} g$ is continuous from $W^{s,p}(K)$ to $W^{s,p}(K)$. Consequently, (6.15) follows from the triangle inequality, (3.10), (3.29), (6.16), and (6.17). The smoothness of the mapping $\mathfrak{I}_1$ (3.2) then gives (3.39).

Step 2: Trace properties (3.38a) and (3.38b). Direct computation shows that (3.38a) and (3.38b) hold.

Step 3: Polynomial preservation. Suppose that $f \in \mathcal{P}_N(\Gamma_1)$, $N \in \mathbb{N}_0$, satisfies $D^l_\Gamma f|_{\partial T} = 0$ for $0 \leq l \leq r-1$. Then, $f \circ \mathfrak{I}_1 = (\omega_1\omega_2\omega_3)^r g$ for some $g \in \mathcal{P}_{N-3r}(T)$, and so $\mathcal{R}^{[1]}_{k,r}(f) = (x_1x_2(1-x_1-x_2-z))^r \mathcal{E}^{[1]}_k(g) \in \mathcal{P}_{N+k}(K)$ thanks to Lemma 3.1. ◻

Appendix A. Properties of spaces with vanishing traces. In this section, we show that smooth functions with vanishing traces are dense in the space $W^{s,p}_{\mathfrak{E}}(T)$ (3.9) and that functions in $W^{s,p}_{\mathfrak{E},r}(T)$ (3.26) satisfy a Hardy inequality.

A.1. A Density result. We begin with a density result for the spaces $W^{s,p}_{\mathfrak{E}}(T)$ defined in subsection 3.2.

LEMMA A.1. *Let $\mathfrak{E} \subseteq \{\gamma_1, \gamma_2, \gamma_3\}$ and define*

$$C^\infty_{\mathfrak{E}}(T) := \left\{ \phi \in C^\infty(\bar{T}) : \bigcup_{\gamma \in \mathfrak{E}} \gamma \cap \operatorname{supp} \phi = \emptyset \right\}.$$

For $1 < p < \infty$ and $0 \leq s < \infty$, the space $C^\infty_{\mathfrak{E}}(T)$ is dense in $W^{s,p}_{\mathfrak{E}}(T)$.

Proof. Let $\mathfrak{E} \subseteq \{\gamma_1, \gamma_2, \gamma_3\}$ and $1 < p < \infty$ be given.
Step 1: $0 \leq s < 1/p$. The space $C^\infty_c(T) \subseteq C^\infty_{\mathfrak{E}}(T)$ is dense in $W^{s,p}(T) = W^{s,p}_{\mathfrak{E}}(T)$ (see e.g. [38, Theorem 1.4.5.2]).
Step 2: $s \geq 1/p$ and $\mathfrak{E} = \{\gamma_1\}$. Let $s = m + \sigma$ with $m \in \mathbb{N}_0$ and $\sigma \in [0, 1)$, and let $f \in W^{s,p}_{\mathfrak{E}}(T)$. For $n \in \mathbb{N}$, we construct a partition of unity on T as follows. Let $\{\boldsymbol{a}_i\}_{i=1}^3$ denote the vertices of T labeled counterclockwise as in Figure 1b and define the following sets:

$$\mathcal{U}_0 := \left\{ \boldsymbol{x} \in T : \operatorname{dist}(\boldsymbol{x}, \partial T) > \frac{1}{2n} \right\},$$

$$\mathcal{U}_{(i-1)(n-1)+j} = \mathcal{U}^{(i)}_j := B\left(\boldsymbol{a}_{i+2} + \frac{j}{n}\mathbf{t}_i, \frac{3}{4n}\right) \cap \bar{T}, \qquad 1 \leq j \leq n-1,\ 1 \leq i \leq 3,$$

$$\mathcal{U}_{3n-3+k} := B\left(\boldsymbol{a}_k, \frac{3}{4n}\right) \cap \bar{T}, \qquad 1 \leq k \leq 3,$$

where we use the notation $B(\boldsymbol{x}, r)$ to denote the ball of radius r centered at $\boldsymbol{x}$. By construction, $T \subset \bigcup_{i=0}^{3n} \mathcal{U}_i$, and so there exists a partition of unity $\{\phi_i \in C^\infty_c(\mathcal{U}_i) : 0 \leq i \leq 3n\}$ satisfying

$$\sum_{i=0}^{3n} \phi_i = 1 \quad \text{and} \quad \|D^k \phi_i\|_{\infty, \mathcal{U}_i} \lesssim_k n^k, \qquad 0 \leq i \leq 3n,\ \forall k \in \mathbb{N}_0.$$

We denote $f_i := \phi_i f$ for $0 \leq i \leq 3n-3$ and set $\mathcal{V}_i := \mathcal{U}_i \cap T$ for $0 \leq i \leq 3n$.

Let $\{\delta_i\}_{i=0}^{3n-3}$ be arbitrary positive constants. The construction proceeds in several parts.
Part (a). The function $f_0 := \phi_0 f$ satisfies

$$D^l f_0|_{\partial \mathcal{U}_0} = 0, \qquad 0 \leq l < s - \frac{1}{p},$$
$$\|\operatorname{dist}(\cdot, \partial \mathcal{U}_0)^{-\sigma} D^m f_0\|_{p, \mathcal{U}_0} \lesssim_{m,p} n^{\frac{1}{p}+m} \|f\|_{m,p,\mathcal{U}_0} < \infty \qquad \text{if } \sigma p = 1.$$

By [38, Theorem 1.4.5.2], there exists a $\psi_0 \in C^\infty_c(\mathcal{U}_0)$ satisfying

$$\delta_0 \geq \|f_0 - \psi_0\|_{s,p,\mathcal{U}_0} + \begin{cases} \|\operatorname{dist}(\cdot, \partial \mathcal{U}_0)^{-\sigma} D^m (f_0 - \psi_0)\|_{p, \mathcal{U}_0} & \text{if } \sigma p = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Since $f \in W^{s,p}(T)$, we apply the same argument to $f_i := \phi_i f$ on $\mathcal{V}_i$, $1 \le i \le n-1$, to show that there exists a $\psi_i \in C_c^\infty(\mathcal{V}_i)$ satisfying

$$\delta_i \ge \|f_i - \psi_i\|_{s,p,\mathcal{V}_i} + \begin{cases} \|\operatorname{dist}(\cdot, \partial\mathcal{V}_i)^{-\sigma} D^m(f_i - \psi_i)\|_{p,\mathcal{V}_i} & \text{if } \sigma p = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Part (b). For $n \le i \le 3n-2$, $f \in W^{s,p}(\mathcal{V}_i)$, and so there exists a $\rho_i \in C^\infty(\bar{\mathcal{V}}_i)$ satisfying $\|f - \rho_i\|_{s,p,\mathcal{V}_i} \le \delta_i n^{m+2}$ thanks to [38, Theorem 1.4.5.2]. Then, the function $\psi_i := \phi_i \rho_i$ satisfies

