SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C6.4
Honour School of Mathematics and Statistics Part C: Paper C6.4
Master of Science in Mathematical Sciences: Paper C6.4

Finite Element Methods for Partial Differential Equations

TRINITY TERM 2023

Tuesday 06 June, 2:30pm to 4:15pm

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

You should ensure that you observe the following points:

- start a new answer booklet for each question which you attempt.
- indicate on the front page of the answer booklet which question you have attempted in that booklet.
- cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.
- hand in your answers in numerical order.

If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.

Do not turn this page until you are told that you may do so

1. Suppose that $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain and let Δ denote the Laplace operator in d space-dimensions. Suppose further that $f \in L^2(\Omega)$ and consider the boundary-value problem

$$\begin{aligned} \Delta^2 u + u = f \qquad & \text{in } \Omega, \\ u = 0 \quad \text{and} \quad \Delta u = 0 \qquad & \text{on } \partial\Omega. \end{aligned} \tag{1}$$

(a) [6 marks] By introducing the substitution $w := -\Delta u$, show that the boundary-value problem (1) can be rewritten as the following system of second-order elliptic boundary-value problems:

$$\begin{aligned} -\Delta u - w = 0 \qquad & \text{in } \Omega, \\ u - \Delta w = f \qquad & \text{in } \Omega, \\ u = 0 \quad \text{and} \quad w = 0 \qquad & \text{on } \partial\Omega. \end{aligned} \tag{2}$$

State the weak formulation of the boundary-value problem (2), and show by applying the Lax–Milgram theorem on the Hilbert space $\mathcal{V} := H_0^1(\Omega) \times H_0^1(\Omega)$ that (2) has a unique weak solution $(u, w) \in \mathcal{V}$.

[*You may assume the Poincaré–Friedrichs inequality, asserting the existence of a positive constant* $K = K(\Omega)$ *such that* $\|v\|^2_{L^2(\Omega)} \leqslant K\|\nabla v\|^2_{L^2(\Omega)}$ *for all* $v \in H_0^1(\Omega)$.]

(b) [8 marks] Now suppose that $\mathcal{U}_h$ and $\mathcal{W}_h$ are finite-dimensional subspaces of $H_0^1(\Omega)$ and define $\mathcal{V}_h := \mathcal{U}_h \times \mathcal{W}_h$. Based on the weak formulation from part (a), formulate a Galerkin approximation over the subspace $\mathcal{V}_h$ of $\mathcal{V}$ to the boundary-value problem (2), and show that it has a unique solution $(u_h, w_h) \in \mathcal{V}_h$.

Show further that

$$\begin{aligned} &\|\nabla(u - u_h)\|^2_{L^2(\Omega)} + \|\nabla(w - w_h)\|^2_{L^2(\Omega)} \\ &\leqslant \left(\|u - u_h\|^2_{H^1(\Omega)} + \|w - w_h\|^2_{H^1(\Omega)}\right)^{\frac{1}{2}} \left(\min_{v_h \in \mathcal{U}_h} \|u - v_h\|^2_{H^1(\Omega)} + \min_{z_h \in \mathcal{W}_h} \|w - z_h\|^2_{H^1(\Omega)}\right)^{\frac{1}{2}}. \end{aligned}$$

Hence deduce that there exists a positive real number C, independent of h, such that

$$\left(\|u - u_h\|^2_{H^1(\Omega)} + \|w - w_h\|^2_{H^1(\Omega)}\right)^{\frac{1}{2}} \leqslant C \left(\min_{v_h \in \mathcal{U}_h} \|u - v_h\|^2_{H^1(\Omega)} + \min_{z_h \in \mathcal{W}_h} \|w - z_h\|^2_{H^1(\Omega)}\right)^{\frac{1}{2}}.$$

(c) [4 marks] Let $\Omega \subset \mathbb{R}^2$ be a bounded open polygonal domain and suppose that $\{\mathcal{M}_h\}_{h>0}$ is a shape-regular mesh sequence on $\overline{\Omega}$, indexed by the mesh size h, consisting of closed triangular cells. Suppose further that $\mathcal{U}_h$ and $\mathcal{W}_h$ consist of continuous piecewise linear functions defined on $\mathcal{M}_h$. Show that, if both u and w belong to $H^2(\Omega) \cap H_0^1(\Omega)$, then there exists a positive real number $C_* = C_*(u, w)$, independent of h, such that

$$\left(\|u - u_h\|^2_{H^1(\Omega)} + \|w - w_h\|^2_{H^1(\Omega)}\right)^{\frac{1}{2}} \leqslant C_* h.$$

