EXISTENCE OF GLOBAL WEAK SOLUTIONS TO COMPRESSIBLE ISENTROPIC FINITELY EXTENSIBLE NONLINEAR BEAD-SPRING CHAIN MODELS FOR DILUTE POLYMERS: THE TWO-DIMENSIONAL CASE

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Abstract. We prove the existence of global-in-time weak solutions to a general class of models that arise from the kinetic theory of dilute solutions of nonhomogeneous polymeric liquids, where the polymer molecules are idealized as bead-spring chains with finitely extensible nonlinear elastic (FENE) type spring potentials. The class of models under consideration involves the unsteady, compressible, isentropic, isothermal Navier–Stokes system in a bounded domain Ω in $\mathbb{R}^d$, $d = 2$, for the density ρ, the velocity $\underset{\sim}{u}$ and the pressure p of the fluid, with an equation of state of the form $p(\rho) = c_p \rho^\gamma$, where c_p is a positive constant and $\gamma > 1$. The right-hand side of the Navier–Stokes momentum equation includes an elastic extra-stress tensor, which is the classical Kramers expression. The elastic extra-stress tensor stems from the random movement of the polymer chains and is defined through the associated probability density function that satisfies a Fokker–Planck-type parabolic equation, a crucial feature of which is the presence of a centre-of-mass diffusion term. This extends the result in our paper [*J.W. Barrett & E. Süli: Existence of global weak solutions to compressible isentropic finitely extensible bead-spring chain models for dilute polymers, Math. Models Methods Appl. Sci., **26** (2016)*], which established the existence of global-in-time weak solutions to the system for $d \in \{2, 3\}$ and $\gamma > \frac{3}{2}$, but the elastic extra-stress tensor required there the addition of a quadratic interaction term to the classical Kramers expression to complete the compactness argument on which the proof was based. We show here that in the case of $d = 2$ and $\gamma > 1$ the existence of global-in-time weak solutions can be proved in the absence of the quadratic interaction term. Our results require no structural assumptions on the drag term in the Fokker–Planck equation; in particular, the drag term need not be corotational. With a nonnegative initial density $\rho_0 \in L^\infty(\Omega)$ for the continuity equation; a square-integrable initial velocity datum $\underset{\sim}{u}_0$ for the Navier–Stokes momentum equation; and a nonnegative initial probability density function ψ_0 for the Fokker–Planck equation, which has finite relative entropy with respect to the Maxwellian M associated with the spring potential in the model, we prove, *via* a limiting procedure on a pressure regularization parameter, the existence of a global-in-time bounded-energy weak solution $t \mapsto (\rho(t), \underset{\sim}{u}(t), \psi(t))$ to the coupled Navier–Stokes–Fokker–Planck system, satisfying the initial condition $(\rho(0), \underset{\sim}{u}(0), \psi(0)) = (\rho_0, \underset{\sim}{u}_0, \psi_0)$.

Keywords: Kinetic polymer models, FENE chain, compressible Navier–Stokes–Fokker–Planck system, variable density, nonhomogeneous dilute polymer

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1. Introduction

In Barrett & Süli [9] we established the existence of global-in-time weak solutions to a large class of bead-spring chain models with finitely extensible nonlinear elastic (FENE) type spring potentials, — a system of nonlinear partial differential equations that arises from the kinetic theory of dilute polymer solutions. We first restate the model considered there, and we then discuss the aims of the extensions of the results of [9] contained in the present paper.

In the model studied in [9], the solvent is a compressible, isentropic, viscous, isothermal Newtonian fluid confined to a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, with boundary $\partial\Omega$. For the sake of simplicity of presentation, Ω is assumed to have a 'solid boundary' $\partial\Omega$; the velocity field $\underset{\sim}{u}$ will then satisfy the no-slip boundary condition $\underset{\sim}{u} = \underset{\sim}{0}$ on $\partial\Omega$. The equations of continuity and balance of linear momentum have the form of the compressible Navier–Stokes equations (cf. Lions [18], Feireisl [14], Novotný & Straškraba [20], or Feireisl & Novotný [15]) in which the *elastic* *extra-stress* tensor $\underset{\approx}{\tau}$ (i.e., the polymeric part of the Cauchy stress tensor) appears as a source term in the conservation of momentum equation:

Given $T \in \mathbb{R}_{>0}$, find $\rho : (\underset{\sim}{x}, t) \in \Omega \times [0,T] \mapsto \rho(\underset{\sim}{x}, t) \in \mathbb{R}_{\geq 0}$ and $\underset{\sim}{u} : (\underset{\sim}{x}, t) \in \overline{\Omega} \times [0,T] \mapsto \underset{\sim}{u}(\underset{\sim}{x}, t) \in \mathbb{R}^d$ such that

$$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla}_x \cdot (\underset{\sim}{u}\, \rho) = 0 \qquad \text{in } \Omega \times (0,T], \tag{1.1a}$$

$$\rho(\underset{\sim}{x}, 0) = \rho_0(\underset{\sim}{x}) \qquad \forall \underset{\sim}{x} \in \Omega, \tag{1.1b}$$

$$\frac{\partial(\rho\, \underset{\sim}{u})}{\partial t} + \underset{\sim}{\nabla}_x \cdot (\rho\, \underset{\sim}{u} \otimes \underset{\sim}{u}) - \underset{\sim}{\nabla}_x \cdot \underset{\approx}{S}(\underset{\sim}{u}, \rho) + \underset{\sim}{\nabla}_x\, p(\rho) = \rho\, \underset{\sim}{f} + \underset{\sim}{\nabla}_x \cdot \underset{\approx}{\tau} \qquad \text{in } \Omega \times (0,T], \tag{1.1c}$$

$$\underset{\sim}{u} = \underset{\sim}{0} \qquad \text{on } \partial\Omega \times (0,T], \tag{1.1d}$$

$$(\rho\, \underset{\sim}{u})(\underset{\sim}{x}, 0) = (\rho_0\, \underset{\sim}{u}_0)(\underset{\sim}{x}) \qquad \forall \underset{\sim}{x} \in \Omega. \tag{1.1e}$$

It is assumed that each of the equations above has been written in its nondimensional form; ρ denotes a nondimensional solvent density, $\underset{\sim}{u}$ is a nondimensional solvent velocity, defined as the velocity field scaled by the characteristic flow speed U_0. Here $\underset{\approx}{S}(\underset{\sim}{u}, \rho)$ is the Newtonian part of the viscous stress tensor defined by

$$\underset{\approx}{S}(\underset{\sim}{u}, \rho) := \mu^S(\rho) \left[\underset{\approx}{D}(\underset{\sim}{u}) - \frac{1}{d} (\underset{\sim}{\nabla}_x \cdot \underset{\sim}{u})\, \underset{\approx}{I} \right] + \mu^B(\rho)\, (\underset{\sim}{\nabla}_x \cdot \underset{\sim}{u})\, \underset{\approx}{I}, \tag{1.2}$$

where $\underset{\approx}{I}$ is the $d \times d$ identity tensor, $\underset{\approx}{D}(\underset{\sim}{v}) := \frac{1}{2} (\underset{\approx}{\nabla}_x\, \underset{\sim}{v} + (\underset{\approx}{\nabla}_x\, \underset{\sim}{v})^{\mathrm{T}})$ is the rate of strain tensor, with $(\underset{\approx}{\nabla}_x\, \underset{\sim}{v})(\underset{\sim}{x}, t) \in \mathbb{R}^{d \times d}$ and $\big(\underset{\approx}{\nabla}_x\, \underset{\sim}{v}\big)_{ij} = \frac{\partial v_i}{\partial x_j}$. The shear viscosity, $\mu^S(\cdot) \in \mathbb{R}_{>0}$, and the bulk viscosity, $\mu^B(\cdot) \in \mathbb{R}_{\geq 0}$, of the solvent are both scaled and, generally, density-dependent.

In addition, p is the nondimensional pressure satisfying the isentropic equation of state

$$p(s) = c_p\, s^\gamma \qquad \forall s \in \mathbb{R}_{\geq 0}, \tag{1.3}$$

where $c_p \in \mathbb{R}_{>0}$ and $\gamma > \frac{d}{2}$. Our analysis also applies, without alterations, to other monotonic equations of state, see Remark 1.1 in [9].

On the right-hand side of (1.1c), $\underset{\sim}{f}$ is the nondimensional density of body forces and $\underset{\approx}{\tau}$ denotes the elastic extra-stress tensor. In a *bead-spring chain model*, consisting of $K + 1$ beads coupled with K elastic springs to represent a polymer chain, $\underset{\approx}{\tau}$ is defined by a version of the *Kramers expression* depending on the probability density function ψ of the (random) conformation vector $\underset{\sim}{q} := (\underset{\sim}{q}_1^{\mathrm{T}}, \dots, \underset{\sim}{q}_K^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{Kd}$ of the chain (see equation (1.10) below), with $\underset{\sim}{q}_i$ representing the d-component conformation/orientation vector of the ith spring, $i = 1, \dots, K$. The Kolmogorov equation satisfied by ψ is a second-order parabolic equation, the Fokker–Planck equation, whose transport coefficients depend on the velocity field $\underset{\sim}{u}$, and the hydrodynamic drag coefficient appearing in the Fokker–Planck equation is, generally, a nonlinear function of the density ρ.

The domain D of admissible conformation vectors $D \subset \mathbb{R}^{Kd}$ will be assumed to be a K-fold Cartesian product $D_1 \times \cdots \times D_K$ of bounded open balls $D_i = B(\underset{\sim}{0}, b_i^{\frac{1}{2}})$ in $\mathbb{R}^d$ centred at $\underset{\sim}{0} \in \mathbb{R}^d$ and of radius $b_i^{\frac{1}{2}}$, with $b_i \in \mathbb{R}_{>0}$, $i = 1, \dots, K$. We consider the spring potentials $U_i \in C^1([0, \frac{b_i}{2}); \mathbb{R}_{\geq 0})$, $i = 1, \dots, K$, and will suppose that $U_i(0) = 0$ and that U_i is unbounded on $[0, \frac{b_i}{2})$ for each $i = 1, \dots, K$; the resulting bead-spring chain models are referred to as FENE (finitely extensible nonlinear elastic) type models; in the case of $K = 1$, the corresponding models are called FENE type *dumbbell models*.

The elastic spring-force $\underset{\sim}{F}_i : D_i \subseteq \mathbb{R}^d \to \mathbb{R}^d$ of the ith spring in the chain is defined by

$$\underset{\sim}{F}_i(\underset{\sim}{q}_i) := U_i'(\tfrac{1}{2}|\underset{\sim}{q}_i|^2)\, \underset{\sim}{q}_i, \qquad \text{with } \underset{\sim}{q}_i \in D_i \text{ and } i = 1, \dots, K. \tag{1.4}$$

The partial Maxwellian M_i, associated with the spring potential U_i, is defined by

$$M_i(\underset{\sim}{q}_i) := \frac{1}{\mathcal{Z}_i} \mathrm{e}^{-U_i(\frac{1}{2}|\underset{\sim}{q}_i|^2)}, \qquad \mathcal{Z}_i := \int_{D_i} \mathrm{e}^{-U_i(\frac{1}{2}|\underset{\sim}{q}_i|^2)}\, \mathrm{d}\underset{\sim}{q}_i, \qquad i = 1, \dots, K.$$

The (total) Maxwellian in the model is then

$$M(\underset{\sim}{q}) := \prod_{i=1}^{K} M_i(\underset{\sim}{q}_i), \qquad \underset{\sim}{q} := (\underset{\sim}{q}_1^{\mathrm{T}}, \dots, \underset{\sim}{q}_K^{\mathrm{T}})^{\mathrm{T}} \in D := \bigtimes_{i=1}^{K} D_i. \tag{1.5}$$

Observe that, for $i = 1, \dots, K$ and $\underset{\sim}{q} \in D$,

$$M(\underset{\sim}{q})\, \underset{\sim}{\nabla}_{q_i} [M(\underset{\sim}{q})]^{-1} = -[M(\underset{\sim}{q})]^{-1}\, \underset{\sim}{\nabla}_{q_i} M(\underset{\sim}{q}) = \underset{\sim}{\nabla}_{q_i} \big(U_i(\tfrac{1}{2}|\underset{\sim}{q}_i|^2)\big) = U_i'(\tfrac{1}{2}|\underset{\sim}{q}_i|^2)\, \underset{\sim}{q}_i, \tag{1.6a}$$

and, by definition,

$$\int_D M(q)\, \mathrm{d}\underset{\sim}{q} = 1. \tag{1.6b}$$

We shall assume that for $i = 1, \dots, K$ there exist constants $c_{ij} > 0$, $j = 1, 2, 3, 4$, and $\theta_i > 1$ such that the spring potential $U_i \in C^1([0, \frac{b_i}{2}); \mathbb{R}_{\geq 0})$ and the associated partial Maxwellian M_i satisfy

$$c_{i1}\, [\mathrm{dist}(\underset{\sim}{q}_i,\, \partial D_i)]^{\theta_i} \leq M_i(\underset{\sim}{q}_i) \leq c_{i2}\, [\mathrm{dist}(\underset{\sim}{q}_i,\, \partial D_i)]^{\theta_i} \qquad \forall \underset{\sim}{q}_i \in D_i, \tag{1.7a}$$

$$c_{i3} \leq \mathrm{dist}(\underset{\sim}{q}_i,\, \partial D_i)\, U_i'(\tfrac{1}{2}|\underset{\sim}{q}_i|^2) \leq c_{i4} \qquad \forall \underset{\sim}{q}_i \in D_i. \tag{1.7b}$$

It follows from (1.7a,b) that (if $\theta_i > 1$, as has been assumed here,)

$$\int_{D_i} \Big[1 + [U_i(\tfrac{1}{2}|\underset{\sim}{q}_i|^2)]^2 + [U_i'(\tfrac{1}{2}|\underset{\sim}{q}_i|^2)]^2\Big]\, M_i(\underset{\sim}{q}_i)\, \mathrm{d}\underset{\sim}{q}_i < \infty, \qquad i = 1, \dots, K. \tag{1.8}$$

Example 1.1. In the classical FENE dumbbell model, introduced by Warner [25], $K = 1$ and the spring force is given by $\underset{\sim}{F}(\underset{\sim}{q}) = (1 - |\underset{\sim}{q}|^2/b)^{-1}\, \underset{\sim}{q}$, with $\underset{\sim}{q} \in D = B(\underset{\sim}{0}, b^{\frac{1}{2}})$, corresponding to $U(s) = -\frac{b}{2}\log\big(1 - \frac{2s}{b}\big)$, $s \in [0, \frac{b}{2})$, $b > 2$. More generally, in a classical FENE bead spring chain, one considers $K+1$ beads linearly coupled with K springs, each with a classical Warner type FENE spring potential. Direct calculations show that the partial Maxwellians M_i and the elastic potentials U_i, $i = 1, \dots, K$, of the classical FENE bead spring chain satisfy the conditions (1.7a,b) with $\theta_i := \frac{b_i}{2}$, provided that $b_i > 2$, $i = 1, \dots, K$. Thus, (1.8) also holds and $b_i > 2$, $i = 1, \dots, K$. Note, however, that (1.8) fails for $b_i \in (0, 2]$, i.e., for $\theta_i \in (0, 1]$, which is why we have assumed in the statement of (1.7a,b) that $\theta_i > 1$ for $i = 1, \dots, K$.

It is interesting to note that in the (equivalent) stochastic version of the classical FENE dumbbell model ($K = 1$) a solution to the system of stochastic differential equations associated with the Fokker–Planck equation exists and has trajectorial uniqueness if, and only if, $\frac{b}{2} \geq 1$; (cf. Jourdain, Lelièvre & Le Bris [16] for details). Thus, in the general class of FENE-type bead-spring chain models considered here, the assumption $\theta_i > 1$, $i = 1, \dots, K$, is the weakest reasonable requirement on the decay-rate of M_i in (1.7a) as $\mathrm{dist}(\underset{\sim}{q}_i, \partial D_i) \to 0$. $\diamond$

The governing equations of the general nonhomogeneous bead-spring chain models with centre-of-mass diffusion considered in [9] are (1.1a–e), where the extra-stress tensor $\underset{\approx}{\tau}$ as the difference of the classical Kramers expression and a quadratic interaction term; i.e.,

$$\underset{\approx}{\tau}(\psi) := \underset{\approx}{\tau}_1(\psi) - \mathfrak{z}\left(\int_D \psi\, \mathrm{d}\underset{\sim}{q}\right)^2 \underset{\approx}{I}, \tag{1.9}$$

with $\mathfrak{z} \in \mathbb{R}_{\geq 0}$. Here, $\underset{\approx}{\tau}_1(\psi)$ is the *Kramers expression*; that is,

$$\underset{\approx}{\tau}_1(\psi) := k\left[\left(\sum_{i=1}^{K} \underset{\approx}{C}_i(\psi)\right) - (K+1)\int_D \psi\, \mathrm{d}\underset{\sim}{q}\, \underset{\approx}{I}\right], \tag{1.10}$$

where $k \in \mathbb{R}_{>0}$, with the first term in the square brackets being due to the K springs and the second to the $K+1$ beads in the bead-spring chain representing the polymer molecule; see Chapter 15 in [10]. Further,

$$\underset{\approx}{C}_i(\psi)(\underset{\sim}{x}, t) := \int_D \psi(\underset{\sim}{x}, \underset{\sim}{q}, t)\, U_i'(\tfrac{1}{2}|\underset{\sim}{q}_i|^2)\, \underset{\sim}{q}_i\, \underset{\sim}{q}_i^{\mathrm{T}}\, \mathrm{d}\underset{\sim}{q}, \qquad i = 1, \dots, K. \tag{1.11}$$

The function $\int_D \psi(\underset{\sim}{x}, \underset{\sim}{q}, t)\,\mathrm{d}\underset{\sim}{q}$, appearing in (1.9) and (1.10), is called the *polymer number density* and will play an important role in our analysis below; we shall therefore define

$$\varrho(\underset{\sim}{x}, t) := \int_D \psi(\underset{\sim}{x}, \underset{\sim}{q}, t)\,\mathrm{d}\underset{\sim}{q}.$$

We alert the reader at this point not to confuse the notation ϱ for the polymer number density with the notation ρ introduced at the beginning of the paper for the fluid density.