$$\|f_i - \psi_i\|_{s,p,\mathcal{V}_i} \lesssim_{s,p} \|\phi_i\|_{m,\infty,\mathcal{V}_i} \|f - \rho_i\|_{m,p,\mathcal{V}_i} + \sum_{l=0}^{m} \|D^l \phi_i D^{m-l}(f - \rho_i)\|_{\sigma,p,\mathcal{V}_i} \lesssim_{s,p} \delta_i,$$

where we used [44, Theorem 6.3] to conclude that

$$\begin{aligned} \|D^l \phi_i D^{m-l}(f - \rho_i)\|_{\sigma,p,\mathcal{V}_i} &\lesssim_{s,p} \|D^l \phi_i\|_{\infty,\mathcal{V}_i}^{1-\sigma} \|D^{l+1}\phi_i\|_{\infty,\mathcal{V}_i}^{\sigma} \|D^{m-l}(f - \rho_i)\|_{p,\mathcal{U}_i} \\ &\qquad + \|D^l \phi_i\|_{\infty,\mathcal{V}_i} \|D^{m-l}(f - \rho_i)\|_{\sigma,p,\mathcal{V}_i} \\ &\le \delta_i. \end{aligned}$$

Moreover, when $\sigma p = 1$, we have

$$\|\omega_1^{-\sigma} D^m(f_i - \psi_i)\|_{p,\mathcal{V}_i} \lesssim_p n^\sigma \|D^m(f_i - \psi_i)\|_{p,\mathcal{V}_i} \lesssim_{s,p} \delta_i.$$

Part (c). For $3n-1 \le i \le 3n$, we will show that

$$\lim_{n\to\infty} \|f_i\|_{s,p,\mathcal{V}_i} = 0 \quad \text{and if } \sigma p = 1, \quad \lim_{n\to\infty} \|\omega_1^{-\sigma} D^m f_i\|_{p,\mathcal{V}_i} = 0. \tag{A.1}$$

Thanks to [44, Theorem 6.3], there holds

$$\begin{aligned} \|f_i\|_{s,p,\mathcal{V}_i} &\lesssim_{s,p} \sum_{j=0}^{m} \sum_{l=0}^{j} \|D^l \phi_i\|_{\infty,\mathcal{V}_i} \|D^{j-l} f\|_{p,\mathcal{V}_i} + \sum_{l=0}^{m} \|D^l \phi_i\|_{\infty,\mathcal{V}_i} \|D^{m-l} f\|_{\sigma,p,\mathcal{V}_i} \\ &\quad + \sum_{l=0}^{m} \|D^l \phi_i\|_{\infty,\mathcal{V}_i}^{1-\sigma} \|D^{l+1}\phi_i\|_{\infty,\mathcal{V}_i}^{\sigma} \|D^{m-l} f\|_{p,\mathcal{U}_i} \\ &\lesssim_{s,p} \sum_{j=0}^{m} \sum_{l=0}^{j} n^l \|D^{j-l} f\|_{p,\mathcal{V}_i} + \sum_{l=0}^{m} \left(n^{l+\sigma} \|D^{m-l} f\|_{p,\mathcal{U}_i} + n^l \|D^{m-l} f\|_{\sigma,p,\mathcal{V}_i} \right). \end{aligned}$$

Similar computations show that for $\sigma p = 1$, there holds

$$\|\omega_1^{-\sigma} D^m f_i\|_{p,\mathcal{V}_i} \lesssim_{s,p} \sum_{j=0}^{m} n^j \|\omega_1^{-\sigma} D^{m-j} f\|_{p,\mathcal{V}_i}.$$

Since $\mathcal{V}_i \cap \gamma_1 \neq \emptyset$, Poincaré's inequality gives

$$\|D^r f\|_{p,\mathcal{V}_i} \lesssim_{r,p} n^{-(s-r)} |D^m f|_{\sigma,p,\mathcal{V}_i} \qquad 0 \le r \le m,$$

and so $\|f_i\|_{s,p,\mathcal{V}_i} \lesssim_{s,p} |D^m f|_{\sigma,p,\mathcal{V}_i}$. Moreover, if $\sigma p = 1$, then $D^{m-j} f \in W_{\mathfrak{E}}^{1,p}(\mathcal{V}_i)$ for $1 \le j \le m$, and so [24, Theorem 5.2], [35, Theorem 3.2], and a standard scaling argument give

$$\begin{aligned} \|\omega_1^{-\sigma} D^{m-j} f\|_{p,\mathcal{V}_i} \lesssim n^{\sigma-1} \|\omega_1^{-1} D^{m-j} f\|_{p,\mathcal{V}_i} &\lesssim_p n^{\sigma-1} \|D^{m-j+1} f\|_{p,\mathcal{V}_i} \\ &\lesssim_{s,p} n^{-j} |D^m f|_{\sigma,p,\mathcal{V}_i}, \end{aligned}$$

and so $\|\omega_1^{-\sigma} D^m f_i\|_{p,\mathcal{V}_i} \lesssim_{s,p} |D^m f|_{\sigma,p,\mathcal{V}_i}$. Equality (A.1) now follows from that fact that $|D^m f|_{\sigma,p,\mathcal{V}_i} \to 0$ as $n \to \infty$ since $|\mathcal{V}_i| \to 0$ as $n \to \infty$.

Part (d). Let $\epsilon > 0$ be given. First, choose n large enough so that

$$\frac{\epsilon}{2} \geq \sum_{i=3n-1}^{3n} \|f_i\|_{s,p,\mathcal{V}_i} + \begin{cases} \sum_{i=3n-1}^{3n} \|\omega_1^{-\sigma} D^k f_i\|_{p,\mathcal{V}_i} & \text{if } \sigma p = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for $\{\delta_i\}_{i=0}^{3n-2}$ chosen sufficiently small, we construct ψ_i as above so that

$$\frac{\epsilon}{2} \geq \sum_{i=0}^{3n-2} \|f_i - \psi_i\|_{s,p,\mathcal{V}_i} + \begin{cases} \sum_{i=3n-1}^{3n} \|\omega_1^{-\sigma} D^m (f_i - \psi_i)\|_{p,\mathcal{V}_i} & \text{if } \sigma p = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\tilde{\psi}_i$ denote the extension of ψ_i by zero to $T \setminus \mathcal{U}_i$, $0 \leq i \leq 3n-2$ and set $\tilde{\psi}_j \equiv 0$ for $3n-1 \leq j \leq 3n$. Then, $\tilde{\psi}_i \in C^\infty_{\mathfrak{E}}(T)$, and the function $\psi = \sum_{i=0}^{3n} \tilde{\psi}_i$ then satisfies $\psi \in C^\infty_{\mathfrak{E}}(T)$ and

