A17939W1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part B: Paper B6.1
Honour School of Mathematics and Computer Science Part B: Paper B6.1
Honour School of Mathematics and Statistics Part B: Paper B6.1

NUMERICAL SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

TRINITY TERM 2023

Tuesday 6 June, 2:30pm to 4:15pm

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

You should ensure that you observe the following points:

- start a new answer booklet for each question which you attempt.
- indicate on the front page of the answer booklet which question you have attempted in that booklet.
- cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.
- hand in your answers in numerical order.

If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.

Do not turn this page until you are told that you may do so

1. Let $f, g \in C([0,1])$ and $A, B \in \mathbb{R}$. Suppose that $u \in C^2([0,1])$ and $v \in C^2([0,1])$ are the solutions of the boundary-value problems, respectively,

$$-u''(x) + u(x) + u^3(x) = f(x), \quad x \in (0,1), \qquad u(0) = A, \quad u(1) = B \tag{1}$$

and

$$-v''(x) + v(x) + v^3(x) = g(x), \quad x \in (0,1), \qquad v(0) = A, \quad v(1) = B. \tag{2}$$

(a) [7 marks] Show that

$$\int_0^1 [(u'(x) - v'(x))^2 + (u(x) - v(x))^2 + (u^3(x) - v^3(x))(u(x) - v(x))] \,\mathrm{d}x$$
$$= \int_0^1 (f(x) - g(x))(u(x) - v(x)) \,\mathrm{d}x.$$

Hence deduce that

$$\int_0^1 (u'(x) - v'(x))^2 \,\mathrm{d}x + \frac{1}{2}\int_0^1 (u(x) - v(x))^2 \,\mathrm{d}x \leqslant \frac{1}{2}\int_0^1 (f(x) - g(x))^2 \,\mathrm{d}x.$$

(b) [7 marks] On a finite difference mesh $\overline{\Omega}_h := \{x_i := ih \,:\, i = 0, \dots, N\}$ of spacing $h := 1/N$, where $N \geqslant 2$, construct finite difference schemes for the numerical solution of the boundary-value problems (1) and (2), where the second derivatives of u and v have been approximated by second-order central difference operators. Denoting by U and V the finite difference approximations on the mesh $\overline{\Omega}_h$ of u and v, respectively, show that

$$\|D_x^- U - D_x^- V]|_h^2 + \frac{1}{2}\|U - V\|_h^2 \leqslant \frac{1}{2}\|f - g\|_h^2,$$

where D_x^- denotes the first-order backward difference operator, and the norms $\|\cdot]|_h$ and $\|\cdot\|_h$ are defined by $\|V]|_h := (V,V]_h^{1/2}$ and $\|V\|_h := (V,V)_h^{1/2}$, respectively, with

$$(V, W]_h := \sum_{i=1}^{N} h V_i W_i \quad \text{and} \quad (V, W)_h := \sum_{i=1}^{N-1} h V_i W_i.$$

(c) [6 marks] Let $\varphi_i := D_x^+ D_x^- u(x_i) - u''(x_i)$ for $i = 1, \dots, N-1$. Show that

$$\|D_x^- U - D_x^- u]|_h^2 + \frac{1}{2}\|U - u\|_h^2 \leqslant \frac{1}{2}\|\varphi\|_h^2.$$

(d) [5 marks] Assuming that $u \in C^4([0,1])$, show that there exists a positive constant C, independent of h, whose value you should specify, such that

$$\|U - u\|_{1,h} \leqslant Ch^2,$$

where $\|\cdot\|_{1,h}$ is a discrete Sobolev norm that you should carefully define.

2. Suppose that $\Omega := (0,1)^2$ and let $\mathbb{N}$ denote the set of all positive integers.

(a) [5 marks] Show that for each pair $(m,n) \in \mathbb{N}\times\mathbb{N}$ there exists a positive real number $\lambda_{m,n}$, which you should find, such that the function $u^{(m,n)}$ defined by

$$u^{(m,n)}(x,y) := \sin(m\pi x)\sin(n\pi y)$$

satisfies the elliptic partial differential equation

$$-\Delta u^{(m,n)}(x,y) + u^{(m,n)}(x,y) = \lambda_{m,n} u^{(m,n)}(x,y), \qquad (x,y) \in \Omega,$$

and the Dirichlet boundary condition $u^{(m,n)}|_{\partial\Omega} = 0$.