[*Bounds on the error between a function* $v \in H^2(\Omega) \cap H_0^1(\Omega)$ *and its continuous piecewise linear interpolant* $I_h v$ *may be used without proof.*]

(d) [7 marks] Let N denote the number of nodes of the mesh $\mathcal{M}_h$ from part (c) that are contained in Ω, and let $\mathcal{U}_h$ and $\mathcal{W}_h$ be as in part (c). Show that if $\mathcal{U}_h = \mathcal{W}_h$, then the finite element approximation of (2) can be restated as a system of linear algebraic equations whose matrix has the form

$$A = \begin{pmatrix} S & -M \\ M & S \end{pmatrix},$$

where the blocks S and M are $N \times N$ symmetric positive definite matrices, whose entries you should carefully define in terms of the basis functions ϕ_i, $i = 1, \ldots, N$, of $\mathcal{U}_h = \mathcal{W}_h$. Show further that the matrix A is positive definite.

2. (a) [5 marks] Consider the degree-2 Lagrange CG_2 element on a nondegenerate triangle K, shown in Fig. 1 below. Let $\mathcal{V} := \mathcal{P}_2(K)$, where

$$\mathcal{P}_2(K) := \operatorname{span}\{x_1^{\alpha_1} x_2^{\alpha_2}|_K \,:\, \alpha_1 + \alpha_2 \leqslant 2,\ \alpha_i \in \mathbb{N}_{\geqslant 0} \text{ for } i = 1, 2\},$$

with (x_1, x_2) signifying the Cartesian coordinates for K and the degrees of freedom, indicated with the black dots at the three vertices and at the midpoints of the edges of K, referring to point evaluation.

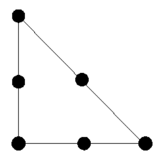

Figure 1: Degrees of freedom for the element CG_2 on the triangle K.

Prove that this element is unisolvent.

(b) [5 marks] Suppose that $\Omega \subset \mathbb{R}^2$ is a triangle. Let Γ_D be the union of two edges (including their end-points) of Ω and let Γ_N denote the remaining edge (excluding its end-points), so that $\Gamma_D \cup \Gamma_N = \partial\Omega$. Let n denote the unit outward normal vector to Γ_N. For $f \in L^2(\Omega)$ consider the elliptic boundary-value problem

$$\begin{aligned} -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_D, \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \Gamma_N. \end{aligned} \tag{3}$$

Show that the boundary-value problem (3) has a unique weak solution u in a suitable subspace $\mathcal{V}$ of $H^1(\Omega)$, which you should carefully define.

(c) [4 marks] Let $\{\mathcal{M}_h\}_{h>0}$ be a shape-regular mesh sequence on $\overline{\Omega}$, indexed by the mesh size h, consisting of closed triangular cells. Suppose further that $\mathcal{V}_h \subset \mathcal{V}$ consists of continuous piecewise quadratic functions v_h; that is, on each triangle K of the mesh, $v_h|_K$ is a linear combination of monic polynomials of the form $x^{\alpha_1} y^{\alpha_2}$, where $\alpha_1, \alpha_2 \in \mathbb{N}_{\geqslant 0}$ with $\alpha_1 + \alpha_2 \leqslant 2$.

Show that the finite element approximation of problem (3) posed on $\mathcal{M}_h$ has a unique solution $u_h \in \mathcal{V}_h$.

(d) [5 marks] Let u and u_h be as in parts (b) and (c) of the question, respectively, and suppose that $u \in H^3(\Omega) \cap \mathcal{V}$.

Show that there exists a positive constant C_1, independent of h, such that

$$\|u - u_h\|_{H^1(\Omega)} \leqslant C_1 h^2 \|u\|_{H^3(\Omega)}.$$

[*Bounds on the error between a function v and its continuous piecewise quadratic interpolant $I_h v \in \mathcal{V}_h$ may be used without proof.*]

(e) [6 marks] Suppose that Ω is an acute triangle, that is, a triangle with three acute angles ($< \pi/2$). By using the Aubin–Nitsche duality argument show further that

$$\|u - u_h\|_{L^2(\Omega)} \leqslant C_2 h^3 \|u\|_{H^3(\Omega)},$$

where C_2 is a positive constant, independent of h.