$$\begin{aligned} {}_{\mathfrak{E}}\|f - \psi\|_{s,p,T} &\leq \sum_{i=0}^{3n} \|f_i - \psi_i\|_{s,p,\mathcal{V}_i} + \begin{cases} \sum_{i=0}^{3n} \|\omega_1^{-\sigma} D^m (f_i - \psi_i)\|_{p,\mathcal{V}_i} & \text{if } \sigma p = 1, \\ 0 & \text{otherwise,} \end{cases} \\ &\lesssim_{s,p} \epsilon, \end{aligned}$$

which shows that $C^\infty_{\mathfrak{E}}(T)$ is dense in $W^{s,p}_{\mathfrak{E}}(T)$.

Step 3: $\mathfrak{E} = \{\gamma_2\}$ or $\mathfrak{E} = \{\gamma_3\}$. If $\mathfrak{E} = \{\gamma_2\}$, the density of $C^\infty_{\gamma_2}(T)$ in $W^{s,p}_{\gamma_2}(T)$ follows from the fact that $f \in W^{s,p}_{\gamma_2}(T)$ if and only if $f \circ \mathfrak{F}_2^{-1} \in W^{s,p}_{\gamma_1}(T)$, where $\mathfrak{F}_2^{-1}(x_1, x_2) = (1 - x_1 - x_2, x_1)$ is the inverse of $\mathfrak{F}_2$ defined in (6.4). The case $\mathfrak{E} = \{\gamma_3\}$ follows from similar arguments using the mapping $\mathfrak{F}_2$.

Step 4: $|\mathfrak{E}| = 2$. Now let $\mathfrak{E} = \{\gamma_1, \gamma_2\}$. The density of $C^\infty_{\mathfrak{E}}(T)$ in $W^{s,p}_{\mathfrak{E}}(T)$ may be shown using a similar construction to the case $\mathfrak{E} = \{\gamma_1\}$. In particular, we apply the construction of Step 1 Part (a) for $0 \leq i \leq 2n-2$ and $i = 3n$, Part (b) for $2n-1 \leq i \leq 3n-3$, Part (c) for $3n-2 \leq i \leq 3n-1$, and proceed analogously as in Part (d). The remaining cases for $|\mathfrak{E}| = 2$ are proved along similar lines.

Step 5: $\mathfrak{E} = \{\gamma_1, \gamma_2, \gamma_3\}$. This case is a restatement of [38, Lemma 1.4.5.2]. □

A.2. Hardy inequalities. First, we construct a bounded averaging operator.

LEMMA A.2. *There exists a linear operator $\mathcal{H}_1$ satisfying the following properties:*

(i) $\mathcal{H}_1$ maps $C(\bar{T})$ boundedly into $C(\bar{T})$, and there holds

$$\mathcal{H}_1(f)(\boldsymbol{x}) = \frac{1}{x_1} \int_0^{x_1} f(u, x_2)\,\mathrm{d}u = \int_0^1 f(ux_1, x_2)\,\mathrm{d}u \qquad \forall \boldsymbol{x} \in T. \tag{A.2}$$

(ii) $\mathcal{H}_1$ maps $W^{s,p}_{\mathfrak{E},r}(T)$ boundedly into $W^{s,p}_{\mathfrak{E},r}(T)$ and for all $p \in (1, \infty)$, $s \in [0, \infty)$, $r \in \mathbb{N}_0$, and $\mathfrak{E} \in \{\emptyset, \{\gamma_1\}, \{\gamma_1, \gamma_2\}\}$. In particular,

$$ {}_{\mathfrak{E},r}\|\mathcal{H}_1(f)\|_{s,p,T} \lesssim_{s,p,r} {}_{\mathfrak{E},r}\|f\|_{s,p,T} \qquad \forall f \in W^{s,p}_{\mathfrak{E},r}(T). \tag{A.3}$$

Proof. **Step 1: Continuity on $C(\bar{T})$.** Let $f \in C(\bar{T})$ and define $\mathcal{H}_1(f)$ by (A.2). Elementary arguments show that $\mathcal{H}_1(f) \in C(\bar{T})$ with $\|\mathcal{H}_1(\phi_i)\|_{\infty,T} \leq \|\phi_i\|_{\infty,T}$.

Step 2: Extension to $W^{s,p}_{\mathfrak{E},r}(T)$ when $\mathfrak{E} = \emptyset$. Let $f \in C^\infty(\bar{T})$ and $1 < p < \infty$. For $\alpha \in \mathbb{N}_0^2$, there holds

$$D^\alpha \mathcal{H}_1(f)(\boldsymbol{x}) = \int_0^1 u^{\alpha_1} (D^\alpha f)(ux_1, x_2)\,\mathrm{d}u = \frac{1}{x_1^{\alpha_1+1}} \int_0^{x_1} u^{\alpha_1} (D^\alpha f)(u, x_2)\,\mathrm{d}u, \tag{A.4}$$

and so $\mathcal{H}_1(f) \in C^\infty(\bar{T})$. Moreover, Hardy's inequality [40, Theorem 327] gives

$$\begin{aligned}\|D^\alpha \mathcal{H}_1(f)\|_{p,T}^p &\leq \int_0^1 \int_0^{1-x_2} \left| \frac{1}{x_1} \int_0^{x_1} |D^\alpha f(u, x_2)| \,\mathrm{d}u \right|^p \mathrm{d}x_1 \,\mathrm{d}x_2 \\ &\leq \left(\frac{p}{p-1} \right)^p \int_0^1 \int_0^{1-x_2} |D^\alpha f(x_1, x_2)|^p \,\mathrm{d}x_1 \,\mathrm{d}x_2.\end{aligned}$$

Consequently, we obtain

$$\|\mathcal{H}_1(f)\|_{s,p,T} \lesssim_{s,p} \|f\|_{s,p,T} \qquad \forall f \in C^\infty(\bar{T}),\ s \in \mathbb{N}_0.$$

Since $C^\infty(\bar{T})$ is dense in $W^{s,p}(T)$ [38, Theorem 1.4.5.2], $\mathcal{H}_1$ can be continuously extended to a linear operator from $W^{s,p}(T)$ into $W^{s,p}(T)$ for $s \in \mathbb{N}_0$. The case for non-integer $s \in (0, \infty)$ follows from interpolation.