The probability density function ψ is a solution of the Fokker–Planck (forward Kolmogorov) equation

$$\frac{\partial \psi}{\partial t} + \underset{\sim}{\nabla}_x \cdot (\underset{\sim}{u}\,\psi) + \sum_{i=1}^{K} \underset{\sim}{\nabla}_{q_i} \cdot \left(\underset{\approx}{\sigma}(\underset{\sim}{u})\,\underset{\sim}{q}_i\,\psi \right)$$

$$= \varepsilon\,\Delta_x \left(\frac{\psi}{\zeta(\rho)} \right) + \frac{1}{4\,\lambda} \sum_{i=1}^{K}\sum_{j=1}^{K} A_{ij}\,\underset{\sim}{\nabla}_{q_i} \cdot \left(M\,\underset{\sim}{\nabla}_{q_j} \left(\frac{\psi}{\zeta(\rho)\,M} \right) \right) \quad \text{in } \Omega \times D \times (0, T], \tag{1.12}$$

with $\underset{\approx}{\sigma}(\underset{\sim}{v}) \equiv \underset{\approx}{\nabla}_x\,\underset{\sim}{v}$ and a, generally, density-dependent scaled drag coefficient $\zeta(\cdot) \in \mathbb{R}_{>0}$. A concise derivation of the Fokker–Planck equation (1.12) can be found in Section 1 of Barrett and Süli [4]. Let $\partial\overline{D}_i := D_1 \times \cdots \times D_{i-1} \times \partial D_i \times D_{i+1} \times \cdots \times D_K$. We impose the following boundary and initial conditions on solutions of (1.12):

$$\left[\frac{1}{4\,\lambda} \sum_{j=1}^{K} A_{ij}\,M\,\underset{\sim}{\nabla}_{q_j} \left(\frac{\psi}{\zeta(\rho)\,M} \right) - \underset{\approx}{\sigma}(\underset{\sim}{u})\,\underset{\sim}{q}_i\,\psi \right] \cdot \frac{\underset{\sim}{q}_i}{|\underset{\sim}{q}_i|} = 0$$

$$\text{on } \Omega \times \partial\overline{D}_i \times (0, T], \ \text{for } i = 1, \dots, K, \tag{1.13a}$$

$$\varepsilon\,\underset{\sim}{\nabla}_x \left(\frac{\psi}{\zeta(\rho)} \right) \cdot \underset{\sim}{n} = 0 \qquad \text{on } \partial\Omega \times D \times (0, T], \tag{1.13b}$$

$$\psi(\cdot, \cdot, 0) = \psi_0(\cdot, \cdot) \geq 0 \qquad \text{on } \Omega \times D, \tag{1.13c}$$

where $\underset{\sim}{q}_i$ is normal to ∂D_i, as D_i is a bounded ball centred at the origin, and $\underset{\sim}{n}$ is normal to $\partial\Omega$.

The nondimensional constant $k > 0$ featuring in (1.10) is a constant multiple of the product of the Boltzmann constant k_B and the absolute temperature $\mathtt{T}$. In (1.12), $\varepsilon \in \mathbb{R}_{>0}$ is the centre-of-mass diffusion coefficient defined as $\varepsilon := (\ell_0/L_0)^2/(4(K+1)\lambda)$ with L_0 a characteristic length-scale of the solvent flow, $\ell_0 := \sqrt{k_B \mathtt{T}/\mathtt{H}}$ signifying the characteristic microscopic length-scale and $\lambda := \frac{\zeta_0 U_0}{4\mathtt{H} L_0}$, where $\zeta_0 \in \mathbb{R}_{>0}$ is a characteristic drag coefficient and $\mathtt{H} \in \mathbb{R}_{>0}$ is a spring-constant. The nondimensional parameter $\lambda \in \mathbb{R}_{>0}$, called the Deborah number (and usually denoted by De), characterizes the elastic relaxation property of the fluid, and $\underset{\approx}{A} = (A_{ij})_{i,j=1}^{K}$ is the symmetric positive definite *Rouse matrix*, or connectivity matrix; for example, $\underset{\approx}{A} = \mathtt{tridiag}\,[-1, 2, -1]$ in the case of a (topologically) linear chain; see, Nitta [19]. Concerning these scalings and notational conventions, we remark that the factor $\frac{1}{4\lambda}$ in equation (1.12) above appears as a factor $\frac{1}{2\lambda}$ in the Fokker–Planck equation in our earlier papers [3, 5, 7, 2].

Definition 1.1. *The collection of equations and structural hypotheses* (1.1a–e)–(1.13a–c) *together with the assumption that the Rouse matrix A is symmetric and positive definite (as is always the case, by definition,) will be referred to throughout the paper as model* (P)*, or as the* compressible FENE-type bead-spring chain model with centre-of-mass diffusion. *It will be assumed throughout the paper that the shear viscosity, $\mu^S \in \mathbb{R}_{>0}$, the bulk viscosity, $\mu^B \in \mathbb{R}_{\geq 0}$, and the drag coefficient, $\zeta \in \mathbb{R}_{>0}$, are independent of the density ρ. For the ease of exposition we shall set $\zeta \equiv 1$.*

A noteworthy feature of equation (1.12) in the model (P) compared to classical Fokker–Planck equations for bead-spring-chain models for dilute polymers appearing in the literature is the presence of the $\underset{\sim}{x}$-dissipative centre-of-mass diffusion term $\varepsilon\,\Delta_x \psi$ on the right-hand side of the Fokker–Planck equation (1.12). We refer to Barrett & Süli [1] for the derivation of (1.12) in the case of $K = 1$ and constant ρ; see also the article by Schieber [22] concerning generalized dumbbell models with centre-of-mass diffusion, and the paper of Degond & Liu [11] for a careful justification of the presence of the centre-of-mass diffusion term through asymptotic analysis. In the case of variable density, viscosity and drag, the derivation of the Fokker–Planck equation with centre-of-mass diffusion appears in [4] (cf. also [5, 6]).

In [9] we proved the existence of global-in-time weak solutions to problem (P) for $d = 2$ or 3, provided that $\gamma > \frac{3}{2}$ in (1.3) and $\mathfrak{z} > 0$ in (1.9). Here, in the case of $d = 2$, we extend these results to $\gamma > 1$ and $\mathfrak{z} = 0$. It is unclear whether for $d = 3$ and $\gamma > \frac{3}{2}$ the existence of global-in-time weak solutions can also be shown to hold when $\mathfrak{z} = 0$. As in [9], we assume a nonnegative initial density $\rho_0 \in L^\infty(\Omega)$ for the continuity equation; a square-integrable initial velocity datum $\underset{\sim}{u}_0$ for the Navier–Stokes momentum equation; and a nonnegative initial probability density function ψ_0 for the Fokker–Planck equation, which has finite relative entropy with respect to the Maxwellian M associated with the spring potential in the model.

In the next section, we introduce a regularized problem (P_κ) depending on a parameter $\kappa \in \mathbb{R}_{>0}$, where the pressure $p(s)$ in (1.3) is replaced by $p_\kappa(s) = p(s) + \kappa\,(s^4 + s^\Gamma)$ with $\Gamma = \max\{\gamma, 8\}$, $\gamma > 1$. We end that section by recalling our notation and a number of auxiliary results. In Section 3 we recall from [9] the existence of a global-in-time weak solution $(\rho_\kappa, \underset{\sim}{u}_\kappa, \psi_\kappa)$ to (P_κ) if $\mathfrak{z} \in \mathbb{R}_{>0}$. On setting $\mathfrak{z} = \kappa$, we then establish a number of a priori bounds on $(\rho_\kappa, \underset{\sim}{u}_\kappa, \psi_\kappa)$ independent of κ. Finally, we pass to the limit $\kappa \to 0_+$ to establish the existence of a global-in-time weak solution $(\rho, \underset{\sim}{u}, \psi)$ to (P) for $d = 2$, with $\mathfrak{z} = 0$ and $\gamma > 1$.

2. The polymer model (P_κ)

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with a Lipschitz-continuous boundary $\partial\Omega$, and suppose that the set $D := D_1 \times \cdots \times D_K$ of admissible conformation vectors $\underset{\sim}{q} := (\underset{\sim}{q}_1^{\mathrm{T}}, \ldots, \underset{\sim}{q}_K^{\mathrm{T}})^{\mathrm{T}}$ in (1.12) is such that D_i, $i = 1, \ldots, K$, is an open ball in $\mathbb{R}^2$ centred at the origin, with boundary ∂D_i and radius $\sqrt{b_i}$, $b_i > 2$; let

$$\partial D := \bigcup_{i=1}^{K} \partial\overline{D}_i, \qquad \text{where} \qquad \partial\overline{D}_i := D_1 \times \cdots \times D_{i-1} \times \partial D_i \times D_{i+1} \times \cdots \times D_K. \tag{2.1}$$

Collecting (1.1a–e), (1.2) and (1.9)–(1.13a–c), with $\mu^S \in \mathbb{R}_{>0}$, $\mu^B \in \mathbb{R}_{\geq 0}$ and $\zeta \equiv 1$, we then consider the following regularized initial-boundary-value problem, dependent on the following given regularization parameter $\kappa \in \mathbb{R}_{>0}$. As has been already emphasized in the Introduction, the centre-of-mass diffusion coefficient $\varepsilon \in \mathbb{R}_{>0}$ is a physical parameter and is regarded as being fixed throughout.

$(\mathbf{P}_{\boldsymbol{\kappa}})$ Find $\rho_\kappa : (\underset{\sim}{x}, t) \in \Omega \times [0, T] \mapsto \rho_\kappa(\underset{\sim}{x}, t) \in \mathbb{R}_{\geq 0}$ and $\underset{\sim}{u}_\kappa : (\underset{\sim}{x}, t) \in \overline{\Omega} \times [0, T] \mapsto \underset{\sim}{u}_\kappa(\underset{\sim}{x}, t) \in \mathbb{R}^2$ such that

$$\frac{\partial \rho_\kappa}{\partial t} + \underset{\sim}{\nabla}_x \cdot (\underset{\sim}{u}_\kappa\, \rho_\kappa) = 0 \qquad \text{in } \Omega \times (0, T], \tag{2.2a}$$

$$\rho_\kappa(\underset{\sim}{x}, 0) = \rho_0(\underset{\sim}{x}) \qquad \forall \underset{\sim}{x} \in \Omega, \tag{2.2b}$$

$$\frac{\partial(\rho_\kappa\, \underset{\sim}{u}_\kappa)}{\partial t} + \underset{\sim}{\nabla}_x \cdot (\rho_\kappa\, \underset{\sim}{u}_\kappa \otimes \underset{\sim}{u}_\kappa) - \underset{\sim}{\nabla}_x \cdot \underset{\approx}{S}(\underset{\sim}{u}_\kappa) + \underset{\sim}{\nabla}_x\, p_\kappa(\rho_\kappa) = \rho_\kappa\, \underset{\sim}{f} + \underset{\sim}{\nabla}_x \cdot \underset{\approx}{\tau}(\psi_\kappa) \qquad \text{in } \Omega \times (0, T], \tag{2.2c}$$

$$\underset{\sim}{u}_\kappa = \underset{\sim}{0} \qquad \text{on } \partial\Omega \times (0, T], \tag{2.2d}$$

$$(\rho_\kappa\, \underset{\sim}{u}_\kappa)(\underset{\sim}{x}, 0) = (\rho_0 \underset{\sim}{u}_0)(\underset{\sim}{x}) \qquad \forall \underset{\sim}{x} \in \Omega, \tag{2.2e}$$

where $\psi_\kappa : (\underset{\sim}{x}, \underset{\sim}{q}, t) \in \overline{\Omega} \times \overline{D} \times [0, T] \mapsto \psi_\kappa(\underset{\sim}{x}, \underset{\sim}{q}, t) \in \mathbb{R}_{\geq 0}$. Here $p_\kappa(\cdot)$ is a regularization of $p(\cdot)$, (1.3), defined by

$$p_\kappa(s) := p(s) + \kappa\,(s^4 + s^\Gamma), \qquad \text{where } \kappa \in \mathbb{R}_{>0},\ \Gamma = \max\{\gamma, 8\} \text{ and } \gamma > 1. \tag{2.3}$$

The Fokker–Planck equation satisfied by ψ_κ is:

$$\frac{\partial \psi_\kappa}{\partial t} + \underset{\sim}{\nabla}_x \cdot \left(\underset{\sim}{u}_\kappa\, \psi_\kappa\right) + \sum_{i=1}^{K} \underset{\sim}{\nabla}_{q_i} \cdot \left(\underset{\approx}{\sigma}(\underset{\sim}{u}_\kappa)\, \underset{\sim}{q}_i\, \psi_\kappa\right)$$

$$= \varepsilon\, \Delta_x\, \psi_\kappa + \frac{1}{4\,\lambda} \sum_{i=1}^{K}\sum_{j=1}^{K} A_{ij}\, \underset{\sim}{\nabla}_{q_i} \cdot \left(M\, \underset{\sim}{\nabla}_{q_j} \left(\frac{\psi_\kappa}{M}\right)\right) \qquad \text{in } \Omega \times D \times (0,T]. \tag{2.4}$$

Here, $\underset{\approx}{\sigma}(\underset{\sim}{v}) \equiv \underset{\approx}{\nabla}_x\, \underset{\sim}{v}$, and

$$\underset{\approx}{A} \in \mathbb{R}^{K\times K} \text{ is symmetric positive definite with smallest eigenvalue } a_0 \in \mathbb{R}_{>0}. \tag{2.5}$$

We impose the following boundary and initial conditions:

$$\left[\frac{1}{4\,\lambda} \sum_{j=1}^{K} A_{ij}\, M\, \underset{\sim}{\nabla}_{q_j}\left(\frac{\psi_\kappa}{M}\right) - \underset{\approx}{\sigma}(\underset{\sim}{u}_\kappa)\, \underset{\sim}{q}_i\, \psi_\kappa\right] \cdot \frac{\underset{\sim}{q}_i}{|\underset{\sim}{q}_i|} = 0$$

$$\text{on } \Omega \times \partial\overline{D}_i \times (0,T], \quad i = 1, \dots, K, \tag{2.6a}$$

$$\varepsilon\, \underset{\sim}{\nabla}_x\, \psi_\kappa \cdot \underset{\sim}{n} = 0 \qquad \text{on } \partial\Omega \times D \times (0,T], \tag{2.6b}$$

$$\psi_\kappa(\underset{\sim}{x}, \underset{\sim}{q}, 0) = \psi_0(\underset{\sim}{x}, \underset{\sim}{q}) \geq 0 \qquad \forall (\underset{\sim}{x}, \underset{\sim}{q}) \in \Omega \times D, \tag{2.6c}$$

where $\underset{\sim}{n}$ is the unit outward normal to $\partial\Omega$. The boundary conditions for ψ_κ on $\partial\Omega \times D \times (0,T]$ and $\Omega \times \partial D \times (0,T]$ have been chosen so as to ensure that

$$\int_{\Omega\times D} \psi_\kappa(\underset{\sim}{x}, \underset{\sim}{q}, t)\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} = \int_{\Omega\times D} \psi_\kappa(\underset{\sim}{x}, \underset{\sim}{q}, 0)\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} \qquad \forall t \in (0,T]. \tag{2.7}$$

Henceforth, we shall write

$$\widehat{\psi}_\kappa = \frac{\psi_\kappa}{M}, \qquad \widehat{\psi}_0 = \frac{\psi_0}{M}. \tag{2.8}$$

We end this section by stating our notation and collecting together some auxiliary results.

2.1. **Notation and Auxiliary Results.** For later purposes, we recall the following Lebesgue interpolation result and the Gagliardo–Nirenberg inequality. Let $1 \leq r \leq \upsilon \leq s < \infty$, then, for any bounded Lipschitz domain $\mathcal{O}$,

$$\|\eta\|_{L^\upsilon(\mathcal{O})} \leq \|\eta\|_{L^r(\mathcal{O})}^{1-\vartheta}\, \|\eta\|_{L^s(\mathcal{O})}^{\vartheta} \qquad \forall \eta \in L^s(\mathcal{O}), \tag{2.9}$$

where $\vartheta = \frac{s(\upsilon - r)}{\upsilon(s-r)}$. Let $r \in [2,\infty)$ and $\vartheta = 2\left(\frac{1}{2} - \frac{1}{r}\right)$. As $\Omega \subset \mathbb{R}^2$, there is a constant $C = C(\Omega, r)$, such that

$$\|\eta\|_{L^r(\Omega)} \leq C\, \|\eta\|_{L^2(\Omega)}^{1-\vartheta}\, \|\eta\|_{H^1(\Omega)}^{\vartheta} \qquad \forall \eta \in H^1(\Omega). \tag{2.10}$$

Let $\mathcal{F} \in C(\mathbb{R}_{>0})$ be defined by $\mathcal{F}(s) := s\,(\log s - 1) + 1$, $s > 0$. As $\lim_{s\to 0_+} \mathcal{F}(s) = 1$, the function $\mathcal{F}$ can be considered to be defined and continuous on $[0,\infty)$, where it is a nonnegative, strictly convex function with $\mathcal{F}(1) = 0$. Hence $\mathcal{F}(s) \geq \mathcal{F}(\mathrm{e}) + (s - \mathrm{e})\mathcal{F}'(\mathrm{e}) = s - \mathrm{e} + 1$, and so it follows that

$$\mathcal{F}(s) \geq [s - \mathrm{e} + 1]_+ \qquad \forall s \geq 0. \tag{2.11}$$

We assume the following:

$$\begin{aligned}
&d = 2, \qquad \partial\Omega \in C^{2,\theta},\ \theta \in (0,1); \qquad \rho_0 \in L^\infty_{\geq 0}(\Omega); \qquad \underset{\sim}{u}_0 \in \underset{\sim}{L}^2(\Omega);\\
&\psi_0 \geq 0 \ \text{ a.e. on } \Omega \times D \text{ with } \mathcal{F}(\widehat{\psi}_0) \in L^1_M(\Omega \times D) \text{ and } \int_D \psi_0(\cdot, \underset{\sim}{q})\, \mathrm{d}\underset{\sim}{q} \in L^\infty_{\geq 0}(\Omega);\\
&\mu^S \in \mathbb{R}_{>0},\ \mu^B \in \mathbb{R}_{\geq 0}; \qquad \text{the Rouse matrix } \underset{\approx}{A} \in \mathbb{R}^{K\times K} \text{ satisfies (2.5)};\\
&p,\, p_\kappa \in C^1(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}) \text{ are defined by (1.3), with } \gamma > 1, \text{ and (2.3)};\\
&\underset{\sim}{f} \in L^2(0,T; \underset{\sim}{L}^\infty(\Omega)) \qquad \text{and} \qquad D_i = B(\underset{\sim}{0}, b_i^{\frac{1}{2}}), \quad \theta_i > 1, \quad i = 1, \dots, K, \quad \text{in (1.7a,b).}
\end{aligned} \tag{2.12}$$

On recalling (1.3) and (2.3), we introduce P, $P_\kappa \in C^1(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$, for $\kappa \in \mathbb{R}_{>0}$, such that

$$\begin{aligned} & s\,P'_{(\kappa)}(s) - P_{(\kappa)}(s) = p_{(\kappa)}(s) \quad \text{and} \quad P_{(\kappa)}(0) = P'_{(\kappa)}(0) = 0 \\ \Rightarrow \quad & P(s) = \frac{p(s)}{\gamma-1} = \frac{c_p}{\gamma-1}\, s^\gamma \qquad \text{and} \qquad P_\kappa(s) = P(s) + \kappa\left(\frac{s^4}{3} + \frac{s^\Gamma}{\Gamma-1}\right). \end{aligned} \tag{2.13}$$

In the first line of (2.13), and henceforth, throughout the rest of the paper, the subscript "$(\cdot)$" means *with and without the subscript* "$\cdot$".