(b) [9 marks] On the finite difference mesh

$$\overline{\Omega}_h := \{(x_i, y_j) \,:\, x_i := ih,\ y_j := jh,\ i,j = 0,\dots,N\}$$

of spacing $h := 1/N$ in both coordinate directions, where $N \geqslant 2$, consider the five-point finite difference scheme

$$-\Delta_h U_{i,j}^{(m,n)} + U_{i,j}^{(m,n)} = \Lambda_{m,n} U_{i,j}^{(m,n)}, \qquad i,j = 1,\dots,N-1, \tag{3}$$

for the numerical solution of the partial differential equation stated in part (a), subject to the boundary condition $U_{0,j}^{(m,n)} = U_{N,j}^{(m,n)} = U_{i,0}^{(m,n)} = U_{i,N}^{(m,n)} = 0$ for all $i,j = 0,\dots,N$, where Δ_h is a finite difference operator that you should carefully define.

Show that for each pair $(m,n) \in \{1,\dots,N-1\} \times \{1,\dots,N-1\}$ there exists a positive real number $\Lambda_{m,n}$, which you should find, such that

$$U_{i,j}^{(m,n)} = \sin(m\pi x_i)\sin(n\pi y_j), \qquad i,j = 0,\dots,N,$$

satisfies the finite difference scheme (3).

(c) [6 marks] Determine a positive real number C, independent of h, such that

$$|\lambda_{1,1} - \Lambda_{1,1}| \leqslant Ch^2.$$

(d) [5 marks] Show further that $\lambda_{N-1,N-1} \geqslant \Lambda_{N-1,N-1}$ for all $N \geqslant 3$, and

$$\lim_{N\to+\infty} (\lambda_{N-1,N-1} - \Lambda_{N-1,N-1}) = +\infty.$$

3. Consider the initial-value problem

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} &= a\frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \quad t > 0, \\ u(x,0) &= u_0(x), \qquad -\infty < x < \infty, \end{aligned} \tag{4}$$

where $a \geqslant 0$ and u_0 is a real-valued, bounded and continuous function of $x \in (-\infty, \infty)$.

Let $\mathbb{Z}$ denote the set of all integers and consider a finite difference mesh of spacing $\Delta x > 0$ in the x-direction and $\Delta t > 0$ in the positive t-direction.

(a) [4 marks] Formulate an implicit finite difference scheme for the numerical solution of the initial-value problem (4), with U_j^m denoting the finite difference approximation of $u(j\Delta x, m\Delta t)$ for $j \in \mathbb{Z}$ and $m = 0, 1, \ldots$, where $\frac{\partial^2 u}{\partial x^2}(j\Delta x, m\Delta t)$ is approximated by $D_x^+ D_x^- U_j^m := (U_{j+1}^m - 2U_j^m + U_{j-1}^m)/(\Delta x)^2$ and where $\frac{\partial u}{\partial x}(j\Delta x, m\Delta t)$ is approximated by $D_x^0 U_j^m := (U_{j+1}^m - U_{j-1}^m)/(2\Delta x)$.

(b) [6 marks] Suppose further that $U_j^0 := u_0(x_j)$, $j \in \mathbb{Z}$, and that

$$\|U^0\|_{\ell_2} := \left(\Delta x \sum_{j \in \mathbb{Z}} |U_j^0|^2 \right)^{1/2}$$

is finite. Show that the implicit scheme from part (a) is unconditionally practically stable in the ℓ_2 norm.

(c) [4 marks] Formulate an explicit finite difference scheme for the numerical solution of the initial-value problem (4), with U_j^m denoting the finite difference approximation of $u(j\Delta x, m\Delta t)$ for $j \in \mathbb{Z}$ and $m = 0, 1, \ldots$, where $\frac{\partial^2 u}{\partial x^2}(j\Delta x, m\Delta t)$ is approximated by $D_x^+ D_x^- U_j^m$ and where $\frac{\partial u}{\partial x}(j\Delta x, m\Delta t)$ is approximated by $D_x^0 U_j^m$.

(d) [6 marks] Let $\nu := \Delta t/\Delta x$, $\mu := \Delta t/(\Delta x)^2$, and suppose that U_j^0 is as in part (b). Show that if $a > 0$ and

$$\nu^2 \leqslant 2a\mu \leqslant 1,$$

then the explicit finite difference scheme from part (c) is practically stable in the ℓ_2 norm.

(e) [5 marks] Show further that if $a = 0$, then there exists a $c > 0$ and a mesh function U^0 with $\|U^0\|_{\ell_2} < \infty$, such that

$$\|U^m\|_{\ell_2} \geqslant (1 + c)^m \|U^0\|_{\ell_2} \quad \text{for all } m \geqslant 1$$

and all $\Delta x > 0$ and $\Delta t > 0$. Hence deduce that if $a = 0$ then there is no choice of $\Delta x > 0$ and $\Delta t > 0$ for which the explicit finite difference scheme from part (c) is practically stable in the ℓ_2 norm.

[The discrete version of Parseval's identity for the semidiscrete Fourier transform may be used without proof.]