Step 3: Inequality (A.3) when $\mathfrak{E} = \{\gamma_1\}$. The case $r = 0$ follows from Step 2, so let $r \in \mathbb{N}$. Assume first the $s \leq r$ and let $f \in C^\infty_{\gamma_1}(T)$. Equation (A.4) shows that $\mathcal{H}_1(f) \in C^\infty_{\gamma_1}(T)$. Moreover, for $s = m + 1/p$, $m \in \mathbb{N}_0$, we apply Hardy's inequality [40, Theorem 327] to obtain

$$\begin{aligned}\|\omega_1^{-\frac{1}{p}} D^\alpha \mathcal{H}_1(f)\|_{p,T}^p &\leq \int_0^1 \int_0^{1-x_2} \left(\frac{1}{x_1} \int_0^{x_1} \frac{1}{u^{\frac{1}{p}}} |D^\alpha f(u, x_2)| \,\mathrm{d}u \right)^p \mathrm{d}x_1 \,\mathrm{d}x_2 \\ &\leq \left(\frac{p}{p-1} \right)^p \|\omega_1^{-\frac{1}{p}} D^\alpha f\|_{p,T}^p \end{aligned} \tag{A.5}$$

for all $\alpha \in \mathbb{N}_0$ with $|\alpha| = m$. Thus, ${}_{\gamma_1}\|\mathcal{H}_1(f)\|_{s,p,T} \lesssim_{s,p} {}_{\gamma_1}\|f\|_{s,p,T}$ for all $f \in C^\infty_{\gamma_1}(T)$. By density (Lemma A.1), $\mathcal{H}_1$ maps $W^{s,p}_{\gamma_1,r}(T)$ boundedly into $W^{s,p}_{\gamma_1,r}(T)$ for all $p \in (1, \infty)$ and $s \in [0, r]$.

Now let $s > r$ and $f \in W^{s,p}_{\gamma_1,r}(T)$. Step 2 and the arguments above show that $\mathcal{H}_1(f) \in W^{s,p}(T) \cap W^{r,p}_{\gamma_1}(T)$, and so $\mathcal{H}_1(f) \in W^{s,p}_{\gamma_1,r}(T)$ if $s - 1/p \notin \mathbb{Z}$ with ${}_{\gamma_1,r}\|\mathcal{H}_1(f)\|_{s,p,T} \lesssim_{s,p} {}_{\gamma_1,r}\|f\|_{s,p,T}$. Now let $s = m + 1/p$ for some $m \in \mathbb{N}$. Then, $f \in C(\bar{T})$ and thanks to Step 1 and (A.5), we have

$$\left\| \omega_1^{-\frac{1}{p}} \frac{\partial^{m-r-1} D^{r-1} \mathcal{H}_1(f)}{\partial x_2^{m-r-1}} \right\|_{p,T} \lesssim_p \left\| \omega_1^{-\frac{1}{p}} \frac{\partial^{m-r-1} D^{r-1} f}{\partial x_2^{m-r-1}} \right\|_{p,T}.$$

Consequently, $\mathcal{H}_1(f) \in W^{s,p}_{\gamma_1,r}(T)$ and (A.3) holds when $\mathfrak{E} = \{\gamma_1\}$.

Step 4: Inequality (A.3) when $\mathfrak{E} = \{\gamma_1, \gamma_2\}$. Again let $r \in \mathbb{N}$. Assume first that $s \leq r$. As above, $\mathcal{H}_1(f) \in C^\infty_{\mathfrak{E}}(T)$ for any $f \in C^\infty_{\mathfrak{E}}(T)$ by (A.4), and for $s = m + 1/p$, $m \in \mathbb{N}_0$, Hardy's inequality [40, Theorem 327] gives

$$\begin{aligned}\|\omega_2^{-\frac{1}{p}} D^\alpha \mathcal{H}_1(f)\|_{p,T}^p &\leq \int_0^1 \frac{1}{x_2} \int_0^{1-x_2} \left(\frac{1}{x_1} \int_0^{x_1} |D^\alpha f(u, x_2)| \,\mathrm{d}u \right)^p \mathrm{d}x_1 \,\mathrm{d}x_2 \\ &\leq \left(\frac{p}{p-1} \right)^p \|\omega_2^{-\frac{1}{p}} D^\alpha f\|_{p,T}^p \end{aligned} \tag{A.6}$$

for all $\alpha \in \mathbb{N}_0$ with $|\alpha| = m$. Inequality (A.3) now follows from Step 2 and (3.10). By density, $\mathcal{H}_1$ maps $W^{s,p}_{\mathfrak{E},r}(T)$ boundedly into $W^{s,p}_{\mathfrak{E},r}(K)$ for all $p \in (1, \infty)$ and $s \in [0, r]$.

Now let $s > r$ and $f \in W^{s,p}_{\mathfrak{E},r}(T)$. Arguing analogously as in Step 3, we have $\mathcal{H}_1(f) \in W^{s,p}_{\mathfrak{E},r}(T)$ if $s - \frac{1}{p} \notin \mathbb{Z}$ with ${}_{\mathfrak{E},r}\|\mathcal{H}_1(f)\|_{s,p,T} \lesssim_{s,p} {}_{\mathfrak{E},r}\|f\|_{s,p,T}$. Moreover, (A.6) gives

$$\left\|\omega_2^{-\frac{1}{p}} \frac{\partial^{m-r-1} D^{r-1} \mathcal{H}_1(f)}{\partial x_1^{m-r-1}}\right\|_{p,T} \lesssim_p \left\|\omega_2^{-\frac{1}{p}} \frac{\partial^{m-r-1} D^{r-1} f}{\partial x_1^{m-r-1}}\right\|_{p,T}.$$

Consequently, $\mathcal{H}_1(f) \in W^{s,p}_{\mathfrak{E},r}(T)$ and (A.3) holds when $\mathfrak{E} = \{\gamma_1, \gamma_2\}$. ∎

Finally, we state and prove various versions of Hardy's inequality.