In (2.12), $L^r_M(\Omega \times D)$, for $r \in [1,\infty)$, denotes the Maxwellian-weighted L^r space over $\Omega \times D$ with norm

$$\|\varphi\|_{L^r_M(\Omega\times D)} := \left\{ \int_{\Omega\times D} M\,|\varphi|^r \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \right\}^{\frac{1}{r}}.$$

Similarly, we introduce $L^r_M(D)$, the Maxwellian-weighted L^r space over D. Letting

$$\|\varphi\|_{H^1_M(\Omega\times D)} := \left\{ \int_{\Omega\times D} M \left[|\varphi|^2 + \left|\underset{\sim}{\nabla}_x \varphi\right|^2 + \left|\underset{\sim}{\nabla}_q \varphi\right|^2 \right] \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \right\}^{\frac{1}{2}}, \tag{2.14}$$

we then set

$$H^1_M(\Omega\times D) := \left\{ \varphi \in L^1_{\mathrm{loc}}(\Omega\times D) : \|\varphi\|_{H^1_M(\Omega\times D)} < \infty \right\}. \tag{2.15}$$

We recall the Aubin–Lions–Simon compactness theorem; see, e.g., Simon [23]. Let $\mathfrak{X}_0$, $\mathfrak{X}$ and $\mathfrak{X}_1$ be Banach spaces with a compact embedding $\mathfrak{X}_0 \hookrightarrow \mathfrak{X}$ and a continuous embedding $\mathfrak{X} \hookrightarrow \mathfrak{X}_1$. Then, for $\varsigma_i \in [1,\infty)$, $i = 0,1$, the embedding

$$\{\, \eta \in L^{\varsigma_0}(0,T;\mathfrak{X}_0) : \frac{\partial \eta}{\partial t} \in L^{\varsigma_1}(0,T;\mathfrak{X}_1) \,\} \hookrightarrow L^{\varsigma_0}(0,T;\mathfrak{X}) \tag{2.16}$$

is compact. We recall also a generalization of the Aubin–Lions–Simon compactness theorem due to Dubinskiĭ [12]; see also Barrett & Süli [8]. Prior to stating Dubinskiĭ's theorem we introduce the necessary prerequisites.

Let $\mathfrak{X}$ be a linear space over the field $\mathbb{R}$ of real numbers, and suppose that $\mathfrak{M}$ is a subset of $\mathfrak{X}$ such that

$$\lambda\eta \in \mathfrak{M} \qquad \forall \lambda \in \mathbb{R}_{\geq 0},\ \forall \eta \in \mathfrak{M}. \tag{2.17}$$

In other words, whenever η is contained in $\mathfrak{M}$, the ray through η from the origin of the linear space $\mathfrak{X}$ is also contained in $\mathfrak{M}$. Note in particular that while any set $\mathfrak{M}$ with property (2.17) must contain the zero element of the linear space $\mathfrak{X}$, the set $\mathfrak{M}$ need not be closed under summation. The linear space $\mathfrak{X}$ will be referred to as the *ambient space* for $\mathfrak{M}$. Suppose further that each element η of a set $\mathfrak{M}$ with property (2.17) is assigned a certain real number, denoted by $[\eta]_{\mathfrak{M}}$, such that:

(i) $[\eta]_{\mathfrak{M}} \geq 0$; and $[\eta]_{\mathfrak{M}} = 0$ if, and only if, $\eta = 0$; and
(ii) $[\lambda\eta]_{\mathfrak{M}} = \lambda[\eta]_{\mathfrak{M}}$ for all $\lambda \in \mathbb{R}_{\geq 0}$ and all $\eta \in \mathfrak{X}$.

We shall then say that $\mathfrak{M}$ is a *seminormed set.* A subset $\mathfrak{B}$ of a seminormed set $\mathfrak{M}$ is said to be *bounded* if there exists a positive constant K_0 such that $[\eta]_{\mathfrak{M}} \leq K_0$ for all $\eta \in \mathfrak{B}$. A seminormed set $\mathfrak{M}$ contained in a normed linear space $\mathfrak{X}$ with norm $\|\cdot\|_{\mathfrak{X}}$ is said to be *embedded in* $\mathfrak{X}$, and we write $\mathfrak{M} \subset \mathfrak{X}$, if there exists a $K_0 \in \mathbb{R}_{>0}$ such that

$$\|\eta\|_{\mathfrak{X}} \leq K_0[\eta]_{\mathfrak{M}} \qquad \forall \eta \in \mathfrak{M}.$$

Thus, bounded subsets of a seminormed set are also bounded subsets of the ambient normed linear space the seminormed set is embedded in. The embedding of a seminormed set $\mathfrak{M}$ into a normed linear space $\mathfrak{X}$ is said to be *compact* if from any bounded, infinite set of elements of $\mathfrak{M}$ one can extract a subsequence that converges in $\mathfrak{X}$.

Theorem 2.1 (Dubinskiĭ's compactness theorem)**.** *Suppose that $\mathfrak{M}$ is a semi-normed set that is compactly embedded into a Banach space $\mathfrak{X}$, which is, in turn, continuously embedded into a Banach space $\mathfrak{X}_1$. Then, for $\varsigma_i \in [1,\infty)$, $i=0,1$, the embedding*

$$\{\, \eta \in L^{\varsigma_0}(0,T;\mathfrak{M}) : \frac{\partial \eta}{\partial t} \in L^{\varsigma_1}(0,T;\mathfrak{X}_1)\,\} \hookrightarrow L^{\varsigma_0}(0,T;\mathfrak{X}) \tag{2.18}$$

is compact.

Let $\mathfrak{X}$ be a Banach space. We shall denote by $C_w([0,T];\mathfrak{X})$ the set of all functions $\eta \in L^\infty(0,T;\mathfrak{X})$ such that $t \in [0,T] \mapsto \langle \varphi, \eta(t)\rangle_{\mathfrak{X}} \in \mathbb{R}$ is continuous on $[0,T]$ for all $\varphi \in \mathfrak{X}'$, the dual space of $\mathfrak{X}$. Here, and throughout, $\langle\cdot,\cdot\rangle_{\mathfrak{X}}$ denotes the duality pairing between $\mathfrak{X}'$ and $\mathfrak{X}$. Whenever $\mathfrak{X}$ has a predual, $\mathfrak{E}$, say, (viz. $\mathfrak{E}' = \mathfrak{X}$), we shall denote by $C_{w*}([0,T];\mathfrak{X})$ the set of all functions $\eta \in L^\infty(0,T;\mathfrak{X})$ such that $t \in [0,T] \mapsto \langle \eta(t), \zeta\rangle_{\mathfrak{E}} \in \mathbb{R}$ is continuous on $[0,T]$ for all $\zeta \in \mathfrak{E}$. We note the following results.

Suppose that $\mathfrak{X}_0$, $\mathfrak{X}_1$ are Banach spaces; then,

$$\mathfrak{X}_0 \hookrightarrow \mathfrak{X}_1 \qquad \Rightarrow \qquad \mathfrak{X}_1' \hookrightarrow \mathfrak{X}_0'. \tag{2.19}$$

In addition, if the first embedding is compact so is the second.

Lemma 2.1. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces.*

(a) *Assume that the space $\mathfrak{X}$ is reflexive and is continuously embedded in the space $\mathfrak{Y}$; then, $L^\infty(0,T;\mathfrak{X}) \cap C_w([0,T];\mathfrak{Y}) = C_w([0,T];\mathfrak{X})$.*

(b) *Assume that $\mathfrak{X}$ has a separable predual $\mathfrak{E}$ and $\mathfrak{Y}$ has a predual $\mathfrak{F}$ such that $\mathfrak{F}$ is continuously embedded in $\mathfrak{E}$; then, $L^\infty(0,T;\mathfrak{X}) \cap C_{w*}([0,T];\mathfrak{Y}) = C_{w*}([0,T];\mathfrak{X})$.*

Proof. Part (a) is due to Strauss [24] (cf. Lions & Magenes [17], Lemma 8.1, Ch. 3, Sec. 8.4); part (b) is proved analogously, *via* the sequential Banach–Alaoglu theorem. □

We note from Lemma 2.1(a) above and Lemma 6.2 in Novotný & Straškraba [20] that if $\{\eta_n\}_{n\in\mathbb{N}}$ is such that

$$\|\eta_n\|_{L^\infty(0,T;L^r(\Omega))} + \left\|\frac{\partial \eta_n}{\partial t}\right\|_{L^\varsigma(0,T;W_0^{1,\upsilon}(\Omega)')} \leq C, \qquad r,\varsigma,\upsilon \in (1,\infty), \tag{2.20a}$$

then there exists a subsequence (not indicated) of $\{\eta_n\}_{n\in\mathbb{N}}$ and an $\eta \in C_w([0,T];L^r(\Omega))$ such that

$$\eta_n \to \eta \qquad \text{in } C_w([0,T];L^r(\Omega)). \tag{2.20b}$$

For $r \in [1,\infty)$, let

$$Z_r := \{\varphi \in L^r_M(\Omega\times D) : \varphi \geq 0 \text{ a.e. on } \Omega \times D\}. \tag{2.21}$$

For r, $s \in (1,\infty)$, let

$$L^r_0(\Omega) := \{\zeta \in L^r(\Omega) : \int_\Omega \zeta \,\mathrm{d}\underset{\sim}{x} = 0\}, \tag{2.22a}$$

$$\underset{\sim}{E}^{r,s}(\Omega) := \{\underset{\sim}{w} \in \underset{\sim}{L}^r(\Omega) : \underset{\sim}{\nabla}_x \cdot \underset{\sim}{w} \in L^s(\Omega)\} \tag{2.22b}$$

$$\text{and} \quad \underset{\sim}{E}^{r,s}_0(\Omega) := \{\underset{\sim}{w} \in \underset{\sim}{E}^{r,s}(\Omega) : \underset{\sim}{w}\cdot\underset{\sim}{n} = 0 \text{ on } \partial\Omega\}. \tag{2.22c}$$

The equality $\underset{\sim}{w}\cdot\underset{\sim}{n} = 0$ on $\partial\Omega$ should be understood in the sense of traces of Sobolev functions, with equality in $W^{1-\frac{\upsilon}{\upsilon'},\upsilon'}(\partial\Omega)'$, where $\frac{1}{\upsilon}+\frac{1}{\upsilon'} = 1$ and $\upsilon = \min\{r,s\}$; cf. Lemma 3.10 in [20].

We now introduce the Bogovskiĭ operator $\underset{\sim}{\mathcal{B}} : L^r_0(\Omega) \to \underset{\sim}{W}^{1,r}_0(\Omega)$, $r \in (1,\infty)$, such that

$$\int_\Omega \left(\underset{\sim}{\nabla}_x \cdot \underset{\sim}{\mathcal{B}}(\zeta) - \zeta\right)\eta \,\mathrm{d}\underset{\sim}{x} = 0 \qquad \forall \eta \in L^{\frac{r}{r-1}}(\Omega); \tag{2.23}$$

which satisfies

$$\|\underset{\sim}{\mathcal{B}}(\zeta)\|_{W^{1,r}(\Omega)} \leq C\,\|\zeta\|_{L^r(\Omega)} \qquad \forall \zeta \in L^r_0(\Omega), \tag{2.24a}$$

$$\|\underset{\sim}{\mathcal{B}}(\underset{\sim}{\nabla}_x \cdot \underset{\sim}{w})\|_{L^r(\Omega)} \leq C\,\|\underset{\sim}{w}\|_{L^r(\Omega)} \qquad \forall \underset{\sim}{w} \in \underset{\sim}{E}^{r,s}_0(\Omega), \tag{2.24b}$$

see Lemma 3.17 in Novotný & Straškraba [20].

We shall require the following regularity result for a parabolic initial-boundary-value problem, with a homogeneous Neumann boundary condition and right-hand side in divergence form:

Lemma 2.2 (Lemma 7.38 in [20]). *Let $\theta \in (0,1]$, $r, s \in (1,\infty)$, and suppose that G is a bounded domain in $\mathbb{R}^d$. Suppose further that*

$$\partial G \in C^{2,\theta}, \qquad \alpha \in \mathbb{R}_{>0}, \qquad z_0 \in L^s(G), \qquad \underset{\sim}{b} \in L^r(0,T;L^s(G)).$$

Then, there exists a unique $z \in L^r(0,T;W^{1,s}(G)) \cap C([0,T];L^s(G))$, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_G z\,\eta \,\mathrm{d}\underset{\sim}{x} + \alpha \int_G \underset{\sim}{\nabla}_x z \cdot \underset{\sim}{\nabla}_x \eta \,\mathrm{d}\underset{\sim}{x} = -\int_G \underset{\sim}{b}\cdot \underset{\sim}{\nabla}_x \eta \,\mathrm{d}\underset{\sim}{x} \quad \text{in } (C_0^\infty[0,T])' \quad \forall \eta \in C^\infty(\overline{G}),$$

$z(\underset{\sim}{x},0) = z_0(\underset{\sim}{x})$ for a.e. $\underset{\sim}{x} \in G$, and

$$\alpha^{1-\frac{1}{r}}\|z\|_{L^\infty(0,T;L^s(G))} + \alpha\|\underset{\sim}{\nabla}_x z\|_{L^r(0,T;L^s(G))} \le C(r,s,G)\big[\alpha^{1-\frac{1}{r}}\|z_0\|_{L^s(G)} + \|\underset{\sim}{b}\|_{L^r(0,T;L^s(G))}\big].$$

Due to the presence of the extra stress term in the momentum equation, we require a modification of the effective viscous flux compactness result, Proposition 7.36 in Novotný & Straškraba [20]. Such results require pseudodifferential operators defined via the Fourier transform $\mathfrak{F}$. We briefly recall the key ideas, and refer to Section 4.4.1 in [20] for the details.

For the sake of clarity, the definitions below, preceding Lemma 2.3, will be stated for the general case of $d \ge 2$; we shall, thereafter, make use of them in the special case of $d = 2$ only. With

$$\mathfrak{S}(\mathbb{R}^d) := \left\{ \eta \in C^\infty(\mathbb{R}^d) : \sup_{\underset{\sim}{x}\in\mathbb{R}^d} \left| x_1^{\varsigma_1}\cdots x_d^{\varsigma_d} \frac{\partial^{|\underset{\sim}{\lambda}|}\eta}{\partial_{x_1}^{\lambda_1}\cdots\partial_{x_d}^{\lambda_d}} \right| \le C(|\underset{\sim}{\varsigma}|,|\underset{\sim}{\lambda}|) \quad \forall \underset{\sim}{\varsigma},\ \underset{\sim}{\lambda} \in \mathbb{N}^d \right\}, \tag{2.25}$$

the space of smooth rapidly decreasing (complex-valued) functions, we introduce the Fourier transform $\mathfrak{F} : \mathfrak{S}(\mathbb{R}^d) \to \mathfrak{S}(\mathbb{R}^d)$, and its inverse $\mathfrak{F}^{-1} : \mathfrak{S}(\mathbb{R}^d) \to \mathfrak{S}(\mathbb{R}^d)$, defined by

$$[\mathfrak{F}(\eta)](\underset{\sim}{y}) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} \mathrm{e}^{-i\underset{\sim}{x}\cdot\underset{\sim}{y}}\,\eta(\underset{\sim}{x})\,\mathrm{d}\underset{\sim}{x} \qquad \text{and} \qquad [\mathfrak{F}^{-1}(\eta)](\underset{\sim}{x}) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} \mathrm{e}^{i\underset{\sim}{x}\cdot\underset{\sim}{y}}\,\eta(\underset{\sim}{y})\,\mathrm{d}\underset{\sim}{y}. \tag{2.26}$$

These are extended to $\mathfrak{F}$, $\mathfrak{F}^{-1} : \mathfrak{S}(\mathbb{R}^d)' \to \mathfrak{S}(\mathbb{R}^d)'$, where $\mathfrak{S}(\mathbb{R}^d)'$, the dual of $\mathfrak{S}(\mathbb{R}^d)$, is the space of tempered distributions, via

$$\langle \mathfrak{F}(\eta),\xi\rangle_{\mathfrak{S}(\mathbb{R}^d)} = \langle \eta, \mathfrak{F}(\xi)\rangle_{\mathfrak{S}(\mathbb{R}^d)} \quad \text{and} \quad \langle \mathfrak{F}^{-1}(\eta),\xi\rangle_{\mathfrak{S}(\mathbb{R}^d)} = \langle \eta, \mathfrak{F}^{-1}(\xi)\rangle_{\mathfrak{S}(\mathbb{R}^d)} \quad \forall \xi \in \mathfrak{S}(\mathbb{R}^d). \tag{2.27}$$

We now define the 'inverse divergence operators' $\mathcal{A}_j : \mathfrak{S}(\mathbb{R}^d) \to \mathfrak{S}(\mathbb{R}^d)'$, $j = 1,\dots,d$, by

$$\mathcal{A}_j(\eta) := -\mathfrak{F}^{-1}\left(\frac{i\,y_j}{|\underset{\sim}{y}|^2}\,[\mathfrak{F}(\eta)](\underset{\sim}{y})\right). \tag{2.28}$$

It follows from Theorems 1.55 and 1.57 in [20] and Sobolev embedding that, for $j = 1,\dots d$,

$$\|\underset{\sim}{\nabla}_x \mathcal{A}_j(\eta)\|_{L^r(\mathbb{R}^d)} \le C(r)\,\|\eta\|_{L^r(\mathbb{R}^d)} \qquad \forall \eta \in \mathfrak{S}(\mathbb{R}^d), \quad r \in (1,\infty), \tag{2.29a}$$

$$\|\mathcal{A}_j(\eta)\|_{L^{\frac{dr}{d-r}}(\mathbb{R}^d)} \le C(r)\,\|\eta\|_{L^r(\mathbb{R}^d)} \qquad \forall \eta \in \mathfrak{S}(\mathbb{R}^d), \quad r \in (1,d). \tag{2.29b}$$

Hence, we deduce from (2.29a,b) that $\mathcal{A}_j$ can be extended to $\mathcal{A}_j : L^r(\mathbb{R}^d) \to D^{1,r}(\mathbb{R}^d)$ for $r \in (1,\infty)$, $j = 1,\dots,d$, where $D^{1,r}(\mathbb{R}^d)$ is a homogeneous Sobolev space; see Section 1.3.6 in [20]. In addition, by duality, $\mathcal{A}_j$ can be extended to $\mathcal{A}_j : D^{1,r}(\mathbb{R}^d)' \to D^{1,r}(\mathbb{R}^d)$ for $r \in (1,\infty)$, $j = 1,\dots,d$; see (4.4.4) in [20]. Moreover, as $\mathcal{A}_j(\eta)$ is real for a real-valued function η, from the Parseval–Plancherel formula we have, for all $\eta \in L^r(\mathbb{R}^d)$ and $\xi \in L^{\frac{r}{r-1}}(\mathbb{R}^d)$, $r \in (1,\infty)$, having compact support that

$$\int_{\mathbb{R}^d} \mathcal{A}_j(\eta)\,\xi \,\mathrm{d}\underset{\sim}{x} = -\int_{\mathbb{R}^d} \eta\,\mathcal{A}_j(\xi)\,\mathrm{d}\underset{\sim}{x}, \qquad j = 1,\dots,d. \tag{2.30}$$

Finally, we introduce the so-called Riesz operator $\mathcal{R}_{kj} : L^r(\mathbb{R}^d) \to L^r(\mathbb{R}^d)$, $r \in (1,\infty)$, defined by

$$\mathcal{R}_{kj}(\eta) = \frac{\partial}{\partial x_k}\mathcal{A}_j(\eta), \qquad j,\ k = 1,\dots,d. \tag{2.31}$$

We note for all $\eta \in L^r(\mathbb{R}^d)$ and $\xi \in L^{\frac{r}{r-1}}(\mathbb{R}^d)$, $r \in (1,\infty)$, that

$$\sum_{j=1}^{d} \mathcal{R}_{jj}(\eta) = \sum_{j=1}^{d} \frac{\partial}{\partial x_j} \mathcal{A}_j(\eta) = \eta, \tag{2.32a}$$

$$\mathcal{R}_{kj}(\eta) = \mathcal{R}_{jk}(\eta) \qquad \text{and} \qquad \int_{\mathbb{R}^d} \mathcal{R}_{jk}(\eta)\, \xi \, \mathrm{d}\underset{\sim}{x} = \int_{\mathbb{R}^d} \eta\, \mathcal{R}_{jk}(\xi)\, \mathrm{d}\underset{\sim}{x}, \qquad j,\, k = 1, \dots, d. \tag{2.32b}$$

Below we use the notation $\underset{\sim}{\mathcal{A}}(\cdot)$ and $\underset{\approx}{\mathcal{R}}(\cdot)$ with components $\mathcal{A}_j(\cdot)$ and $\mathcal{R}_{kj}(\cdot)$, $j, k = 1, \dots, d$, respectively, and with $d = 2$. We shall adopt the convention that whenever any of these operators is applied to a function or a distribution that has been defined on $\Omega \subset \mathbb{R}^2$ only, it is tacitly understood that the function or distribution in question has been extended by 0 from Ω to the whole of $\mathbb{R}^2$.