[*You may use without proof the following result. If Ω is an acute triangle and $g \in L^2(\Omega)$, then there exists a unique $w \in H^2(\Omega)$ such that $\Delta w = g$ in Ω, $w|_{\Gamma_D} = 0$, $\frac{\partial w}{\partial n}|_{\Gamma_N} = 0$, and there exists a positive constant C_3 such that $\|w\|_{H^2(\Omega)} \leqslant C_3 \|g\|_{L^2(\Omega)}$.*]

3. Suppose that $\Omega \subset \mathbb{R}^3$ is a Lipschitz domain, $f \in L^2(\Omega; \mathbb{R}^3)$, and μ and λ are positive constants. Consider the functional $J : H^1(\Omega; \mathbb{R}^3) \to \mathbb{R}$ defined by

$$J(v) = \frac{1}{2} a(v, v) - \ell(v), \qquad v \in H^1(\Omega; \mathbb{R}^3),$$

where the bilinear functional $a(\cdot, \cdot)$ and the linear functional $\ell(\cdot)$ are defined, respectively, by

$$a(w, v) := \int_\Omega 2\mu\, \varepsilon(w(x)) : \varepsilon(v(x)) + \lambda \operatorname{div}(w(x)) \operatorname{div}(v(x))\, \mathrm{d}x, \quad \ell(v) := \int_\Omega f(x) \cdot v(x)\, \mathrm{d}x.$$

For an $\mathbb{R}^3$-valued vector function $v = (v_1, v_2, v_3)^{\mathrm{T}} \in H^1(\Omega; \mathbb{R}^3)$, $\varepsilon(v) \in L^2(\Omega; \mathbb{R}^{3\times 3})$ is the $\mathbb{R}^{3\times 3}$-valued matrix function such that

$$(\varepsilon(v))_{i,j} := \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right), \qquad i, j = 1, 2, 3.$$

Finally, for two matrices $A, B \in \mathbb{R}^{3\times 3}$ the Frobenius inner product ":" is defined by

$$A : B := \sum_{i,j=1}^{3} A_{i,j} B_{i,j}$$

and "$\cdot$" signifies the dot-product in $\mathbb{R}^3$.

(a) [6 marks] Show that J is a convex functional on $H^1(\Omega; \mathbb{R}^3)$; that is, for each pair of functions $v, w \in H^1(\Omega; \mathbb{R}^3)$ and for each $\theta \in [0, 1]$,

$$J(\theta v + (1 - \theta) w) \leqslant \theta J(v) + (1 - \theta) J(w).$$

(b) [8 marks] Suppose that $u \in H^1_0(\Omega; \mathbb{R}^3)$ is such that $J(v) \geqslant J(u)$ for all $v \in H^1_0(\Omega; \mathbb{R}^3)$. Show that u is then the unique weak solution of the boundary-value problem

$$\begin{aligned} -2\mu \operatorname{div}(\varepsilon(u)) - \lambda \operatorname{grad}(\operatorname{div} u) &= f \qquad \text{on } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{4}$$

with the divergence $\operatorname{div}(\varepsilon(u))$ of the matrix function $\varepsilon(u)$ taken row-wise.

(c) [5 marks] Suppose that $\mathcal{V}_h$ is a finite-dimensional subspace of $H^1_0(\Omega; \mathbb{R}^3)$.
Show that there exists a unique $u_h \in \mathcal{V}_h$ such that

$$a(u_h, v_h) = \ell(v_h) \qquad \forall\, v_h \in \mathcal{V}_h.$$

Show further that

$$J(v_h) \geqslant J(u_h) \geqslant J(u) \qquad \text{for all } v_h \in \mathcal{V}_h.$$

(d) [6 marks] Consider the *energy norm* $\|\cdot\|_a$ on $H^1_0(\Omega; \mathbb{R}^3)$ defined by $\|v\|_a := [a(v, v)]^{\frac{1}{2}}$.
Verify that $\|\cdot\|_a$ is indeed a norm.
Show further that u_h is the best approximation from $\mathcal{V}_h$ to u in the energy norm $\|\cdot\|_a$ in the sense that

$$\|u - u_h\|_a = \min_{v_h \in \mathcal{V}_h} \|u - v_h\|_a.$$

[*The Poincaré–Friedrichs inequality (cf. part (a) of Question 1) and the following result, known as Korn's inequality, may be used without proof. There exists a positive constant $C = C(\Omega)$ such that*

$$\sum_{i,j=1}^{3} \left\| \frac{\partial v_i}{\partial x_j} \right\|^2_{L^2(\Omega)} \leqslant C \sum_{i,j=1}^{3} \left\| (\varepsilon(v))_{i,j} - \frac{1}{3} \delta_{i,j} \operatorname{div}(v) \right\|^2_{L^2(\Omega)} \qquad \textit{for all } v \in H^1_0(\Omega; \mathbb{R}^3).\ ]$$