THEOREM A.3. *Let $1 < p < \infty$ and $\emptyset \neq \mathfrak{E} \subseteq \{\gamma_1, \gamma_2\}$. For $0 \leq s < \infty$ and $i \in \{1, 2\}$ such that $\gamma_i \in \mathfrak{E}$, the mapping $f \mapsto \omega_i^{-1} f$ is bounded (i) $W^{s+1,p}(T) \cap W^{1,p}_{\mathfrak{E}}(T)$ to $W^{s,p}(T)$, and (ii) $W^{s+1,p}_{\mathfrak{E}}(T)$ to $W^{s,p}_{\mathfrak{E}}(T)$, and there holds*

$$\|\omega_i^{-1} f\|_{s,p,T} \lesssim_{s,p} \|\partial_i f\|_{s,p,T} \qquad \forall f \in W^{s+1,p}(T) \cap W^{1,p}_{\mathfrak{E}}(T), \tag{A.7}$$

$$_{\mathfrak{E}}\|\omega_i^{-1} f\|_{s,p,T} \lesssim_{s,p} {}_{\mathfrak{E}}\|\partial_i f\|_{s,p,T} \qquad \forall f \in W^{s+1,p}_{\mathfrak{E}}(T). \tag{A.8}$$

Additionally, for $0 \leq s < 1$ and $i \in \{1, 2\}$, the mapping $f \mapsto \omega_i^{-s} f$ is bounded $W^{s,p}_{\gamma_i}(T)$ to $L^p(T)$, and there holds

$$\|\omega_i^{-s} f\|_{p,T} \lesssim_{s,p} {}_{\gamma_i}\|f\|_{s,p,T} \qquad \forall f \in W^{s,p}_{\gamma_i}(T). \tag{A.9}$$

Proof. Let $1 < p < \infty$ be given.

Step 1: Inequalities (A.7) and (A.8) when $\mathfrak{E} \in \{\{\gamma_1\}, \{\gamma_1, \gamma_2\}\}$. Thanks to the fundamental theorem of calculus, there holds

$$f(\boldsymbol{x}) = \int_0^{x_1} (\partial_1 f)(u, x_2)\, \mathrm{d}u = x_1 \mathcal{H}_1(\partial_1 f) \qquad \forall \boldsymbol{x} \in T,\ \forall f \in C^\infty_{\gamma_1}(T). \tag{A.10}$$

By density (Lemma A.1), (A.10) holds for a.e. $\boldsymbol{x} \in T$ for all $f \in W^{1,p}_{\mathfrak{E}}(T)$. Lemma A.2 and (A.10) then show that the mapping $f \mapsto \omega_i^{-1} f$ is bounded (i) from $W^{s+1,p}(T) \cap W^{1,p}_{\mathfrak{E}}(T)$ to $W^{s,p}(T)$, and (ii) from $W^{s+1,p}_{\mathfrak{E}}(T)$ to $W^{s,p}_{\mathfrak{E}}(T)$ provided that $\gamma_i \in \mathfrak{E}$. Inequalities (A.7) and (A.8) now follow from (A.3) and (A.10).

Step 2: Inequalities (A.7) and (A.8) when $\mathfrak{E} = \{2\}$. Note that $f \in W^{s+1,p}(T) \cap W^{1,p}_{\gamma_2}(T)$ if and only if $g := f \circ \mathfrak{F}_1 \in W^{s+1,p}(T) \cap W^{1,p}_{\gamma_1}(T)$ and $f \in W^{s+1}_{\gamma_2}(T)$ if and only if $g \in W^{s+1}_{\gamma_1}(T)$, where $\mathfrak{F}_1$ is defined in (6.3). Inequalities (A.7) and (A.8) then follow from Step 1.

Step 3: Inequality (A.9) with $i = 1$. Now let $0 \leq s < 1$. For $sp = 1$, (A.9) follows immediately from the definition of the norm. In the case $sp < 1$, the proof of Theorem 1.4.4.4 in [38] gives

$$\|\omega_i^{-s} f\|_{p,T} \leq \|\operatorname{dist}(\cdot, \partial T)^{-s} f\|_{p,T} \lesssim_{s,p} \|f\|_{s,p,T} = {}_{\gamma_1}\|f\|_{s,p,T} \qquad \forall f \in W^{s,p}_{\gamma_1}(T).$$

Finally, let $sp > 1$ and let $f \in W^{s,p}_{\gamma_1}(T)$ be given. We denote by $\tilde{f} \in W^{s,p}(T)$ any extension of f to $\mathbb{R}^2$ satisfying $\|\tilde{f}\|_{s,p,\mathbb{R}^2} \lesssim_{s,p} \|f\|_{s,p,T}$ (see e.g. [34] or [44, Theorem 8.4]). Thanks to Theorem 6.79, inequality (6.58), and Remark 6.80 of [44], there holds

$$\int_0^\infty \int_{\mathbb{R}} \frac{|\tilde{f}(x_1, x_2) - \tilde{f}(0, x_2)|^p}{x_1^{sp}}\, \mathrm{d}x_2\, \mathrm{d}x_1 \lesssim_{s,p} |\tilde{f}|^p_{s,p,\mathbb{R}_+ \times \mathbb{R}}.$$

Since $f|_{\gamma_1} = \tilde{f}|_{\gamma_1} = 0$, we obtain

$$\|\omega_1^{-s} f\|_{p,T} = \|\omega_1^{-s}(\tilde{f} - \tilde{f}(0, \cdot))\|_{p,T} \leq \|\omega_1^{-s}(\tilde{f} - \tilde{f}(0, \cdot))\|_{p,\mathbb{R}_+ \times \mathbb{R}} \lesssim_{s,p} |\tilde{f}|_{s,p,\mathbb{R}_+ \times \mathbb{R}},$$

and so $\|\omega_1^{-s} f\|_{p,T} \lesssim_{s,p} \|f\|_{s,p,T}$, which completes the proof of (A.9) for $i = 1$.

Step 4: Inequality (A.9) with $i = 2$. This can be reduced to the case $i = 1$ using similar arguments as in Step 2 using the mapping $\mathfrak{F}_1$ defined in (6.3). ∎

Corollary A.4. *Let $1 < p < \infty$, $0 \le s < \infty$, $1 \le i \le 2$, and $r_1, r_2 \in \mathbb{N}_0$ with $r_i \ge 1$. Then, for all $f \in W^{s+1,p}_{\gamma_1,r_1}(T) \cap W^{s+1,p}_{\gamma_2,r_2}(T)$, there holds $\omega_i^{-1} f \in W^{s,p}_{\gamma_i,r_i-1}(T) \cap W^{s,p}_{\gamma_j,r_j}(T)$ and*

$$\text{(A.11a)} \qquad {}_{\gamma_i,r_i-1}\|\omega_i^{-1} f\|_{s,p,T} \lesssim_{s,p,r_1,r_2} {}_{\gamma_i,r_i-1}\|\partial_i f\|_{s,p,T} \le {}_{\gamma_i,r_i}\|f\|_{s+1,p,T},$$

$$\text{(A.11b)} \qquad {}_{\gamma_j,r_j}\|\omega_i^{-1} f\|_{s,p} \lesssim_{s,p,r_1,r_2} {}_{\gamma_j,r_j}\|\partial_i f\|_{s,p,T} \le {}_{\gamma_j,r_j}\|f\|_{s+1,p,T},$$

where $1 \le j \le 2$, $j \ne i$.