We now have the following modification of Proposition 7.36 in Novotný & Straškraba [20], which is adequate for our purposes.

Lemma 2.3. *Given $\{(g_n, \underset{\sim}{u}_n, \underset{\sim}{m}_n, p_n, \underset{\approx}{\tau}_n, f_n, \underset{\sim}{F}_n)\}_{n\in\mathbb{N}}$, we assume for any $\zeta \in C_0^\infty(\Omega)$ that, as $n \to \infty$,*

$$g_n \to g \qquad \textit{in } C_w([0,T]; L^q(\Omega)), \qquad \textit{weakly } (\text{-}\star) \textit{ in } L^\omega(\Omega_T), \tag{2.33a}$$

$$\underset{\sim}{u}_n \to \underset{\sim}{u} \qquad \textit{weakly in } L^2(0,T; \underset{\sim}{H}^1_0(\Omega)), \tag{2.33b}$$

$$\underset{\sim}{m}_n \to \underset{\sim}{m} \qquad \textit{in } C_w([0,T]; \underset{\sim}{L}^z(\Omega)), \tag{2.33c}$$

$$p_n \to p \qquad \textit{weakly in } L^r(\Omega_T), \tag{2.33d}$$

$$\underset{\approx}{\tau}_n \to \underset{\approx}{\tau} \qquad \textit{strongly in } L^1(0,T; \underset{\approx}{L}^{\frac{q}{q-1}}(\Omega)), \tag{2.33e}$$

$$f_n \to f \qquad \textit{weakly in } L^2(0,T; H^1(\Omega)'), \tag{2.33f}$$

$$\underset{\sim}{\mathcal{A}}(\zeta\, f_n) \to \underset{\sim}{\mathcal{A}}(\zeta\, f) \qquad \textit{strongly in } L^2(0,T; \underset{\sim}{L}^{\frac{z}{z-1}}(\Omega)), \tag{2.33g}$$

$$\underset{\sim}{F}_n \to \underset{\sim}{F} \qquad \textit{weakly in } \underset{\sim}{L}^s(\Omega_T), \tag{2.33h}$$

where $q \in (2,\infty)$, $r,\, s \in (1,\infty)$, $\omega \in \left[\max\{2, \frac{r}{r-1}\}, \infty\right]$ and $z \in \left(\frac{q}{q-1}, \infty\right)$.

In addition, suppose that

$$\frac{\partial g_n}{\partial t} + \underset{\sim}{\nabla}_x \cdot (\underset{\sim}{u}_n\, g_n) = f_n \qquad \textit{in } C_0^\infty(\Omega_T)', \tag{2.34a}$$

$$\begin{aligned} &\frac{\partial \underset{\sim}{m}_n}{\partial t} + \underset{\sim}{\nabla}_x \cdot (\underset{\sim}{m}_n \otimes \underset{\sim}{u}_n) - \mu\, \Delta_x\, \underset{\sim}{u}_n - (\mu + \lambda)\, \underset{\sim}{\nabla}_x\, (\underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}_n) + \underset{\sim}{\nabla}_x\, p_n \\ &\qquad = \underset{\sim}{F}_n + \underset{\sim}{\nabla}_x \cdot \underset{\approx}{\tau}_n \qquad \textit{in } \underset{\sim}{C}_0^\infty(\Omega_T)'. \end{aligned} \tag{2.34b}$$

It then follows that, for any $\zeta \in C_0^\infty(\Omega)$ and $\eta \in C_0^\infty(0,T)$,

$$\lim_{n\to\infty} \int_0^T \eta \left(\int_\Omega \zeta\, g_n\, [p_n - (2\mu + \lambda)\, \underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}_n]\, \mathrm{d}\underset{\sim}{x} \right) \mathrm{d}t = \int_0^T \eta \left(\int_\Omega \zeta\, g\, [p - (2\mu + \lambda)\, \underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}]\, \mathrm{d}\underset{\sim}{x} \right) \mathrm{d}t. \tag{2.35}$$

Proof. We adapt the proof of Proposition 7.36 in [20] which is for $d = 3$, see also Lemma 5.6 in [9], by just pointing out the key differences. As $q > d = 2$, then $q^\star$, the Sobolev conjugate of q in the notation (1.3.64) of [20], is such that $q^\star = \infty$. Hence our restrictions on r, s, ω satisfy the restrictions of Proposition 7.36 in [20] and Lemma 5.6 in [9]. The restriction on z in [20] and Lemma 5.6 in [9], $z \in (\frac{6q}{5q-6}, \infty)$, gives rise to $\frac{zq}{z+q} > \frac{6}{5}$ and hence, on recalling (2.19), that $L^{\frac{zq}{z+q}}(\Omega) \hookrightarrow H^1(\Omega)'$ is compact. As $d = 2$ here, this can be relaxed to $\frac{zq}{z+q} > 1$ yielding our restriction on z, $z \in (\frac{q}{q-1}, \infty)$.

With any $\widetilde{\zeta} \in C_0^\infty(\Omega)$, it follows from (2.34a) and properties (2.30)–(2.32a,b) of $\mathcal{A}_j$ and $\mathcal{R}_{kj}$ that, for $i = 1, 2$,

$$\frac{\partial}{\partial t}\mathcal{A}_i(\widetilde{\zeta}\, g_n) + \sum_{j=1}^{2} \mathcal{R}_{ij}(\widetilde{\zeta}\, g_n\, u_n^j) = \mathcal{A}_i(\widetilde{\zeta}\, f_n) + \mathcal{A}_i(g_n\, \underset{\sim}{u}_n \cdot \underset{\sim}{\nabla}_x\, \widetilde{\zeta}) \qquad \text{in } C_0^\infty(\Omega_T)', \tag{2.36}$$

where we adopt the notation u_n^j for the j^{th} component of $\underset{\sim}{u}_n$. With any $\zeta, \widetilde{\zeta} \in C_0^\infty(\Omega)$ and $\eta \in C_0^\infty(0,T)$, we now consider $\eta\,\zeta\,\underset{\sim}{\mathcal{A}}(\widetilde{\zeta}\, g_n)$ as a test function for (2.34b). It follows from (2.33a) and (2.29a,b) that $\underset{\sim}{\mathcal{A}}(\widetilde{\zeta}\, g_n) \in L^\infty(0,T;\underset{\sim}{W}^{1,q}(\Omega)) \cap L^\omega(0,T;\underset{\sim}{W}^{1,\omega}(\Omega))$, and hence $\underset{\sim}{\mathcal{A}}(\widetilde{\zeta}\, g_n) \in \underset{\sim}{L}^\infty(\Omega_T)$ as $q > 2$. As $d = 2$, (2.33a–c) yield that

$$\begin{aligned} g_n\, \underset{\sim}{u}_n &\in L^2(0,T;\underset{\sim}{L}^{s_1}(\Omega)) \qquad \forall s_1 \in [1,q) \\ \text{and} \qquad \underset{\sim}{m}_n \otimes \underset{\sim}{u}_n &\in L^2(0,T;\underset{\approx}{L}^{s_2}(\Omega)) \qquad \forall s_2 \in [1,z). \end{aligned}$$

As $z \in (\frac{q}{q-1}, \infty)$, and therefore $\frac{z}{z-1} \in (1,q)$, it follows from (2.36), (2.29a,b), (2.31) and (2.33g) that

$$\frac{\partial}{\partial t}\underset{\sim}{\mathcal{A}}(\widetilde{\zeta}\, g_n) \in L^2(0,T;\underset{\sim}{L}^{\frac{z}{z-1}}(\Omega)).$$

Noting the above and (2.33b–e,h), we see that $\eta\,\zeta\,\underset{\sim}{\mathcal{A}}(\widetilde{\zeta}\, g_n)$ is a valid test function for (2.34b). The rest of the proof is the same as that of Lemma 5.1 in [9]. □

We need also the following variation of Lemma 2.3 for later use in Section 3.

Corollary 2.1. *The results of Lemma 2.3 hold with the assumptions (2.33f,g) replaced by*

$$f_n \to f \qquad \textit{weakly in } L^2(\Omega_T), \qquad \textit{as } n \to \infty. \tag{2.37}$$

Proof. The proof is a simple adaption of the proof of Corollary 5.1 in [9]. We just highlight the differences. One can still pass to the limit $n \to \infty$ in (2.34a) using (2.37) in place of (2.33f,g). We deduce from (2.37), (2.29b) and (2.30) that for any $\varsigma \in [1,\infty)$

$$\underset{\sim}{\mathcal{A}}(\zeta\, f_n) \to \underset{\sim}{\mathcal{A}}(\zeta\, f) \qquad \text{weakly in } L^2(0,T;\underset{\sim}{L}^\varsigma(\Omega)), \qquad \text{as } n \to \infty. \tag{2.38}$$

As $\frac{z}{z-1} < q$, (2.38) ensures that one can still deduce from (2.36) that $\eta\,\zeta\,\underset{\sim}{\mathcal{A}}(\widetilde{\zeta}\, g_n)$ is a valid test function for (2.34b). The only other change to note is that, it still follows from (2.33c), (2.29a,b) and Sobolev embedding, as $d = 2$ and $z > 1$, that

$$\underset{\sim}{\mathcal{A}}(\widetilde{\zeta}\, m_n) \to \underset{\sim}{\mathcal{A}}(\widetilde{\zeta}\, m) \qquad \text{weakly in } L^\infty(0,T;\underset{\sim}{W}^{1,z}(\Omega)), \qquad \text{strongly in } L^\upsilon(0,T;\underset{\sim}{L}^2(\Omega)), \tag{2.39}$$

where $\upsilon \in [1,\infty)$. The rest of the proof is the same as the proof of Corollary 5.1 in [9]. □

3. Existence of a solution to (P)

First, we state an existence result for problem (P_κ), (2.2a–e)–(2.6a–c), on recalling (2.8).

Theorem 3.1. *Under the assumptions (2.12) and the definitions (1.2), (1.10), (1.11) and (2.13) with $\Gamma = \max\{\gamma, 8\}$, there exists a global weak solution $(\rho_\kappa, \underset{\sim}{u}_\kappa, \widehat{\psi}_\kappa)$ to problem (P_κ) for any κ, $\mathfrak{z} \in \mathbb{R}_{>0}$, in the sense that*

$$\rho_\kappa \in C_w([0,T];L^\Gamma_{\geq 0}(\Omega)) \cap H^1(0,T;H^1(\Omega)') \cap L^{\Gamma+1}(\Omega_T), \qquad \underset{\sim}{u}_\kappa \in L^2(0,T;\underset{\sim}{H}^1_0(\Omega)), \tag{3.1a}$$

$$\rho_\kappa\, \underset{\sim}{u}_\kappa \in C_w([0,T];L^{\frac{2\Gamma}{\Gamma+1}}(\Omega)) \cap W^{1,\frac{\Gamma+1}{\Gamma}}(0,T;\underset{\sim}{W}^{1,\Gamma+1}_0(\Omega)'), \tag{3.1b}$$

$$\widehat{\psi}_\kappa \in L^\upsilon(0,T;Z_1) \cap H^1(0,T;M^{-1}H^s(\Omega \times D)'), \qquad \mathcal{F}(\widehat{\psi}_\kappa) \in L^\infty(0,T;L^1_M(\Omega \times D)),$$

$$\sqrt{\widehat{\psi}_\kappa} \in L^2(0,T;H^1_M(\Omega \times D)) \qquad \textit{and} \qquad \underset{\approx}{\tau}_1(M\,\widehat{\psi}_\kappa) \in \underset{\approx}{L}^r(\Omega_T), \tag{3.1c}$$

where $\upsilon \in [1,\infty)$, $s > K+2$ and $r \in [1, \frac{8}{5})$, satisfy

$$\int_0^T \left\langle \frac{\partial \rho_\kappa}{\partial t}, \eta \right\rangle_{H^1(\Omega)} \mathrm{d}t - \int_0^T \int_\Omega \rho_\kappa\, \underset{\sim}{u}_\kappa \cdot \underset{\sim}{\nabla}_x \eta \, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t = 0 \qquad \forall \eta \in L^2(0,T;H^1(\Omega)), \tag{3.2a}$$

with $\rho_\kappa(\cdot,0) = \rho_0(\cdot)$,

$$\int_0^T \left\langle \frac{\partial(\rho_\kappa \underset{\sim}{u}_\kappa)}{\partial t}, \underset{\sim}{w} \right\rangle_{W_0^{1,\Gamma+1}(\Omega)} \mathrm{d}t + \int_0^T \int_\Omega \left[\underset{\approx}{S}(\underset{\sim}{u}_\kappa) - \rho_\kappa \underset{\sim}{u}_\kappa \otimes \underset{\sim}{u}_\kappa - p_\kappa(\rho_\kappa) \underset{\approx}{I} \right] : \underset{\approx}{\nabla}_x \underset{\sim}{w} \, \mathrm{d}\underset{\sim}{x} \, \mathrm{d}t$$

$$= \int_0^T \int_\Omega \left[\rho_\kappa \underset{\sim}{f} \cdot \underset{\sim}{w} - \left(\underset{\approx}{\tau}_1(M \widehat{\psi}_\kappa) - \mathfrak{z}\, \varrho_\kappa^2 \underset{\approx}{I} \right) : \underset{\approx}{\nabla}_x \underset{\sim}{w} \right] \mathrm{d}\underset{\sim}{x} \, \mathrm{d}t$$

$$\forall \underset{\sim}{w} \in L^{\Gamma+1}(0,T; \underset{\sim}{W}_0^{1,\Gamma+1}(\Omega)), \tag{3.2b}$$

with $(\rho_\kappa \underset{\sim}{u}_\kappa)(\cdot,0) = (\rho_0 \underset{\sim}{u}_0)(\cdot)$, *and*

$$\int_0^T \left\langle M \frac{\partial \widehat{\psi}_\kappa}{\partial t}, \varphi \right\rangle_{H^s(\Omega\times D)} \mathrm{d}t + \frac{1}{4\lambda} \sum_{i=1}^K \sum_{j=1}^K A_{ij} \int_0^T \int_{\Omega\times D} M \underset{\sim}{\nabla}_{q_j} \widehat{\psi}_\kappa \cdot \underset{\sim}{\nabla}_{q_i} \varphi \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \, \mathrm{d}t$$

$$+ \int_0^T \int_{\Omega\times D} M \left[\varepsilon \underset{\sim}{\nabla}_x \widehat{\psi}_\kappa - \underset{\sim}{u}_\kappa \widehat{\psi}_\kappa \right] \cdot \underset{\sim}{\nabla}_x \varphi \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \, \mathrm{d}t$$

$$- \int_0^T \int_{\Omega\times D} M \sum_{i=1}^K \left[\underset{\approx}{\sigma}(\underset{\sim}{u}_\kappa) \underset{\sim}{q}_i \right] \widehat{\psi}_\kappa \cdot \underset{\sim}{\nabla}_{q_i} \varphi \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \, \mathrm{d}t = 0 \quad \forall \varphi \in L^2(0,T; H^s(\Omega \times D)), \tag{3.2c}$$

with $\widehat{\psi}_\kappa(\cdot,0) = \widehat{\psi}_0(\cdot)$. *Here*

$$\varrho_\kappa = \int_D M \widehat{\psi}_\kappa \, \mathrm{d}\underset{\sim}{q} \in L^\infty(0,T; L^2_{\geq 0}(\Omega)) \cap L^2(0,T; H^1(\Omega)). \tag{3.3}$$

In addition, $(\rho_\kappa, \underset{\sim}{u}_\kappa, \widehat{\psi}_\kappa, \varrho_\kappa)$ *satisfy, for a.a.* $t' \in (0,T)$,

$$\frac{1}{2} \int_\Omega \rho_\kappa(t') \, |\underset{\sim}{u}_\kappa(t')|^2 \, \mathrm{d}\underset{\sim}{x} + \int_\Omega P_\kappa(\rho_\kappa(t')) \, \mathrm{d}\underset{\sim}{x} + k \int_{\Omega\times D} M \, \mathcal{F}(\widehat{\psi}_\kappa(t')) \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x}$$

$$+ \mu^S c_0 \int_0^{t'} \|\underset{\sim}{u}_\kappa\|^2_{H^1(\Omega)} \, \mathrm{d}t + k \int_0^{t'} \int_{\Omega\times D} M \left[\frac{a_0}{2\lambda} \left| \underset{\sim}{\nabla}_q \sqrt{\widehat{\psi}_\kappa} \right|^2 + 2\varepsilon \left| \underset{\sim}{\nabla}_x \sqrt{\widehat{\psi}_\kappa} \right|^2 \right] \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \, \mathrm{d}t$$

$$+ \mathfrak{z} \, \|\varrho_\kappa(t')\|^2_{L^2(\Omega)} + 2\, \mathfrak{z}\, \varepsilon \int_0^{t'} \|\underset{\sim}{\nabla}_x \varrho_\kappa\|^2_{L^2(\Omega)} \, \mathrm{d}t$$

$$\leq \mathrm{e}^{t'} \left[\frac{1}{2} \int_\Omega \rho_0 \, |\underset{\sim}{u}_0|^2 \, \mathrm{d}\underset{\sim}{x} + \int_\Omega P_\kappa(\rho_0) \, \mathrm{d}\underset{\sim}{x} + k \int_{\Omega\times D} M \, \mathcal{F}(\widehat{\psi}_0) \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \right.$$

$$\left. + \mathfrak{z} \int_\Omega \left(\int_D M \widehat{\psi}_0 \, \mathrm{d}\underset{\sim}{q} \right)^2 \mathrm{d}\underset{\sim}{x} + \frac{1}{2} \int_0^{t'} \|\underset{\sim}{f}\|^2_{L^\infty(\Omega)} \, \mathrm{d}t \int_\Omega \rho_0 \, \mathrm{d}\underset{\sim}{x} \right]$$

$$\leq C, \tag{3.4}$$

where $C \in \mathbb{R}_{>0}$ *is independent of* κ, $\mathfrak{z} \in (0,1]$.