Proof. Let $f \in W^{s+1,p}_{\gamma_1,r_1+1}(T) \cap W^{s+1,p}_{\gamma_2,r_2}(T)$ be given. By definition, there holds $\partial_1 f \in W^{s,p}_{\gamma_1,r_1}(T) \cap W^{s,p}_{\gamma_2,r_2}(T)$ since $\mathbf{t}_{\gamma_2} = [0,1]^T$. Thanks to identity (A.10), which was shown in the proof of Theorem A.3 to hold for all $f \in W^{1,p}_{\gamma_1}(T)$, the result for $i = 1$ follows from Lemma A.2. The case $i = 2$ can be reduced to the case $i = 1$ using similar arguments as in the proof of Theorem A.3 using the mapping $\mathfrak{F}_1$ (6.3). ∎

Appendix B. Equivalent Boundary Norm. We begin with a result that states necessary and sufficient conditions for a function defined on two faces $\Gamma_i \cup \Gamma_j \subset \partial K$ to belong to $W^{s,p}(\Gamma_i \cup \Gamma_j)$.

Lemma B.1. *Let $0 \le s < 1$, $1 < p < \infty$, and $1 \le i < j \le 4$. Then, $f \in L^p(\Gamma_i \cup \Gamma_j)$ satisfies $f \in W^{s,p}(\Gamma_i \cup \Gamma_j)$ if and only if*

(i) $f_i \in W^{s,p}(\Gamma_i)$ and $f_j \in W^{s,p}(\Gamma_j)$;

(ii) if $s > 1/p$, then $f_i|_{\gamma_{ij}} = f_j|_{\gamma_{ij}}$; and

(iii) if $s = 1/p$, then $I^p_{ij}(f_i, f_j) < \infty$,

where $\mathcal{I}^p_{ij}(\cdot,\cdot)$ is defined in (2.2). Additionally,

$$\|f\|^p_{s,p,\Gamma_i\cup\Gamma_j} \approx_{s,p} |||f|||^p_{s,p,\Gamma_i\cup\Gamma_j} := \|f_i\|^p_{s,p,\Gamma_i} + \|f_j\|^p_{s,p,\Gamma_j} + \begin{cases} \mathcal{I}^p_{ij}(f_i, f_j) & \text{if } sp = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $0 \le s < 1$, $1 < p < \infty$, and $1 \le i < j \le 4$ be given.

Step 1: $f \in W^{s,p}(\Gamma_i \cup \Gamma_j) \implies$ (i-iii). Assume first that $f \in W^{s,p}(\Gamma_i \cup \Gamma_j)$. Condition (i) follows from the definition of the norms, and in particular, $\|f_i\|^p_{s,p,\Gamma_i} + \|f_j\|^p_{s,p,\Gamma_j} \le \|f\|^p_{s,p,\Gamma_i\cup\Gamma_j}$. If $s > 1/p$, then the trace theorem shows that f has a well-defined trace on γ_{ij}, and so (ii) holds.

We now show that condition (iii) is satisfied. There holds

$$|\boldsymbol{x} - \boldsymbol{y}|^2 \approx ([\boldsymbol{F}^{-1}_{ij}(\boldsymbol{x}) - \boldsymbol{F}^{-1}_{ji}(\boldsymbol{y})] \cdot \mathrm{e}_1)^2 + ([\boldsymbol{F}^{-1}_{ij}(\boldsymbol{x}) + \boldsymbol{F}^{-1}_{ji}(\boldsymbol{y})] \cdot \mathrm{e}_2)^2$$

for all $(\boldsymbol{x}, \boldsymbol{y}) \in \Gamma_i \times \Gamma_j$, where $\boldsymbol{F}_{ij} : T \to \Gamma_i$ and $\boldsymbol{F}_{ji} : T \to \Gamma_j$ are defined in (2.1), and so

(B.1)

$$\begin{aligned} &\int_{\Gamma_i}\int_{\Gamma_j} \frac{|f_i(\boldsymbol{x}) - f_j(\boldsymbol{y})|^p}{|\boldsymbol{x} - \boldsymbol{y}|^{sp+2}} \,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \\ &\quad\approx_{s,p} \int_{\Gamma_i}\int_{\Gamma_j} \frac{|f_i(\boldsymbol{x}) - f_j(\boldsymbol{y})|^p}{|([\boldsymbol{F}^{-1}_{ij}(\boldsymbol{x}) - \boldsymbol{F}^{-1}_{ji}(\boldsymbol{y})] \cdot \mathrm{e}_1)^2 + ([\boldsymbol{F}^{-1}_{ij}(\boldsymbol{x}) + \boldsymbol{F}^{-1}_{ji}(\boldsymbol{y})] \cdot \mathrm{e}_2)^2|^{\frac{sp+2}{2}}} \,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \\ &\quad\overset{\substack{\boldsymbol{u}=\boldsymbol{F}^{-1}_{ij}(\boldsymbol{x}) \\ \boldsymbol{v}=\boldsymbol{F}^{-1}_{ji}(\boldsymbol{y})}}{\approx_{s,p}} \int_T\int_T \frac{|f_i \circ \boldsymbol{F}_{ij}(\boldsymbol{u}) - f_j \circ \boldsymbol{F}_{ji}(\boldsymbol{v})|^p}{|\boldsymbol{u}_r - \boldsymbol{v}|^{sp+2}} \,\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{u}, \end{aligned}$$

where $\boldsymbol{u}_r := (u_1, -u_2)$. Now let $s = 1/p$. Applying the triangle inequality in conjunction with the above inequality, we obtain

$$\int_T \int_T \frac{|f_i \circ \boldsymbol{F}_{ij}(\boldsymbol{x}) - f_j \circ \boldsymbol{F}_{ji}(\boldsymbol{x})|^p}{|\boldsymbol{x}_r - \boldsymbol{y}|^3} \, \mathrm{d}\boldsymbol{y} \, \mathrm{d}\boldsymbol{x} \lesssim_p \|f\|^p_{s,p,\Gamma_i \cup \Gamma_j}.$$

Assume, for the moment, that the following holds.

$$B(\boldsymbol{x}) := \int_T \frac{x_2}{|\boldsymbol{x}_r - \boldsymbol{y}|^3} \, \mathrm{d}\boldsymbol{y} \gtrsim 1 \qquad \forall \boldsymbol{x} \in T. \tag{B.2}$$