Proof. In [9], after going through several regularization steps, we finally established in Theorem 5.1, via Lemmas 5.1, 5.2 and 5.4, the results (3.1a–c)–(3.4) under the stated assumptions, except that in contrast with the discussion herein, with $d = 2$ and $\gamma > 1$, the proofs there were for $d = 2$ or 3 and $\gamma > \frac{3}{2}$. Due to the form of the regularized pressure $p_\kappa(\cdot)$, see (2.3) and (1.3), it is easy to check that the above existence result for (P_κ) is not effected by relaxing the lower bound on γ to $\gamma > 1$. This lower bound only plays a crucial role in establishing an existence result for (P); see Section 6 of [9] where $\mathfrak{z} > 0$ and $d = 3$ is allowed. $\square$

From now on, we consider problem (P_κ) with the specific choice $\mathfrak{z} = \kappa$ for any $\kappa \in (0,1]$. Throughout this section, $C \in \mathbb{R}_{>0}$ will denote a generic constant independent of κ. In addition, $s' \in (1,\infty)$ will denote the conjugate of $s \in (1,\infty)$, i.e. $\frac{1}{s} + \frac{1}{s'} = 1$.

Lemma 3.1. *Under the assumptions of Theorem 3.1 we have that, for all $\kappa \in (0,1]$ with $\mathfrak{z} = \kappa$, there exists a $C \in \mathbb{R}_{>0}$, independent of κ, such that*

$$\|\rho_\kappa\|_{L^\infty(0,T;L^\gamma(\Omega))} + \|\underset{\sim}{u}_\kappa\|_{L^2(0,T;H^1(\Omega))} + \kappa^{\frac{1}{\Gamma}} \|\rho_\kappa\|_{L^\infty(0,T;L^\Gamma(\Omega))} + \left\|\sqrt{\rho_\kappa}\, \underset{\sim}{u}_\kappa\right\|_{L^\infty(0,T;L^2(\Omega))}$$

$$+ \|\rho_\kappa\, \underset{\sim}{u}_\kappa\|_{L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega))} + \|\rho_\kappa\, \underset{\sim}{u}_\kappa\|_{L^2(0,T;L^{s_1}(\Omega))} + \left\|\rho_\kappa\, |\underset{\sim}{u}_\kappa|^2\right\|_{L^2(0,T;L^{s_2}(\Omega))} \leq C, \tag{3.5a}$$

$$\left\|\frac{\partial \rho_\kappa}{\partial t}\right\|_{L^2(0,T;W^{1,s_1'}(\Omega)')} \leq C, \tag{3.5b}$$

where $s_1 \in (1,\gamma)$ and $s_2 \in [1, \frac{2\gamma}{1+\gamma})$.

Hence, there exist functions

$$\underset{\sim}{u} \in L^2(0,T;\underset{\sim}{H}^1_0(\Omega)) \qquad \text{and} \qquad \rho \in C_w([0,T];L^\gamma_{\geq 0}(\Omega)) \cap H^1(0,T;W^{1,s_1'}(\Omega)') \tag{3.6}$$

and for a subsequence of $\{(\rho_\kappa, \underset{\sim}{u}_\kappa, \widehat{\psi}_\kappa)\}_{\kappa \in (0,1]}$ it follows that, as $\kappa \to 0_+$,

$$\underset{\sim}{u}_\kappa \to \underset{\sim}{u} \qquad \text{weakly in } L^2(0,T;\underset{\sim}{H}^1_0(\Omega)), \tag{3.7}$$

$$\rho_\kappa \to \rho \qquad \text{in } C_w([0,T];L^\gamma(\Omega)), \qquad \text{weakly in } H^1(0,T;W^{1,s_1'}(\Omega)'), \tag{3.8a}$$

$$\rho_\kappa \to \rho \qquad \text{strongly in } L^2(0,T;W^{1,\gamma'}(\Omega)'), \tag{3.8b}$$

and, for any nonnegative $\eta \in C[0,T]$,

$$\int_0^T \left(\int_\Omega P(\rho)\, \mathrm{d}\underset{\sim}{x}\right) \eta\, \mathrm{d}t \leq \liminf_{\kappa \to 0_+} \int_0^T \left(\int_\Omega P(\rho_\kappa)\, \mathrm{d}\underset{\sim}{x}\right) \eta\, \mathrm{d}t. \tag{3.8c}$$

Proof. The first four bounds in (3.5a) follow immediately from (3.4). The fifth bound in (3.5a) follows from the first and fourth bounds. The sixth and seventh bounds in (3.5a) follow from combining the first and fifth bounds, respectively, with the second bound and the Sobolev embedding result (2.10). The bound (3.5b) follows immediately from (3.2a) and the sixth bound in (3.5a).

The result (3.6) and the convergence results (3.7) and (3.8a,b) follow immediately from (3.5a,b), (2.20a,b), (2.16) and (2.19).

Finally, it follows for any nonnegative $\eta \in C[0,T]$, on noting (2.13) and the convexity of $P(\cdot)$, that

$$\begin{aligned}\int_0^T \left(\int_\Omega P_\kappa(\rho_\kappa)\, \mathrm{d}\underset{\sim}{x}\right) \eta\, \mathrm{d}t &\geq \int_0^T \left(\int_\Omega P(\rho_\kappa)\, \mathrm{d}\underset{\sim}{x}\right) \eta\, \mathrm{d}t \\ &\geq \int_0^T \left(\int_\Omega [P(\rho) + P'(\rho_\kappa)\,(\rho_\kappa - \rho)]\, \mathrm{d}\underset{\sim}{x}\right) \eta\, \mathrm{d}t.\end{aligned} \tag{3.9}$$

This yields the desired result (3.8c) on noting (3.8a) and that $P'(\rho_\kappa) \in L^1(0,T;L^{\frac{\gamma}{\gamma-1}}(\Omega))$. $\square$

As $\mathfrak{z} = \kappa$, bounds, independent of κ, on ϱ_κ do not follow directly from the bounds on ϱ_κ in (3.4); we shall therefore proceed as follows. By choosing $\varphi = (\underset{\sim}{x}, \underset{\sim}{q}, t) = \phi(\underset{\sim}{x}, t) \otimes 1(\underset{\sim}{q})$, for any $\phi \in L^2(0,T;C^\infty(\overline{\Omega}))$, in (3.2c) yields, on noting (3.3) and (2.12), that

$$\int_0^T \left\langle \frac{\partial \varrho_\kappa}{\partial t}, \phi \right\rangle_{C^\infty(\overline{\Omega})} \mathrm{d}t + \int_0^T\!\!\int_\Omega \left[\varepsilon\, \underset{\sim}{\nabla}_x \varrho_\kappa - \underset{\sim}{u}_\kappa\, \varrho_\kappa\right] \cdot\, \underset{\sim}{\nabla}_x \phi\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t = 0 \qquad \forall \phi \in L^2(0,T;C^\infty(\overline{\Omega})), \tag{3.10a}$$

with

$$\varrho_\kappa(\cdot,0) = \int_D M(\underset{\sim}{q})\, \widehat{\psi}_0(\cdot,\underset{\sim}{q})\, \mathrm{d}\underset{\sim}{q} = \int_D \psi_0(\cdot,\underset{\sim}{q})\, \mathrm{d}\underset{\sim}{q} \in L^\infty_{\geq 0}(\Omega). \tag{3.10b}$$

Thus we arrive at the following result.

Lemma 3.2. *Under the assumptions of Lemma 3.1 there exists a $C \in \mathbb{R}_{>0}$, independent of κ, such that*

$$\|\widehat{\psi}_\kappa\|_{L^\infty(0,T;L^1_M(\Omega\times D))} + \left\|\underset{\sim}{\nabla}_q \sqrt{\widehat{\psi}_\kappa}\right\|_{L^2(0,T;L^2_M(\Omega\times D))} + \left\|\underset{\sim}{\nabla}_x \sqrt{\widehat{\psi}_\kappa}\right\|_{L^2(0,T;L^2_M(\Omega\times D))} \leq C, \tag{3.11a}$$

$$\|\varrho_\kappa\|_{L^\infty(0,T;L^{2-\delta}(\Omega))} + \|\varrho_\kappa\|_{L^2(0,T;W^{1,2-\delta}(\Omega))} + \|\varrho_\kappa\|_{L^2(0,T;L^{\frac{1}{\delta}}(\Omega))} + \|\varrho_\kappa\|_{L^{4-\delta}(\Omega_T)} \leq C, \tag{3.11b}$$

$$\|\underset{\approx}{\tau}_1(M\,\widehat{\psi}_\kappa)\|_{L^1(0,T;L^{2-\delta}(\Omega))} + \|\underset{\approx}{\tau}_1(M\,\widehat{\psi}_\kappa)\|_{L^{\frac{8}{5}-\delta}(\Omega_T)} \leq C, \tag{3.11c}$$

where $\delta \in (0, \frac{3}{5}]$.

Proof. The first bound in (3.11a) follows from the third bound in (3.4) and (2.11). The second and third bounds in (3.11a) follow immediately from the fifth bound in (3.4).

Choosing $\phi(\underset{\sim}{x},t) = 1(\underset{\sim}{x}) \otimes \chi_{[0,t]}$, where $\chi_{[0,t]}$ is the characteristic function of the interval $[0,t]$, for any $t \in [0,T]$ in (3.10a) yields, on noting (3.10b) and the nonnegativity of ϱ_κ, that

$$\int_\Omega \varrho_\kappa(\underset{\sim}{x},t)\,\mathrm{d}\underset{\sim}{x} = \int_\Omega \varrho_\kappa(\underset{\sim}{x},0)\,\mathrm{d}\underset{\sim}{x} \qquad \forall t \in [0,T] \qquad \Rightarrow \qquad \|\varrho_\kappa\|_{L^\infty(0,T;L^1(\Omega))} \leq C. \tag{3.12}$$

It follows from (3.3) that

$$|\underset{\sim}{\nabla}_x \varrho_\kappa| = 2\left|\int_D M\,\sqrt{\widehat{\psi}_\kappa}\,\underset{\sim}{\nabla}_x \sqrt{\widehat{\psi}_\kappa}\,\mathrm{d}\underset{\sim}{q}\right| \leq 2\,\varrho_\kappa^{\frac{1}{2}}\,\|\underset{\sim}{\nabla}_x \sqrt{\widehat{\psi}_\kappa}\|_{L^2_M(D)} \qquad \text{a.e. in } \Omega_T. \tag{3.13}$$

We deduce from Sobolev embedding, (3.13), (3.12) and (3.11a) that

$$\begin{aligned} \|\varrho_\kappa\|_{L^2(0,T;L^2(\Omega))} &\leq C\,\|\varrho_\kappa\|_{L^2(0,T;W^{1,1}(\Omega))} \\ &\leq C\,\|\varrho_\kappa\|_{L^2(0,T;L^1(\Omega))} + C\,\|\varrho_\kappa\|^{\frac{1}{2}}_{L^\infty(0,T;L^1(\Omega))}\left\|\underset{\sim}{\nabla}_x \sqrt{\widehat{\psi}_\kappa}\right\|_{L^2(0,T;L^2_M(\Omega\times D))} \leq C. \end{aligned} \tag{3.14}$$

It follows from (3.12), (3.14) and (2.9) that

$$\|\varrho_\kappa\|_{L^{\frac{\upsilon}{\upsilon-1}}(0,T;L^\upsilon(\Omega))} \leq C \quad \forall \upsilon \in (1,2] \qquad \Rightarrow \qquad \|\varrho_\kappa\|_{L^{2+\delta}(0,T;L^{2-\delta}(\Omega)} \leq C \quad \forall \delta \in [0,1). \tag{3.15}$$

One can deduce from (3.15), the second bound in (3.5a) and the Sobolev embedding result (2.10) that

$$\|\varrho_\kappa\,\underset{\sim}{u}_\kappa\|_{L^{1+\frac{\delta}{6}}(0,T;L^{2(1-\delta)}(\Omega))} \leq C \qquad \forall \delta \in (0, \tfrac{1}{2}]. \tag{3.16}$$

One can now apply Lemma 2.2 to (3.10a,b) with $\underset{\sim}{b} = -\varrho_\kappa\,\underset{\sim}{u}_\kappa$, to obtain, on noting (3.16) and (3.10b) and relabelling δ, that

$$\|\varrho_\kappa\|_{L^\infty(0,T;L^{2-\delta}(\Omega))} \leq C \qquad \forall \delta \in (0,1]. \tag{3.17}$$

Similarly to (3.16), one can deduce from (3.17), the second bound in (3.5a) and the Sobolev embedding result (2.10) that

$$\|\varrho_\kappa\,\underset{\sim}{u}_\kappa\|_{L^2(0,T;L^{2-\delta}(\Omega))} \leq C \qquad \forall \delta \in (0,1]. \tag{3.18}$$

Similarly to (3.17), on applying Lemma 2.2 to (3.10a,b) and now noting (3.18), we obtain that

$$\|\underset{\sim}{\nabla}_x \varrho_\kappa\|_{L^2(0,T;L^{2-\delta}(\Omega))} \leq C \qquad \forall \delta \in (0,1]. \tag{3.19}$$

The bounds (3.17) and (3.19) yield the first two bounds in (3.11b). We have from Sobolev embedding, for $r \in [1,2)$, that

$$\|\eta\|_{L^{\frac{2r}{2-r}}(\Omega)} \leq C\,\|\eta\|_{W^{1,r}(\Omega)} \qquad \forall \eta \in W^{1,r}(\Omega). \tag{3.20}$$

The third bound in (3.11b) then follows from (3.20) and the second bound in (3.11b). Furthermore, we have from (2.9), for $r \in [1,2)$, $\upsilon = 2\,r$ and $s = \frac{2r}{2-r}$ yielding $\upsilon\,\vartheta = 2$, and (3.20) that

$$\|\eta\|^{2r}_{L^{2r}(\Omega)} \leq \|\eta\|^{2(r-1)}_{L^r(\Omega)}\,\|\eta\|^2_{L^{\frac{2r}{2-r}}(\Omega)} \leq C\,\|\eta\|^{2(r-1)}_{L^r(\Omega)}\,\|\eta\|^2_{W^{1,r}(\Omega)} \qquad \forall \eta \in W^{1,r}(\Omega). \tag{3.21}$$

Hence (3.21) and the first two bounds in (3.11b) yield the fourth bound in (3.11b).

Now we turn our attention to the bound (3.11c). First, we deduce from (1.11), (1.6a) and as $M = 0$ on ∂D that, for $i = 1, \ldots, K$ and a.e. $(\underset{\sim}{x}, t) \in \Omega_T$,

$$\underset{\approx}{C}_i(M\,\widehat{\psi}_\kappa) = -\int_D (\underset{\sim}{\nabla}_{q_i} M)\,\underset{\sim}{q}_i^{\mathrm{T}}\,\widehat{\psi}_\kappa \,\mathrm{d}\underset{\sim}{q} = \int_D M\,(\underset{\sim}{\nabla}_{q_i}\widehat{\psi}_\kappa)\,\underset{\sim}{q}_i^{\mathrm{T}}\,\mathrm{d}\underset{\sim}{q} + \left(\int_D M\,\widehat{\psi}_\kappa \,\mathrm{d}\underset{\sim}{q}\right)\underset{\approx}{I}. \tag{3.22}$$

Hence, for $r \in [1, 2)$, on noting (3.3) and that $\underset{\sim}{\nabla}_{q_i}\widehat{\psi}_\kappa = 2\sqrt{\widehat{\psi}_\kappa}\,\underset{\sim}{\nabla}_{q_i}\sqrt{\widehat{\psi}_\kappa}$, we have for a.a. $t \in (0, T)$ that

$$\begin{aligned}\|\underset{\approx}{C}_i(M\,\widehat{\psi}_\kappa)\|_{L^r(\Omega)} &\leq C\left[\int_\Omega \varrho_\kappa^{\frac{r}{2}}\left(\int_D M\left|\underset{\sim}{\nabla}_{q_i}\sqrt{\widehat{\psi}_\kappa}\right|^2 \mathrm{d}\underset{\sim}{q}\right)^{\frac{r}{2}} \mathrm{d}\underset{\sim}{x} + \int_\Omega \varrho_\kappa^r \,\mathrm{d}\underset{\sim}{x}\right]^{\frac{1}{r}} \\ &\leq C\left[\left\|\underset{\sim}{\nabla}_{q_i}\sqrt{\widehat{\psi}_\kappa}\right\|_{L^2_M(\Omega\times D)} \|\varrho_\kappa\|^{\frac{1}{2}}_{L^{\frac{r}{2-r}}(\Omega)} + \|\varrho_\kappa\|_{L^r(\Omega)}\right].\end{aligned} \tag{3.23}$$

Hence, for r, $s \in [1, 2)$, it follows that

$$\begin{aligned}&\|\underset{\approx}{C}_i(M\,\widehat{\psi}_\kappa)\|_{L^s(0,T;L^r(\Omega))} \\ &\qquad\leq C\left[\left\|\underset{\sim}{\nabla}_{q_i}\sqrt{\widehat{\psi}_\kappa}\right\|_{L^2(0,T;L^2_M(\Omega\times D))} \|\varrho_\kappa\|^{\frac{1}{2}}_{L^{\frac{s}{2-s}}(0,T;L^{\frac{r}{2-r}}(\Omega))} + \|\varrho_\kappa\|_{L^s(0,T;L^r(\Omega))}\right].\end{aligned} \tag{3.24}$$

We deduce from (3.24) and (3.11a) that, for $i = 1, \ldots, K$,

$$\|\underset{\approx}{C}_i(M\,\widehat{\psi}_\kappa)\|_{L^s(0,T;L^r(\Omega))} \leq C \qquad \text{if} \quad \|\varrho_\kappa\|_{L^{\frac{s}{2-s}}(0,T;L^{\frac{r}{2-r}}(\Omega))} \leq C, \tag{3.25}$$

where r, $s \in [1, 2)$. The desired results (3.11c) then follow from (1.10), (3.25) and the third and fourth bounds in (3.11b). $\square$

Next, we bound the time derivative of $\widehat{\psi}_\kappa$.

Lemma 3.3. *There exists a $C \in \mathbb{R}_{>0}$, independent of κ, such that*

$$\left\|M\,\frac{\partial\widehat{\psi}_\kappa}{\partial t}\right\|_{L^{\frac{6}{5}}(0,T;H^s(\Omega\times D)')} \leq C, \tag{3.26}$$

where $s > K + 2$.

Proof. It follows from (3.2c) and (3.11a,b) that, for any $\varphi \in L^6(0, T; W^{1,\infty}(\Omega \times D))$,

$$\begin{aligned}&\left|\int_0^T\int_{\Omega\times D} M\,\frac{\partial\widehat{\psi}_\kappa}{\partial t}\varphi \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t\right| \\ &\leq 2\varepsilon\left|\int_0^T\int_{\Omega\times D} M\,\sqrt{\widehat{\psi}_\kappa}\,\underset{\sim}{\nabla}_x\sqrt{\widehat{\psi}_\kappa}\cdot\underset{\sim}{\nabla}_x\varphi \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t\right| \\ &\qquad + \frac{1}{2\,\lambda}\left|\sum_{i=1}^K\sum_{j=1}^K A_{ij}\int_0^T\int_{\Omega\times D} M\,\sqrt{\widehat{\psi}_\kappa}\,\underset{\sim}{\nabla}_{q_j}\sqrt{\widehat{\psi}_\kappa}\cdot\underset{\sim}{\nabla}_{q_i}\varphi \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t\right| \\ &\qquad + \left|\int_0^T\int_{\Omega\times D} M\,\widehat{\psi}_\kappa\,\underset{\sim}{u}_\kappa\cdot\underset{\sim}{\nabla}_x\varphi \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t\right| \\ &\qquad + \left|\int_0^T\int_{\Omega\times D} M\sum_{i=1}^K\widehat{\psi}_\kappa\left[\underset{\approx}{\sigma}(\underset{\sim}{u}_\kappa)\,\underset{\sim}{q}_i\right]\cdot\underset{\sim}{\nabla}_{q_i}\varphi \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t\right|\end{aligned}$$

$$\leq C\,\|\varrho_\kappa\|_{L^3(0,T;L^2(\Omega))}\left[\left\|\underset{\sim}{\nabla}_x\sqrt{\widehat{\psi}_\kappa}\right\|_{L^2(0,T;L^2_M(\Omega\times D))}\right.$$
$$\left.+\left\|\underset{\sim}{\nabla}_q\sqrt{\widehat{\psi}_\kappa}\right\|_{L^2(0,T;L^2_M(\Omega\times D))}+\|\underset{\sim}{u}_\kappa\|_{L^2(0,T;H^1(\Omega))}\right]\|\varphi\|_{L^6(0,T;W^{1,\infty}(\Omega\times D))}$$
$$\leq C\,\|\varphi\|_{L^6(0,T;W^{1,\infty}(\Omega\times D))}. \tag{3.27}$$

The desired result (3.26) then follows on noting that $H^s(\Omega\times D)\hookrightarrow W^{1,\infty}(\Omega\times D)$ for $s>K+2$, as $d=2$. $\square$

Following on from Lemma 3.1 we have the next lemma.

Lemma 3.4. *There exists a function*

$$\widehat{\psi}\in L^\upsilon(0,T;Z_1)\cap H^1(0,T;M^{-1}(H^s(\Omega\times D))'), \tag{3.28a}$$

where $\upsilon\in[1,\infty)$ and $s>K+2$, with finite relative entropy and Fisher information,

$$\mathcal{F}(\widehat{\psi})\in L^\infty(0,T;L^1_M(\Omega\times D))\qquad\text{and}\qquad\sqrt{\widehat{\psi}}\in L^2(0,T;H^1_M(\Omega\times D)), \tag{3.28b}$$

such that, for a further subsequence of the subsequence of $\{(\rho_\kappa,\,\underset{\sim}{u}_\kappa,\,\widehat{\psi}_\kappa)\}_{\kappa\in(0,1]}$ in Lemma 3.1, we have the following convergence results, as $\kappa\to0_+$,

$$M^{\frac12}\,\underset{\sim}{\nabla}_x\sqrt{\widehat{\psi}_\kappa}\to M^{\frac12}\,\underset{\sim}{\nabla}_x\sqrt{\widehat{\psi}}\qquad\textit{weakly in } L^2(0,T;\underset{\sim}{L}^2(\Omega\times D)), \tag{3.29a}$$

$$M^{\frac12}\,\underset{\sim}{\nabla}_q\sqrt{\widehat{\psi}_\kappa}\to M^{\frac12}\,\underset{\sim}{\nabla}_q\sqrt{\widehat{\psi}}\qquad\textit{weakly in } L^2(0,T;\underset{\sim}{L}^2(\Omega\times D)), \tag{3.29b}$$

$$M\,\frac{\partial\widehat{\psi}_\kappa}{\partial t}\to M\,\frac{\partial\widehat{\psi}}{\partial t}\qquad\textit{weakly in } L^{\frac65}(0,T;H^s(\Omega\times D)'), \tag{3.29c}$$

$$\widehat{\psi}_\kappa\to\widehat{\psi}\qquad\textit{strongly in } L^\upsilon(0,T;L^1_M(\Omega\times D)), \tag{3.29d}$$

$$\underset{\approx}{\tau}_1(M\,\widehat{\psi}_\kappa)\to\underset{\approx}{\tau}_1(M\,\widehat{\psi})\qquad\textit{strongly in } \underset{\approx}{L}^{\frac85-\delta}(\Omega_T)\cap L^1(0,T;\underset{\approx}{L}^{2-\delta}(\Omega)), \tag{3.29e}$$

where $\delta\in(0,\frac35]$, and, for a.a. $t\in(0,T)$,

$$\int_{\Omega\times D}M(\underset{\sim}{q})\,\mathcal{F}(\widehat{\psi}(\underset{\sim}{x},\underset{\sim}{q},t))\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\leq\liminf_{\kappa\to0_+}\int_{\Omega\times D}M(\underset{\sim}{q})\,\mathcal{F}(\widehat{\psi}_\kappa(\underset{\sim}{x},\underset{\sim}{q},t))\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}. \tag{3.29f}$$

In addition, we have that

$$\varrho:=\int_D M\,\widehat{\psi}\,\mathrm{d}\underset{\sim}{q}\in L^\infty(0,T;L^{2-\delta}(\Omega))\cap L^2(0,T;W^{2-\delta}(\Omega)), \tag{3.30}$$

and, as $\kappa\to0_+$,

$$\varrho_\kappa\to\varrho\qquad\textit{weakly-}\star\textit{ in } L^\infty(0,T;L^{2-\delta}(\Omega)),\qquad\textit{weakly in } L^2(0,T;W^{1,2-\delta}(\Omega)), \tag{3.31a}$$

$$\varrho_\kappa\to\varrho\qquad\textit{strongly in } L^r(0,T;L^\varsigma(\Omega)), \tag{3.31b}$$

where $r\in[1,\frac{2\varsigma}{\varsigma-1})$ and $\varsigma\in(1,\infty)$.

Proof. In order to prove the strong convergence result (3.29d), we will apply Dubinskiĭ's compactness result (2.18) with $\mathfrak{X}=L^1_M(\Omega\times D)$, $\mathfrak{X}_1=M^{-1}\,H^s(\Omega\times D)'$ and

$$\mathfrak{M}=\left\{\varphi\in Z_1:\int_{\Omega\times D}M\,\left[|\underset{\sim}{\nabla}_q\sqrt{\varphi}|^2+|\underset{\sim}{\nabla}_x\sqrt{\varphi}|^2\right]\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}<\infty\right\}. \tag{3.32}$$

See Section 5 in [3] for the proof of the compactness of the embedding $\mathfrak{M}\hookrightarrow\mathfrak{X}$, and the continuity of the embedding $\mathfrak{X}\hookrightarrow\mathfrak{X}_1$. Hence, the desired result (3.29d) for $\upsilon\in[1,2]$ follows from (2.18) with $\varsigma_0=2$, $\varsigma_1=\frac65$, and the stated choices of $\mathfrak{M}$, $\mathfrak{X}$ and $\mathfrak{X}_1$ above, on noting (3.11a) and (3.26). The desired result (3.29d) for $\upsilon\in(2,\infty)$ then follows from (3.29d) for $\upsilon=1$, the first bound in (3.11a) and an interpolation result; see Lemma 5.1 in [3]. The weak convergence result (3.29c) follows immediately from (3.26). The weak convergence results (3.29a,b) follow immediately from (3.11a), on noting an argument similar to that in the proof of Lemma 3.3 in [3] in order to identify the limit. The result (3.29f) follows from (3.29d), Fatou's lemma and the third bound in (3.4), see (6.46) in [3] for details. In addition, the convergence results (3.29a–d,f) yield the desired results (3.28a,b).

The results (3.31a) for some limit function ϱ follow immediately from the bounds (3.11b). The fact that $\varrho = \int_D M\,\widehat{\psi}\,\mathrm{d}\underset{\sim}{q}$ follows as (3.3) and (3.29d) yield, for $\upsilon \in [1,\infty)$, that, as $\kappa \to 0_+$,

$$\varrho_\kappa \to \varrho \qquad \text{strongly in } L^\upsilon(0,T;L^1(\Omega)). \tag{3.33}$$

Hence, we have the desired result (3.30). The desired strong convergence result (3.31b) follows from (3.33), (3.11b), and the fact that (2.9) yields, for $\varkappa \in (1,\infty)$,

$$\|\eta\|_{L^r(0,T;L^\varsigma(\Omega))} \leq \|\eta\|^{1-\vartheta}_{L^\varkappa(0,T;L^1(\Omega))}\,\|\eta\|^{\vartheta}_{L^2(0,T;L^\varkappa(\Omega))} \qquad \forall \eta \in L^2(0,T;L^\varkappa(\Omega)), \tag{3.34}$$

where $\varsigma \in (1,\varkappa)$, $\vartheta = \frac{\varkappa(\varsigma-1)}{\varsigma(\varkappa-1)}$ and $r = \frac{2\varkappa}{2(1-\vartheta)+\varkappa\vartheta}$.

Finally, we need to prove (3.29e). One can deduce from (3.11a), (3.29d) and (1.7a,b), on adapting the argument (4.106)–(4.117) in the proof of Lemma 4.8 in [9], that, as $\kappa \to 0_+$,

$$\underset{\approx}{\tau}_1(M\,\widehat{\psi}_\kappa) \to \underset{\approx}{\tau}_1(M\,\widehat{\psi}) \qquad \text{strongly in } \underset{\approx}{L}^1(\Omega_T). \tag{3.35}$$

This, together with (3.11c) and (2.9), yields the desired result (3.29e). $\square$

It follows from (3.2b), (3.5a), (3.11b,c), (2.12), on noting that $\gamma > 1$, and (2.10) that, for any $\underset{\sim}{w} \in L^\infty(0,T;\underset{\sim}{W}^{1,r}_0(\Omega))$ with $r = \max\{\Gamma+1,\upsilon\}$ and $\upsilon > \max\{\frac{\gamma}{\gamma-1},2\}$,

$$\begin{aligned}
&\left|\int_0^T \left\langle \frac{\partial(\rho_\kappa\,\underset{\sim}{u}_\kappa)}{\partial t}, \underset{\sim}{w}\right\rangle_{W^{1,\Gamma+1}_0(\Omega)} \mathrm{d}t - \int_0^T\int_\Omega p_\kappa(\rho_\kappa)\,\underset{\sim}{\nabla}_x\cdot\underset{\sim}{w}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t\right| \\
&\quad\leq C\,\|\rho_\kappa\|_{L^\infty(0,T;L^\gamma(\Omega))}\,\|\underset{\sim}{u}_\kappa\|^2_{L^2(0,T;L^{\frac{2\gamma\upsilon}{(\gamma-1)\upsilon-\gamma}}(\Omega))}\,\|\underset{\sim}{w}\|_{L^\infty(0,T;W^{1,\upsilon}(\Omega))} \\
&\qquad + C\,\|\underset{\sim}{u}_\kappa\|_{L^2(0,T;H^1(\Omega))}\,\|\underset{\sim}{w}\|_{L^2(0,T;H^1(\Omega))} \\
&\qquad + C\left[\|\underset{\approx}{\tau}_1(M\,\widehat{\psi}_\kappa)\|_{L^1(0,T;L^{\frac{\upsilon}{\upsilon-1}}(\Omega))} + \|\varrho_\kappa\|^2_{L^2(0,T;L^{\frac{2\upsilon}{\upsilon-1}}(\Omega))}\right]\|\underset{\approx}{\nabla}_x\underset{\sim}{w}\|_{L^\infty(0,T;L^\upsilon(\Omega))} \\
&\qquad + \|\rho_\kappa\|_{L^\infty(0,T;L^\gamma(\Omega))}\,\|\underset{\sim}{f}\|_{L^2(0,T;L^\infty(\Omega))}\,\|\underset{\sim}{w}\|_{L^2(0,T;L^3(\Omega))} \\
&\quad\leq C\,\|\underset{\sim}{w}\|_{L^\infty(0,T;W^{1,\upsilon}(\Omega))}.
\end{aligned} \tag{3.36}$$

We deduce from (3.36) with $\underset{\sim}{w} = \eta\,\underset{\sim}{v}$, where $\eta \in C^\infty_0(0,T)$ and $\underset{\sim}{v} \in L^\infty(0,T;\underset{\sim}{W}^{1,\upsilon}_0(\Omega)) \cap H^1(0,T;\underset{\sim}{L}^{s_1'}(\Omega))$, for $\upsilon > \max\{\frac{\gamma}{\gamma-1},2\}$ and $s_1 \in (1,\gamma)$, on noting (3.5a) and (2.10), as $\gamma > 1$, that

$$\begin{aligned}
&\left|\int_0^T \eta\int_\Omega p_\kappa(\rho_\kappa)\,\underset{\sim}{\nabla}_x\cdot\underset{\sim}{v}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t\right| \\
&\quad\leq \left|\int_0^T\int_\Omega \rho_\kappa\,\underset{\sim}{u}_\kappa\cdot\frac{\partial(\eta\,\underset{\sim}{v})}{\partial t}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t\right| + C\,\|\eta\|_{L^\infty(0,T)}\,\|\underset{\sim}{v}\|_{L^\infty(I;W^{1,\upsilon}(\Omega))} \\
&\quad\leq C\,\|\rho_\kappa\,\underset{\sim}{u}_\kappa\|_{L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega))}\left\|\frac{\mathrm{d}\eta}{\mathrm{d}t}\right\|_{L^1(0,T)}\|\underset{\sim}{v}\|_{L^\infty(I;L^{\frac{2\gamma}{\gamma-1}}(\Omega))} \\
&\qquad + C\,\|\eta\|_{L^\infty(0,T)}\left[\|\rho_\kappa\,\underset{\sim}{u}_\kappa\|_{L^2(0,T;L^{s_1}(\Omega))}\left\|\frac{\partial\underset{\sim}{v}}{\partial t}\right\|_{L^2(I;L^{s_1'}(\Omega))} + \|\underset{\sim}{v}\|_{L^\infty(I;W^{1,\upsilon}(\Omega))}\right] \\
&\quad\leq C\left[\left\|\frac{\mathrm{d}\eta}{\mathrm{d}t}\right\|_{L^1(0,T)} + \|\eta\|_{L^\infty(0,T)}\right]\left[\|\underset{\sim}{v}\|_{L^\infty(I;W^{1,\upsilon}(\Omega))} + \left\|\frac{\partial\underset{\sim}{v}}{\partial t}\right\|_{L^2(I;L^{s_1'}(\Omega))}\right],
\end{aligned} \tag{3.37}$$

where $I = \mathrm{supp}(\eta) \subset (0,T)$. With $\upsilon = \upsilon(\gamma)$ thus defined, let

$$\vartheta(\gamma) := \frac{\gamma}{\upsilon(\gamma)} = \begin{cases} \gamma - 1 & \text{for } 1 < \gamma \le 2, \\ \frac{\gamma}{2} & \text{for } 2 \le \gamma. \end{cases} \tag{3.38}$$

With $\vartheta(\gamma) \in \mathbb{R}_{>0}$ defined as above and $\ell \in \mathbb{N}$, we now introduce $\mathfrak{b} : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ and $\mathfrak{b}_\ell : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ such that

$$\mathfrak{b}(s) := s^\vartheta \qquad \text{and} \qquad \mathfrak{b}_\ell(s) := \begin{cases} \mathfrak{b}(s) & \text{for } 0 \le s \le \ell, \\ \mathfrak{b}(\ell) & \text{for } \ell \le s. \end{cases} \tag{3.39}$$

We note from (3.39), (3.5a) and (2.10) that, for $\upsilon(\gamma) > \max\{\frac{\gamma}{\gamma-1}, 2\}$ as in (3.37) and $\vartheta(\gamma)$ as in (3.38), we have, for $\upsilon_1 \in [1, \upsilon(\gamma))$, that

$$\|\mathfrak{b}_\ell(\rho_\kappa)\|_{L^\infty(0,T;L^\upsilon(\Omega))} \le \|\rho_\kappa^\vartheta\|_{L^\infty(0,T;L^\upsilon(\Omega))} \le \|\rho_\kappa\|^\vartheta_{L^\infty(0,T;L^\gamma(\Omega))} \le C, \tag{3.40a}$$

$$\|\mathfrak{b}_\ell(\rho_\kappa)\,\underset{\sim}{u}_\kappa\|_{L^2(0,T;L^{\upsilon_1}(\Omega))} \le \|\rho_\kappa\|^\vartheta_{L^\infty(0,T;L^\gamma(\Omega))}\,\|\underset{\sim}{u}_\kappa\|_{L^2(0,T;H^1(\Omega))} \le C, \tag{3.40b}$$

$$\|\mathfrak{b}_\ell(\rho_\kappa)\,\underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}_\kappa\|_{L^2(0,T;L^{\frac{2\gamma}{\gamma+2\vartheta}}(\Omega))} \le \|\rho_\kappa\|^\vartheta_{L^\infty(0,T;L^\gamma(\Omega))}\,\|\underset{\sim}{u}_\kappa\|_{L^2(0,T;H^1(\Omega))} \le C, \tag{3.40c}$$

where $C \in \mathbb{R}_{>0}$ is independent of κ, ϑ and ℓ.