Then, we obtain the following bound for $I^p_{ij}(f_i, f_j)$ defined in (2.2):

$$I^p_{ij}(f_i, f_j) \lesssim \int_T \int_T \frac{|f_i \circ \boldsymbol{F}_{ij}(\boldsymbol{x}) - f_j \circ \boldsymbol{F}_{ji}(\boldsymbol{x})|^p}{|\boldsymbol{x}_r - \boldsymbol{y}|^3} \, \mathrm{d}\boldsymbol{y} \, \mathrm{d}\boldsymbol{x} \lesssim_p \|f\|^p_{s,p,\Gamma}.$$

Thus, f satisfies (iii) and we have shown that for all $f \in W^{s,p}(\Gamma_i \cup \Gamma_j)$ there holds

$$\|f\|_{s,p,\Gamma_i \cup \Gamma_j} \gtrsim_{s,p} |||f|||_{s,p,\Gamma_i \cup \Gamma_j}. \tag{B.3}$$

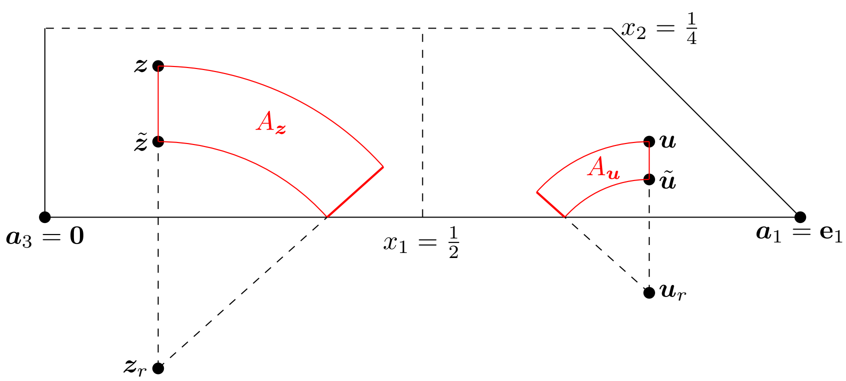

Fig. 2: Annular regions of integration (red) in the proof of (B.2), where $\tilde{\boldsymbol{z}} := (z_1, z_2/2)$ and $\tilde{\boldsymbol{u}} := (u_1, u_2/2)$.

We now turn to the proof of (B.2). First note that B is well-defined on the half plane $\mathbb{R}^2_+ = \{\boldsymbol{x} \in \mathbb{R}^2 : x_2 > 0\}$ and is a continuous function with $B(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \in \mathbb{R}^2_+$, and so it suffices to show (B.2) for $\boldsymbol{x} \in T$ with $x_2 < 1/4$. Let $(\rho, \theta) \in [0, \infty) \times [0, 2\pi)$ denote polar coordinates centered at $\boldsymbol{x}_r$ and let $A_{\boldsymbol{x}}$ denote the following annulus centered at $\boldsymbol{x}_r$ (pictured in Figure 2):

$$A_{\boldsymbol{x}} = \begin{cases} \left\{\boldsymbol{y} \in \mathbb{R}^2 : \frac{3x_2}{2} < \rho < 2x_2, \ \frac{\pi}{2} - \cos^{-1}\left(\frac{2}{3}\right) < \theta < \frac{\pi}{2}\right\} & \text{if } x_1 \le \frac{1}{2}, \\ \left\{\boldsymbol{y} \in \mathbb{R}^2 : \frac{3x_2}{2} < \rho < 2x_2, \ \frac{\pi}{2} < \theta < \frac{\pi}{2} + \cos^{-1}\left(\frac{2}{3}\right)\right\} & \text{if } x_1 > \frac{1}{2}. \end{cases}$$

Then, one may readily verify that $A_{\boldsymbol{x}} \subset T$, and so

$$B(\boldsymbol{x}) \ge \int_{A_{\boldsymbol{x}}} \frac{x_2}{|\boldsymbol{x}_r - \boldsymbol{y}|^3} \, \mathrm{d}\boldsymbol{y} = x_2 \cos^{-1}\left(\frac{2}{3}\right) \int_{\frac{3x_2}{2}}^{2x_2} \frac{1}{\rho^2} \, \mathrm{d}\rho = \frac{1}{6} \cdot \cos^{-1}\left(\frac{2}{3}\right),$$

which completes the proof of (B.2).

Step 2: (i-iii) $\implies f \in W^{s,p}(\Gamma_i \cup \Gamma_j)$. Now assume that $f \in L^p(\Gamma_i \cup \Gamma_j)$ satisfies (i-iii). Thanks to the triangle inequality, there holds

$$\begin{aligned}\text{(B.4)}\quad &\int_T \int_T \frac{|f_i \circ \boldsymbol{F}_{ij}(\boldsymbol{x}) - f_j \circ \boldsymbol{F}_{ji}(\boldsymbol{y})|^p}{|\boldsymbol{x}_r - \boldsymbol{y}|^{sp+2}} \,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \\ &\qquad \lesssim_{s,p} \int_T \int_T \frac{|f_i \circ \boldsymbol{F}_{ij}(\boldsymbol{x}) - f_j \circ \boldsymbol{F}_{ji}(\boldsymbol{x})|^p}{|\boldsymbol{x}_r - \boldsymbol{y}|^{sp+2}} \,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \\ &\qquad\qquad + \int_T \int_T \frac{|f_j \circ \boldsymbol{F}_{ji}(\boldsymbol{x}) - f_j \circ \boldsymbol{F}_{ji}(\boldsymbol{y})|^p}{|\boldsymbol{x} - \boldsymbol{y}|^{sp+2}} \,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

where we used that $|\boldsymbol{x}_r - \boldsymbol{y}| \geq |\boldsymbol{x} - \boldsymbol{y}|$. To bound the second term, we perform a simple change of variables.

$$\begin{aligned}\int_T \int_T \frac{|f_j \circ \boldsymbol{F}_{ji}(\boldsymbol{x}) - f_j \circ \boldsymbol{F}_{ji}(\boldsymbol{y})|^p}{|\boldsymbol{x} - \boldsymbol{y}|^{sp+2}} \,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} &\lesssim_{s,p} \int_T \int_T \frac{|f_j(\boldsymbol{x}) - f_j(\boldsymbol{y})|^p}{|\boldsymbol{F}_{ji}^{-1}(\boldsymbol{x}) - \boldsymbol{F}_{ji}^{-1}(\boldsymbol{y})|^{sp+2}} \,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \\ &\lesssim_{s,p} |||f|||^p_{s,p,\Gamma_i \cup \Gamma_j}.\end{aligned}$$