As $\Gamma > 2$, it follows from (3.5a) and (3.2a), on extending ρ_κ and $\underset{\sim}{u}_\kappa$ from Ω to $\mathbb{R}^d$ by zero, that

$$\frac{\partial \rho_\kappa}{\partial t} + \underset{\sim}{\nabla}_x \cdot (\rho_\kappa\,\underset{\sim}{u}_\kappa) = 0 \qquad \text{in } C_0^\infty(\mathbb{R}^d \times (0,T))', \tag{3.41}$$

see Lemmas 6.8 in Novotný & Straškraba [20]. Applying Lemma 6.11 in [20] to (3.41), we have the renormalized equation, for any $\ell \in \mathbb{N}$,

$$\frac{\partial \mathfrak{b}_\ell(\rho_\kappa)}{\partial t} + \underset{\sim}{\nabla}_x \cdot (\mathfrak{b}_\ell(\rho_\kappa)\,\underset{\sim}{u}_\kappa) + (\rho_\kappa\,(\mathfrak{b}_\ell)'_+(\rho_\kappa) - \mathfrak{b}_\ell(\rho_\kappa)\,)\,\underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}_\kappa = 0 \qquad \text{in } C_0^\infty(\mathbb{R}^d \times (0,T))', \tag{3.42}$$

where $(\mathfrak{b}_\ell)'_+(\cdot)$ is the right-derivative of $\mathfrak{b}_\ell(\cdot)$ satisfying

$$(\mathfrak{b}_\ell)'_+(s) = \begin{cases} \mathfrak{b}'(s) & \text{for } 0 \le s < \ell, \\ 0 & \text{for } \ell \le s. \end{cases} \tag{3.43}$$

For any $\delta \in (0, \frac{T}{2})$, we now introduce the Friedrichs mollifier, with respect to the time variable, $\mathcal{S}_\delta : L^1(0,T;L^q(\Omega)) \to C^\infty(\delta, T-\delta; L^q(\Omega))$, $q \in [1,\infty]$,

$$\mathcal{S}_\delta(\eta)(\underset{\sim}{x}, t) = \frac{1}{\delta}\int_0^T \omega\left(\frac{t-s}{\delta}\right)\eta(\underset{\sim}{x}, s)\,\mathrm{d}s \qquad \text{a.e. in } \Omega \times (\delta, T-\delta), \tag{3.44}$$

where $\omega \in C_0^\infty(\mathbb{R})$, $\omega \ge 0$, $\mathrm{supp}(\omega) \subset (-1,1)$ and $\int_{\mathbb{R}} \omega\,\mathrm{d}s = 1$. It follows from (3.42) and (3.44) that

$$\frac{\partial \mathcal{S}_\delta(\mathfrak{b}_\ell(\rho_\kappa))}{\partial t} + \underset{\sim}{\nabla}_x \cdot [\mathcal{S}_\delta(\mathfrak{b}_\ell(\rho_\kappa)\,\underset{\sim}{u}_\kappa)] + \mathcal{S}_\delta([\rho_\kappa\,(\mathfrak{b}_\ell)'_+(\rho_\kappa) - \mathfrak{b}_\ell(\rho_\kappa)\,]\,\underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}_\kappa) = 0 \qquad \text{in } C_0^\infty(\mathbb{R}^d \times (\delta, T-\delta))'. \tag{3.45}$$

In addition, it follows from (3.44), (3.39), (3.5a), (2.10), (3.43) and (3.45), for $\varkappa \in (1,\infty)$, that

$$\begin{aligned} &\mathcal{S}_\delta(\mathfrak{b}_\ell(\rho_\kappa)) \in C^\infty(\delta, T-\delta; L^\infty(\mathbb{R}^d)), \quad \mathcal{S}_\delta(\mathfrak{b}_\ell(\rho_\kappa)\,\underset{\sim}{u}_\kappa) \in C^\infty(\delta, T-\delta; \underset{\sim}{L}^\varkappa(\mathbb{R}^d)), \\ &\mathcal{S}_\delta([\rho_\kappa\,(\mathfrak{b}_\ell)'_+(\rho_\kappa) - \mathfrak{b}_\ell(\rho_\kappa)\,]\,\underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}_\kappa), \quad \underset{\sim}{\nabla}_x \cdot [\mathcal{S}_\delta(\mathfrak{b}_\ell(\rho_\kappa)\,\underset{\sim}{u}_\kappa)] \in C^\infty(\delta, T-\delta; L^2(\mathbb{R}^d)). \end{aligned} \tag{3.46}$$

One can deduce from $\underset{\sim}{u}_\kappa \in L^2(0,T;\underset{\sim}{H}^1_0(\Omega))$ and (3.46), for $\varkappa \in (1,\infty)$, that

$$\mathcal{S}_\delta(\mathfrak{b}_\ell(\rho_\kappa)\,\underset{\sim}{u}_\kappa) \in C^\infty(\delta, T-\delta; \underset{\sim}{E}_0^{\varkappa,2}(\Omega)), \tag{3.47}$$

where we recall (2.22c). We note from (2.24a), (3.44), (3.39) and (3.40a) that $\underset{\sim}{B}((I-\fint)[\mathcal{S}_\delta(\mathfrak{b}_\ell(\rho_\kappa))]) \in L^\infty(\delta,T-\delta;\underset{\sim}{W}^{1,r}_0(\Omega))$, $r\in[1,\infty)$, and, for $\upsilon(\gamma)>\max\{\frac{2\gamma}{\gamma-1},2\}$ as in (3.37), that

$$\begin{aligned}\|\underset{\sim}{\mathcal{B}}((I-\fint)[\mathcal{S}_\delta(\mathfrak{b}_\ell(\rho_\kappa))])\|_{L^\infty(\delta,T-\delta;W^{1,\upsilon}(\Omega))} &\le C\,\|\mathcal{S}_\delta(\mathfrak{b}_\ell(\rho_\kappa))\|_{L^\infty(\delta,T-\delta;L^\upsilon(\Omega))}\\ &\le C\,\|\mathfrak{b}_\ell(\rho_\kappa)\|_{L^\infty(0,T;L^\upsilon(\Omega))}\le C,\end{aligned}\tag{3.48}$$

where $C\in\mathbb{R}_{>0}$ is independent of κ,ϑ,ℓ and δ. In addition, we have from (3.45), (3.20) with $r=\frac{2\gamma}{\gamma+2\vartheta}$ yielding $\frac{2r}{2-r}=\frac{\gamma}{\vartheta}=\upsilon$ on recalling (3.38), (3.47), (2.24a,b), (3.44), (3.39), (3.43) and (3.40b,c), for $\upsilon_1\in[1,\upsilon)$, that

$$\begin{aligned}&\left\|\frac{\partial}{\partial t}\underset{\sim}{\mathcal{B}}((I-\fint)[\mathcal{S}_\delta(\mathfrak{b}_\ell(\rho_\kappa))])\right\|_{L^2(\delta,T-\delta;L^{\upsilon_1}(\Omega))}\\
&\quad\le \|\underset{\sim}{\mathcal{B}}(\underset{\sim}{\nabla}_x\cdot[\mathcal{S}_\delta(\mathfrak{b}_\ell(\rho_\kappa)\,\underset{\sim}{u}_\kappa)])\|_{L^2(\delta,T-\delta;L^{\upsilon_1}(\Omega))}\\
&\qquad\quad+\|\underset{\sim}{\mathcal{B}}((I-\fint)[\mathcal{S}_\delta([\rho_\kappa\,(\mathfrak{b}_\ell)'_+(\rho_\kappa)-\mathfrak{b}_\ell(\rho_\kappa)]\,\underset{\sim}{\nabla}_x\cdot\underset{\sim}{u}_\kappa)])\|_{L^2(\delta,T-\delta;L^{\upsilon_1}(\Omega))}\\
&\quad\le \|\underset{\sim}{\mathcal{B}}(\underset{\sim}{\nabla}_x\cdot[\mathcal{S}_\delta(\mathfrak{b}_\ell(\rho_\kappa)\,\underset{\sim}{u}_\kappa)])\|_{L^2(\delta,T-\delta;L^{\upsilon_1}(\Omega))}\\
&\qquad\quad+\|\underset{\sim}{\mathcal{B}}((I-\fint)[\mathcal{S}_\delta([\rho_\kappa\,(\mathfrak{b}_\ell)'_+(\rho_\kappa)-\mathfrak{b}_\ell(\rho_\kappa)]\,\underset{\sim}{\nabla}_x\cdot\underset{\sim}{u}_\kappa)])\|_{L^2(\delta,T-\delta;W^{1,\frac{2\gamma}{\gamma+2\vartheta}}(\Omega))}\\
&\quad\le C\left[\|\mathfrak{b}_\ell(\rho_\kappa)\,\underset{\sim}{u}_\kappa\|_{L^2(0,T;L^{\upsilon_1}(\Omega))}+\|[\rho_\kappa\,(\mathfrak{b}_\ell)'_+(\rho_\kappa)-\mathfrak{b}_\ell(\rho_\kappa)]\,\underset{\sim}{\nabla}_x\cdot\underset{\sim}{u}_\kappa\|_{L^2(0,T;L^{\frac{2\gamma}{\gamma+2\vartheta}}(\Omega))}\right]\\
&\quad\le C\left[\|\mathfrak{b}_\ell(\rho_\kappa)\,\underset{\sim}{u}_\kappa\|_{L^2(0,T;L^{\upsilon_1}(\Omega))}+\|\mathfrak{b}_\ell(\rho_\kappa)\,\underset{\sim}{\nabla}_x\cdot\underset{\sim}{u}_\kappa\|_{L^2(0,T;L^{\frac{2\gamma}{\gamma+2\vartheta}}(\Omega))}\right]\le C,\end{aligned}\tag{3.49}$$

where $C\in\mathbb{R}_{>0}$ is independent of κ,ϑ,ℓ and δ. We now have the following result.

Lemma 3.5. *With $s_1\in(1,\gamma)$, $\upsilon(\gamma)>\max\{s_1',2\}>\max\{\frac{\gamma}{\gamma-1},2\}$ and $\vartheta(\gamma)$ as defined in (3.38), we have that*

$$\|\rho_\kappa\|_{L^{\gamma+\vartheta}(\Omega_T)}+\kappa^{\frac{1}{4+\vartheta}}\,\|\rho_\kappa\|_{L^{4+\vartheta}(\Omega_T)}+\kappa^{\frac{1}{\Gamma+\vartheta}}\,\|\rho_\kappa\|_{L^{\Gamma+\vartheta}(\Omega_T)}\le C,\tag{3.50}$$

where $C\in\mathbb{R}_{>0}$ is independent of κ.

Proof. For any $\ell\in\mathbb{N}$ and $\delta\in(0,\frac{T}{2})$, we choose $\underset{\sim}{v}=\underset{\sim}{B}((I-\fint)[\mathcal{S}_\delta(\mathfrak{b}_\ell(\rho_\kappa))])\in L^\infty(\delta,T-\delta,\underset{\sim}{W}^{1,r}_0(\Omega))$, any $r\in[1,\infty)$, and $\eta\in C^\infty_0(0,T)$, with $\mathrm{supp}(\eta)\subset(\delta,T-\delta)$, in (3.37) to obtain, on noting (2.23), (3.1a), (2.3), (3.48), (3.49) as $s_1'<\upsilon$, and (3.5a), that

$$\begin{aligned}&\left|\int_0^T\eta\int_\Omega p_\kappa(\rho_\kappa)\,\mathcal{S}_\delta(\mathfrak{b}_\ell(\rho_\kappa))\,\mathrm{d}\underset{\sim}{x}\;\mathrm{d}t\right|\\
&\qquad\le C\left[\left\|\frac{\mathrm{d}\eta}{\mathrm{d}t}\right\|_{L^1(0,T)}+\|\eta\|_{L^\infty(0,T)}\right]\left[1+\|p_\kappa(\rho_\kappa)\|_{L^1(\Omega_T)}\,\|\fint\mathcal{S}_\delta(\mathfrak{b}_\ell(\rho_\kappa))\|_{L^\infty(\delta,T-\delta)}\right]\\
&\qquad\le C\left[\left\|\frac{\mathrm{d}\eta}{\mathrm{d}t}\right\|_{L^1(0,T)}+\|\eta\|_{L^\infty(0,T)}\right].\end{aligned}\tag{3.51}$$

We now consider (3.51) with $\eta=\eta_\ell\in C^\infty_0(0,T)$ with $\mathrm{supp}(\eta_\ell)\subset(\frac{1}{\ell},T-\frac{1}{\ell})$, $\ell\in\mathbb{N}$ with $\ell>\frac{4}{T}$, where $\eta_\ell\in[0,1]$ with $\eta_\ell(t)=1$ for $t\in[\frac{2}{\ell},T-\frac{2}{\ell}]$ and $\|\frac{\mathrm{d}\eta_\ell}{\mathrm{d}t}\|_{L^\infty(0,T)}\le 2\ell$ yielding $\|\frac{\mathrm{d}\eta_\ell}{\mathrm{d}t}\|_{L^1(0,T)}\le 4$. For a fixed ℓ, we now let $\delta\to 0$ in (3.51) and using the standard convergence properties of mollifiers we obtain that

$$\left|\int_0^T\eta_\ell\int_\Omega p_\kappa(\rho_\kappa)\,\mathfrak{b}_\ell(\rho_\kappa)\,\mathrm{d}\underset{\sim}{x}\;\mathrm{d}t\right|\le C,\tag{3.52}$$

where $C\in\mathbb{R}$ is independent of ℓ and κ. Letting $\ell\to\infty$ in (3.52), and noting that $\eta_\ell\to 1$ pointwise in $(0,T)$, $\mathfrak{b}_\ell(\rho_\kappa)\to\mathfrak{b}(\rho_\kappa)=\rho_\kappa^\vartheta$ pointwise in Ω_T and Fatou's lemma, we obtain that

$$\int_0^T\int_\Omega p_\kappa(\rho_\kappa)\,\rho_\kappa^\vartheta\,\mathrm{d}\underset{\sim}{x}\;\mathrm{d}t\le C.\tag{3.53}$$

Hence the desired result (3.50) follows from (3.53), (2.3) and (1.3). $\square$

Similarly to (3.36), it follows from (3.2b), (3.5a), (3.11b,c) and (2.12), since $\gamma > 1$, that, for any $\underset{\sim}{w} \in L^{\Gamma+1}(0,T;\underset{\sim}{W}_0^{1,\varkappa}(\Omega))$ with $\varkappa = \max\{\Gamma+1, s_2'\}$ and $s_2 \in (1, \frac{2\gamma}{1+\gamma})$,

$$\begin{aligned}
&\left| \int_0^T \left\langle \frac{\partial(\rho_\kappa \underset{\sim}{u}_\kappa)}{\partial t}, \underset{\sim}{w} \right\rangle_{W_0^{1,\Gamma+1}(\Omega)} \mathrm{d}t - \int_0^T \int_\Omega p_\kappa(\rho_\kappa) \underset{\sim}{\nabla}_x \cdot \underset{\sim}{w} \,\mathrm{d}\underset{\sim}{x} \,\mathrm{d}t \right| \\
&\qquad \leq C \left[\|\rho_\kappa |\underset{\sim}{u}_\kappa|^2\|_{L^2(0,T;L^{s_2}(\Omega))} + \|\underset{\sim}{u}_\kappa\|_{L^2(0,T;H^1(\Omega))} \right] \|\underset{\sim}{w}\|_{L^2(0,T;W^{1,s_2'}(\Omega))} \\
&\qquad\quad + C \left[\|\underset{\approx}{\tau}_1(M\,\widehat{\psi}_\kappa)\|_{L^{\frac{3}{2}}(\Omega_T)} + \|\varrho_\kappa\|^2_{L^3(\Omega_T)} \right] \|\underset{\approx}{\nabla}_x \underset{\sim}{w}\|_{L^3(\Omega_T)} \\
&\qquad\quad + \|\rho_\kappa\|_{L^\infty(0,T;L^\gamma(\Omega))} \|\underset{\sim}{f}\|_{L^2(0,T;L^\infty(\Omega))} \|\underset{\sim}{w}\|_{L^2(0,T;L^3(\Omega))} \\
&\qquad \leq C \|\underset{\sim}{w}\|_{L^3(0,T;W^{1,s_3}(\Omega))},
\end{aligned} \tag{3.54}$$

where $s_3 = \max\{3, s_2'\}$. We now have the following result.

Lemma 3.6. *There exists a $C \in \mathbb{R}_{>0}$, independent of κ, such that*

$$\left\| \frac{\partial(\rho_\kappa \underset{\sim}{u}_\kappa)}{\partial t} \right\|_{L^{\frac{\Gamma+\vartheta}{\Gamma}}(0,T;W_0^{1,r}(\Omega)')} \leq C, \tag{3.55}$$

where $\vartheta(\gamma)$ is defined as in (3.38), $r = \max\{\frac{\Gamma+\vartheta}{\vartheta}, s_2'\}$, $s_1 \in (1,\gamma)$, and $s_2 \in (1, \frac{2\gamma}{1+\gamma})$.

Hence, for a further subsequence of the subsequence of Lemma 3.1, it follows that, as $\kappa \to 0_+$,

$$\rho_\kappa \underset{\sim}{u}_\kappa \to \rho \underset{\sim}{u} \qquad \text{weakly in } W^{1,\frac{\Gamma+\vartheta}{\Gamma}}(0,T;\underset{\sim}{W}_0^{1,r}(\Omega)'), \tag{3.56a}$$

$$\rho_\kappa \underset{\sim}{u}_\kappa \to \rho \underset{\sim}{u} \qquad \text{in } C_w([0,T];\underset{\sim}{L}^{\frac{2\gamma}{\gamma+1}}(\Omega)), \qquad \text{strongly in } L^2(0,T;\underset{\sim}{H}^1(\Omega)'), \tag{3.56b}$$

$$\rho_\kappa \underset{\sim}{u}_\kappa \to \rho \underset{\sim}{u} \qquad \text{weakly in } L^2(0,T;\underset{\sim}{L}^{s_1}(\Omega)), \tag{3.56c}$$

$$\rho_\kappa \underset{\sim}{u}_\kappa \otimes \underset{\sim}{u}_\kappa \to \rho \underset{\sim}{u} \otimes \underset{\sim}{u} \qquad \text{weakly in } L^2(0,T;\underset{\approx}{L}^{s_2}(\Omega)), \tag{3.56d}$$

$$\rho_\kappa \to \rho \qquad \text{weakly in } L^{\gamma+\vartheta}(\Omega_T), \tag{3.56e}$$

$$\rho_\kappa^\gamma \to \overline{\rho^\gamma} \qquad \text{weakly in } L^{\frac{\gamma+\vartheta}{\gamma}}(\Omega_T), \tag{3.56f}$$

$$\kappa\,(\rho_\kappa^4 + \rho_\kappa^\Gamma) \to 0 \qquad \text{weakly in } L^{\frac{\Gamma+\vartheta}{\Gamma}}(\Omega_T), \tag{3.56g}$$

where $\overline{\rho^\gamma} \in L_{\geq 0}^{\frac{\gamma+\vartheta}{\gamma}}(\Omega_T)$ remains to be identified.

Proof. It follows from (3.54), (2.3) and (3.50) that, for all $\underset{\sim}{w} \in L^{\frac{\Gamma+\vartheta}{\vartheta}}(0,T;\underset{\sim}{W}_0^{1,r}(\Omega))$,

$$\begin{aligned}
&\left| \int_0^T \left\langle \frac{\partial(\rho_\kappa \underset{\sim}{u}_\kappa)}{\partial t}, \underset{\sim}{w} \right\rangle_{W_0^{1,r}(\Omega)} \mathrm{d}t \right| \\
&\qquad \leq C \|\underset{\sim}{w}\|_{L^3(0,T;W^{1,s_3}(\Omega))} + \|p_\kappa(\rho_\kappa)\|_{L^{\frac{\Gamma+\vartheta}{\Gamma}}(\Omega_T)} \|\underset{\sim}{w}\|_{L^{\frac{\Gamma+\vartheta}{\vartheta}}(0,T;W^{1,\frac{\Gamma+\vartheta}{\vartheta}}(\Omega))} \\
&\qquad \leq C \|\underset{\sim}{w}\|_{L^{\frac{\Gamma+\vartheta}{\vartheta}}(0,T;W^{1,r}(\Omega))},
\end{aligned} \tag{3.57}$$

where we have noted from (3.38) that $\frac{\Gamma+\vartheta}{\vartheta} \geq \frac{\gamma+\vartheta}{\vartheta} = \upsilon + 1 > 3$. The desired result (3.55) then follows from (3.57).

The results (3.56a–d) follow from (3.5a), (3.55), (2.20a,b), (2.16), (3.8b), (3.7) and (2.19). The results (3.56e–g) follow immediately from (3.50) and (2.3). $\square$

We now have the following result.