For the first term, there holds

$$\int_T \frac{\mathrm{d}\boldsymbol{y}}{|\boldsymbol{x}_r - \boldsymbol{y}|^{sp+2}} \leq \int_{\mathbb{R}^2 \setminus B(\boldsymbol{x}_r, x_2)} \frac{\mathrm{d}\boldsymbol{y}}{|\boldsymbol{x}_r - \boldsymbol{y}|^{sp+2}} = 2\pi \int_{x_2}^{\infty} \frac{\mathrm{d}\rho}{\rho^{sp+1}} = \frac{2\pi}{sp} x_2^{-sp},$$

for all $\boldsymbol{x} \in T$, and so

$$\int_T \int_T \frac{|f_i \circ \boldsymbol{F}_{ij}(\boldsymbol{x}) - f_j \circ \boldsymbol{F}_{ji}(\boldsymbol{x})|^p}{|\boldsymbol{x}_r - \boldsymbol{y}|^{sp+2}} \,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \lesssim_{s,p} \int_T \frac{|f_i \circ \boldsymbol{F}_{ij}(\boldsymbol{x}) - f_j \circ \boldsymbol{F}_{ji}(\boldsymbol{x})|^p}{x_2^{sp}} \,\mathrm{d}\boldsymbol{x}.$$

Thanks to conditions (ii)-(iii), the function $g = f_i \circ \boldsymbol{F}_{ij} - f_j \circ \boldsymbol{F}_{ji}$ belongs to $W^{s,p}_{\gamma_2}(T)$ and applying (A.8) and the triangle inequality gives

$$\int_T \frac{|f_i \circ \boldsymbol{F}_{ij}(\boldsymbol{x}) - f_j \circ \boldsymbol{F}_{ji}(\boldsymbol{x})|^p}{x_2^{sp}} \,\mathrm{d}\boldsymbol{x} \lesssim_{s,p} |||f|||^p_{s,p,\Gamma_i \cup \Gamma_j}$$

and so

$$\int_T \int_T \frac{|f_i \circ \boldsymbol{F}_{ij}(\boldsymbol{x}) - f_j \circ \boldsymbol{F}_{ji}(\boldsymbol{y})|^p}{|\boldsymbol{x}_r - \boldsymbol{y}|^{sp+2}} \,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \lesssim_{s,p} |||f|||^p_{s,p,\Gamma_i \cup \Gamma_j}.$$

Then, the reverse inequality of (B.3) immediately follows from (B.1), and so $f \in W^{s,p}(\Gamma_i \cup \Gamma_j)$ and the result follows. ∎

On noting that $f \in W^{s,p}(\partial K)$ if and only if $f \in L^p(\partial K)$ and $f|_{\Gamma_i \cup \Gamma_j} \in W^{s,p}(\Gamma_i \cup \Gamma_j)$ for all $1 \leq i < j \leq 4$ since

$$|f|^p_{s,p,\partial K} = \sum_{1 \leq i,j \leq 4} \int_{\Gamma_i} \int_{\Gamma_j} \frac{|f_i(\boldsymbol{x}) - f_j(\boldsymbol{y})|^p}{|\boldsymbol{x} - \boldsymbol{y}|^{sp+2}} \,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \approx_{s,p} \sum_{1 \leq i < j \leq 4} |f|^p_{s,p,\Gamma_i \cup \Gamma_j},$$

the following result is an immediate consequence of Lemma B.1.

COROLLARY B.2. *Let $0 \leq s < 1$ and $1 < p < \infty$. Then, $f \in L^p(\partial K)$ satisfies $f \in W^{s,p}(\partial K)$ if and only if*

(i) $f_i \in W^{s,p}(\Gamma_i)$ for $1 \leq i \leq 4$;

(ii) if $s > 1/p$, then $f_i|_{\gamma_{ij}} = f_j|_{\gamma_{ij}}$ for all $1 \le i < j \le 4$; and
(iii) if $s = 1/p$, then $I^p_{ij}(f_i, f_j) < \infty$ for all $1 \le i < j \le 4$.
Additionally, (2.3) *holds.*

Appendix C. Partial fractions decomposition.

LEMMA C.1. *For all $\beta \in \mathbb{N}_0^3$ with $|\beta| \ge 2$, there holds*

$$\frac{1}{\omega_1^{\beta_1}\omega_2^{\beta_2}\omega_3^{\beta_3}} = \sum_{\substack{\alpha\in\mathbb{N}_0^3 \\ \alpha_j \le \beta_j \\ |\alpha|\ge 2}} \left(\frac{c_{\alpha,1}}{\omega_1^{\alpha_1}\omega_2^{\alpha_2}} + \frac{c_{\alpha,2}}{\omega_1^{\alpha_1}\omega_3^{\alpha_3}} + \frac{c_{\alpha,3}}{\omega_2^{\alpha_2}\omega_3^{\alpha_3}} \right) \qquad \text{in } T, \tag{C.1}$$

where $\{c_{\alpha,j}\}$ are suitable positive constants.

Proof. We proceed by induction on $|\beta|$. The case $|\beta| = 2$ is trivially true. Assume that (C.1) holds for all $\beta \in \mathbb{N}_0^3$ with $|\beta| = r \ge 2$. Let $\beta \in \mathbb{N}_0^3$ with $|\beta| = r + 1$. If $\beta_j = 0$ for some $j \in \{1, 2, 3\}$, then (C.1) is trivially true, so assume that $\beta_j > 0$ for $1 \le j \le 3$. Then,

$$\begin{aligned}
\frac{1}{\omega_1^{\beta_1}\omega_2^{\beta_2}\omega_3^{\beta_3}} &= \frac{1}{\omega_1\omega_2\omega_3} \cdot \frac{1}{\omega_1^{\beta_1-1}\omega_2^{\beta_2-1}\omega_3^{\beta_3-1}} \\
&= \left(\frac{1}{\omega_1\omega_2} + \frac{1}{\omega_1\omega_3} + \frac{1}{\omega_2\omega_3} \right) \frac{1}{\omega_1^{\beta_1-1}\omega_2^{\beta_2-1}\omega_3^{\beta_3-1}} \\
&= \frac{1}{\omega_1^{\beta_1}\omega_2^{\beta_2}\omega_3^{\beta_3-1}} + \frac{1}{\omega_1^{\beta_1}\omega_2^{\beta_2-1}\omega_3^{\beta_3}} + \frac{1}{\omega_1^{\beta_1-1}\omega_2^{\beta_2}\omega_3^{\beta_3}}.
\end{aligned}$$

By assumption, each of the three terms above is of the form (C.1), which completes the proof. ∎

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