Lemma 3.7. *The triple $(\rho, \underset{\sim}{u}, \widehat{\psi})$, defined as in Lemmas 3.1 and 3.4, satisfies*

$$\int_0^T \left\langle \frac{\partial \rho}{\partial t}, \eta \right\rangle_{W^{1,s_1'}(\Omega)} \mathrm{d}t - \int_0^T \int_\Omega \rho \underset{\sim}{u} \cdot \underset{\sim}{\nabla}_x \eta \,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t = 0 \qquad \forall \eta \in L^2(0,T;W^{1,s_1'}(\Omega)), \tag{3.58a}$$

with $\rho(\cdot,0)=\rho_0(\cdot)$ *and* $s_1\in(1,\gamma)$,

$$\int_0^T \left\langle \frac{\partial(\rho\,\underset{\sim}{u})}{\partial t},\underset{\sim}{w}\right\rangle_{W_0^{1,r}(\Omega)} \mathrm{d}t + \int_0^T\!\!\int_\Omega \left[\underset{\approx}{S}(\underset{\sim}{u}) - \rho\,\underset{\sim}{u}\otimes\underset{\sim}{u} - c_p\,\overline{\rho^\gamma}\,\underset{\approx}{I}\right] : \underset{\approx}{\nabla}_x\,\underset{\sim}{w}\;\mathrm{d}\underset{\sim}{x}\;\mathrm{d}t$$
$$= \int_0^T\!\!\int_\Omega \left[\rho\,\underset{\sim}{f}\cdot\underset{\sim}{w} - \underset{\approx}{\tau}_1(M\,\widehat{\psi}) : \underset{\approx}{\nabla}_x\,\underset{\sim}{w}\right]\mathrm{d}\underset{\sim}{x}\;\mathrm{d}t \quad \forall\underset{\sim}{w}\in L^{\frac{\gamma+\vartheta}{\vartheta}}(0,T;\underset{\sim}{W}_0^{1,r}(\Omega)), \tag{3.58b}$$

with $(\rho\,\underset{\sim}{u})(\cdot,0)=(\rho_0\,\underset{\sim}{u}_0)(\cdot)$, $\vartheta(\gamma)$ *defined as in (3.38),* $r=\max\{\frac{\gamma+\vartheta}{\vartheta},s_2'\}$ *and* $s_2\in(1,\frac{2\gamma}{1+\gamma})$, *and*

$$\int_0^T \left\langle M\,\frac{\partial\widehat{\psi}}{\partial t},\varphi\right\rangle_{H^s(\Omega\times D)} \mathrm{d}t + \frac{1}{4\,\lambda}\sum_{i=1}^K\sum_{j=1}^K A_{ij}\int_0^T\!\!\int_{\Omega\times D} M\,\underset{\sim}{\nabla}_{q_j}\widehat{\psi}\cdot\underset{\sim}{\nabla}_{q_i}\varphi\;\mathrm{d}\underset{\sim}{q}\;\mathrm{d}\underset{\sim}{x}\;\mathrm{d}t$$
$$+\int_0^T\!\!\int_{\Omega\times D} M\left[\varepsilon\,\underset{\sim}{\nabla}_x\widehat{\psi} - \underset{\sim}{u}\,\widehat{\psi}\right]\cdot\underset{\sim}{\nabla}_x\varphi\;\mathrm{d}\underset{\sim}{q}\;\mathrm{d}\underset{\sim}{x}\;\mathrm{d}t$$
$$-\int_0^T\!\!\int_{\Omega\times D} M\sum_{i=1}^K\left[\underset{\approx}{\sigma}(\underset{\sim}{u})\,\underset{\sim}{q}_i\right]\widehat{\psi}\cdot\underset{\sim}{\nabla}_{q_i}\varphi\;\mathrm{d}\underset{\sim}{q}\;\mathrm{d}\underset{\sim}{x}\;\mathrm{d}t = 0 \quad \forall\varphi\in L^6(0,T;H^s(\Omega\times D)), \tag{3.58c}$$

with $\widehat{\psi}(\cdot,0)=\widehat{\psi}_0(\cdot)$ *and* $s>K+2$.

In addition, the triple $(\rho,\underset{\sim}{u},\widehat{\psi})$ *satisfies, for a.a.* $t'\in(0,T)$,

$$\frac12\int_\Omega \rho(t')\,|\underset{\sim}{u}(t')|^2\;\mathrm{d}\underset{\sim}{x} + \int_\Omega P(\rho(t'))\,\mathrm{d}\underset{\sim}{x} + k\int_{\Omega\times D} M\,\mathcal{F}(\widehat{\psi}(t'))\,\mathrm{d}\underset{\sim}{q}\;\mathrm{d}\underset{\sim}{x}$$
$$+\mu^S c_0\int_0^{t'}\|\underset{\sim}{u}\|^2_{H^1(\Omega)}\,\mathrm{d}t + k\int_0^{t'}\!\!\int_{\Omega\times D} M\left[\frac{a_0}{2\lambda}\left|\underset{\sim}{\nabla}_q\sqrt{\widehat{\psi}}\right|^2 + 2\varepsilon\left|\underset{\sim}{\nabla}_x\sqrt{\widehat{\psi}}\right|^2\right]\mathrm{d}\underset{\sim}{q}\;\mathrm{d}\underset{\sim}{x}\;\mathrm{d}t$$
$$\le \mathrm{e}^{t'}\left[\frac12\int_\Omega\rho_0\,|\underset{\sim}{u}_0|^2\,\mathrm{d}\underset{\sim}{x} + \int_\Omega P(\rho_0)\,\mathrm{d}\underset{\sim}{x} + k\int_{\Omega\times D} M\,\mathcal{F}(\widehat{\psi}_0)\,\mathrm{d}\underset{\sim}{q}\;\mathrm{d}\underset{\sim}{x}\right.$$
$$\left. + \frac12\int_0^{t'}\|\underset{\sim}{f}\|^2_{L^\infty(\Omega)}\,\mathrm{d}t\int_\Omega\rho_0\,\mathrm{d}\underset{\sim}{x}\right]. \tag{3.59}$$

Proof. Passing to the limit $\kappa\to 0_+$ for the subsequence of Lemma 3.6 in (3.2a) yields (3.58a) subject to the stated initial condition, on noting (3.8a) and (3.56c).

Passing to the limit $\kappa\to 0_+$ for the subsequence of Lemma 3.6 in (3.2b), by recalling that $\mathfrak{z}=\kappa$ for any $\underset{\sim}{w}\in C_0^\infty(\Omega_T)$ yields (3.58b) for any $\underset{\sim}{w}\in C_0^\infty(\Omega_T)$ subject to the stated initial condition, on noting (3.7), (3.8a), (3.56a,b,d,f,g), (3.29e) and (3.31a,b). The desired result (3.58b) for any $\underset{\sim}{w}\in L^{\frac{\gamma+\vartheta}{\vartheta}}(0,T;\underset{\sim}{W}_0^{1,s_3}(\Omega))$ then follows from (3.54), (3.56f) and noting from (3.38) that $r\ge\frac{\gamma+\vartheta}{\vartheta}=\upsilon+1>3$.

Passing to the limit $\kappa\to 0_+$ for the subsequence of Lemma 3.6 in (3.2c) yields (3.58c) subject to the stated initial condition, on noting (3.7), (3.29a–d), (3.11b) and (2.10); see (4.120)–(4.124) in [9] for a similar argument.

Finally, we deduce (3.59) from (3.4) by multiplying (3.4) by any nonnegative $\eta\in C_0^\infty(0,T)$, integrating over $(0,T)$, and passing to the limit $\kappa\to 0_+$ using the results (3.56d), (3.8c), (3.29a,b,f) and (3.7). The desired result (3.59) then follows from the well-known variant of du Bois-Reymond's lemma according to which, if $\phi\in L^1(0,T)$, then

$$\int_0^T\phi\,\eta\,\mathrm{d}t\ge 0\quad\forall\eta\in C_0^\infty(0,T)\text{ with }\eta\ge 0\text{ on }(0,T)\quad\Rightarrow\quad\phi\ge 0\quad\text{a.e. in }(0,T). \tag{3.60}$$

□

We need to identify $\overline{\rho^\gamma}$ in (3.58b) and (3.56f). Similarly to (3.39), with $\ell \in \mathbb{N}$, we now introduce $\mathfrak{t} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $\mathfrak{t}_\ell : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\mathfrak{t}(s) := s \qquad \text{and} \qquad \mathfrak{t}_\ell(s) := \begin{cases} \mathfrak{t}(s) & \text{for } 0 \leq s \leq \ell, \\ \mathfrak{t}(\ell) & \text{for } \ell \leq s. \end{cases} \tag{3.61}$$

Then, similarly to (3.42), we have the renormalized equation, for any $\ell \in \mathbb{N}$,

$$\frac{\partial \mathfrak{t}_\ell(\rho_\kappa)}{\partial t} + \underset{\sim}{\nabla}_x \cdot (\mathfrak{t}_\ell(\rho_\kappa)\, \underset{\sim}{u}_\kappa) + (\rho_\kappa\, (\mathfrak{t}_\ell)'_+(\rho_\kappa) - \mathfrak{t}_\ell(\rho_\kappa)\,)\, \underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}_\kappa = 0 \qquad \text{in } C_0^\infty(\mathbb{R}^d \times (0,T))', \tag{3.62}$$

where $(\mathfrak{t}_\ell)'_+(\cdot)$ is defined similarly to (3.43). It follows from (3.61), (3.5a) and (3.62) that, for any fixed $\ell \in \mathbb{N}$,

$$\|\mathfrak{t}_\ell(\rho_\kappa)\|_{L^\infty(\Omega_T)} + \|(\rho_\kappa\, (\mathfrak{t}_\ell)'_+(\rho_\kappa) - \mathfrak{t}_\ell(\rho_\kappa))\, \underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}_\kappa\|_{L^2(\Omega_T)} + \left\| \frac{\partial \mathfrak{t}_\ell(\rho_\kappa)}{\partial t} \right\|_{L^2(0,T;H^1(\Omega)')} \leq C(\ell). \tag{3.63}$$

In order to identify $\overline{\rho^\gamma}$ in (3.58b) and (3.56f), we now apply Corollary 2.1 with (2.34a,b) being (3.62) and (3.2b) with $\mathfrak{z} = \kappa$ so that $\mu = \frac{\mu^S}{2}$ and $\lambda = \mu^B - \frac{\mu^S}{2}$, $g_n = \mathfrak{t}_\ell(\rho_\kappa)$ for a fixed $\ell \in \mathbb{N}$, $\underset{\sim}{u}_n = \underset{\sim}{u}_\kappa$, $\underset{\sim}{m}_n = \rho_\kappa\, \underset{\sim}{u}_\kappa$, $p_n = p_\kappa(\rho_\kappa)$, $\underset{\approx}{\tau}_n = \underset{\approx}{\tau}_1(M\,\widehat{\psi}_\kappa) - \kappa\, \varrho_\kappa^2\, \underset{\approx}{I}$, $f_n = -(\rho_\kappa\, (\mathfrak{t}_\ell)'_+(\rho_\kappa) - \mathfrak{t}_\ell(\rho_\kappa))\, \underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}_\kappa$ and $\underset{\sim}{F}_n = \rho_\kappa\, \underset{\sim}{f}$. With $\{(\rho_\kappa, \underset{\sim}{u}_\kappa, \widehat{\psi}_\kappa)\}_{\kappa>0}$ being the subsequence (not indicated) of Lemma 3.6, we have that (2.33a–d) hold with $g = \overline{\mathfrak{t}_\ell(\rho)}$, $\underset{\sim}{u} = \underset{\sim}{u}$, $\underset{\sim}{m} = \rho\, \underset{\sim}{u}$ and $p = c_p\, \overline{\rho^\gamma}$ for any $q \in [1,\infty)$, $\omega = \infty$, $z = \frac{2\gamma}{\gamma+1}$ and $r = \frac{\Gamma+\vartheta}{\Gamma}$, on recalling (3.63), (2.20a,b), (3.7), (3.56b) and (3.56f,g). As $\gamma > 1$, we can choose $q > \frac{2\gamma}{\gamma-1}$ so that $z = \frac{2\gamma}{\gamma+1} > \frac{q}{q-1}$. Hence, the constraints on q, r, ω and z in Lemma 2.3 hold. In addition, the result (2.33e) holds with $\underset{\approx}{\tau} = \underset{\approx}{\tau}_1(M\,\widehat{\psi})$ on recalling (3.29e) and (3.31a,b). The result (2.33h) with $\underset{\sim}{F} = \rho\, \underset{\sim}{f}$ and $s = \min\{\gamma, 2\}$ follows from (3.8a) and (2.12). Finally, the result (2.37) holds with $f = -\overline{(\rho\, (\mathfrak{t}_\ell)'_+(\rho) - \mathfrak{t}_\ell(\rho))\, \underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}}$ on recalling (3.63). Hence, we obtain from (2.35) for the subsequence of Lemma 3.6 that, for any fixed $\ell \in \mathbb{N}$ and for all $\zeta \in C_0^\infty(\Omega)$ and $\eta \in C_0^\infty(0,T)$,

$$\begin{aligned} &\lim_{\kappa \to 0_+} \int_0^T \eta \left(\int_\Omega \zeta\, \mathfrak{t}_\ell(\rho_\kappa)\, [p_\kappa(\rho_\kappa) - \mu^\star\, \underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}_\kappa]\, \mathrm{d}\underset{\sim}{x} \right) \mathrm{d}t \\ &\qquad\qquad = \int_0^T \eta \left(\int_\Omega \zeta\, \overline{\mathfrak{t}_\ell(\rho)}\, [c_p\, \overline{\rho^\gamma} - \mu^\star\, \underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}]\, \mathrm{d}\underset{\sim}{x} \right) \mathrm{d}t, \end{aligned} \tag{3.64}$$

where $\mu^\star := \frac{1}{2}\, \mu^S + \mu^B$.

We deduce from (3.64), (3.61), (2.3), (1.3), (3.56f,g) and (3.7) that, for any fixed $\ell \in \mathbb{N}$,

$$c_p \left[\overline{\mathfrak{t}_\ell(\rho)\, \rho^\gamma} - \overline{\mathfrak{t}_\ell(\rho)}\, \overline{\rho^\gamma} \right] = \mu^\star \left[\overline{\mathfrak{t}_\ell(\rho)\, \underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}} - \overline{\mathfrak{t}_\ell(\rho)}\, \underset{\sim}{\nabla}_x \cdot \underset{\sim}{u} \right] \qquad \text{a.e. in } \Omega_T, \tag{3.65}$$

where, as $\kappa \to 0_+$,

$$\mathfrak{t}_\ell(\rho_\kappa)\, \rho_\kappa^\gamma \to \overline{\mathfrak{t}_\ell(\rho)\, \rho^\gamma} \qquad \text{weakly in } L^{\frac{\gamma+\vartheta}{\gamma}}(\Omega_T), \tag{3.66a}$$

$$\text{and} \qquad \mathfrak{t}_\ell(\rho_\kappa)\, \underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}_\kappa \to \overline{\mathfrak{t}_\ell(\rho)\, \underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}} \qquad \text{weakly in } L^2(\Omega_T). \tag{3.66b}$$

We now follow the argument (6.40)–(6.48a,b) in [9], which is a brief summary of the discussion in Sections 7.10.2–7.10.5 in Novotný & Straškraba [20], to deduce that $\overline{p(\rho)} = p(\rho)$. First, it follows from (3.56f) and (3.66a) that, for any fixed $\ell \in \mathbb{N}$,

$$\begin{aligned} &\int_{\Omega_T} \left[\overline{\mathfrak{t}_\ell(\rho)\, \rho^\gamma} - \overline{\mathfrak{t}_\ell(\rho)}\, \overline{\rho^\gamma} \right] \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t \\ &\quad = \lim_{\kappa \to 0_+} \left[\int_{\Omega_T} (\mathfrak{t}_\ell(\rho_\kappa) - \mathfrak{t}_\ell(\rho))\, (\rho_\kappa^\gamma - \rho^\gamma)\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t + \int_{\Omega_T} (\mathfrak{t}_\ell(\rho) - \overline{\mathfrak{t}_\ell(\rho)})\, (\overline{\rho^\gamma} - \rho^\gamma)\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t \right] \\ &\quad \geq \limsup_{\kappa \to 0_+} \int_{\Omega_T} |\mathfrak{t}_\ell(\rho_\kappa) - \mathfrak{t}_\ell(\rho)|^{\gamma+1}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t, \end{aligned} \tag{3.67}$$

where we have noted that the second term on the second line is nonnegative as the function $\mathfrak{t}_\ell : s \in [0,\infty) \mapsto \mathfrak{t}_\ell(s)$ is concave and the function $s \in [0,\infty) \mapsto s^\gamma$ is convex.

We now introduce $\mathfrak{L}_\ell : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $\ell \in \mathbb{N}$, such that $s\,\mathfrak{L}_\ell'(s) - \mathfrak{L}_\ell(s) = \mathfrak{t}_\ell(s)$ for all $s \in [0,\infty)$, so that

$$\mathfrak{L}_\ell(s) := \begin{cases} \mathfrak{L}(s) := s\log s & \text{for } 0 \leq s \leq \ell, \\ s\log\ell + s - \ell & \text{for } \ell \leq s. \end{cases} \tag{3.68}$$

Next, one deduces, via renormalization, from (3.58a) and (3.2a) that, for any fixed $\ell \in \mathbb{N}$ and for any $t' \in (0,T]$,

$$\int_\Omega [\mathfrak{L}_\ell(\rho)(t') - \mathfrak{L}_\ell(\rho_0)]\, \mathrm{d}\underset{\sim}{x} = -\int_0^{t'}\int_\Omega \mathfrak{t}_\ell(\rho)\, \underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t, \tag{3.69a}$$

$$\int_\Omega [\mathfrak{L}_\ell(\rho_\kappa)(t') - \mathfrak{L}_\ell(\rho_0)]\, \mathrm{d}\underset{\sim}{x} = -\int_0^{t'}\int_\Omega \mathfrak{t}_\ell(\rho_\kappa)\, \underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}_\kappa\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t. \tag{3.69b}$$

Although proving (3.69b) is straightforward as $\rho_\kappa \in C_w([0,T]; L^\Gamma_{\geq 0}(\Omega))$, the proof of (3.69a) is not, since $\rho \in C_w([0,T]; L^\gamma_{\geq 0}(\Omega))$, and so ρ may not be in $L^2(\Omega_T)$ as $\gamma > 1$. Nevertheless, (3.69a) can still be established by adapting Lemma 7.57 in [20], as its proof only requires $\gamma > 1$ and $\vartheta(\gamma) > 0$. Subtracting (3.69a) from (3.69b), and passing to the limit $\kappa \to 0_+$, one deduces from (3.66b) that, for any fixed $\ell \in \mathbb{N}$ and for any $t' \in (0,T]$,

$$\int_\Omega \left[\overline{\mathfrak{L}_\ell(\rho)(t')} - \mathfrak{L}_\ell(\rho)(t')\right] \mathrm{d}\underset{\sim}{x} = \int_0^{t'}\int_\Omega \left[\mathfrak{t}_\ell(\rho)\, \underset{\sim}{\nabla}_x \cdot \underset{\sim}{u} - \overline{\mathfrak{t}_\ell(\rho)\, \underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}}\right] \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t, \tag{3.70}$$

where, on noting (3.8a) and the convexity of $\mathfrak{L}_\ell$,

$$\mathfrak{L}_\ell(\rho_\kappa)(t') \to \overline{\mathfrak{L}_\ell(\rho)(t')} \geq \mathfrak{L}_\ell(\rho)(t') \qquad \text{weakly in } L^\gamma(\Omega), \qquad \text{as } \kappa \to 0_+. \tag{3.71}$$

It follows from (3.67), (3.65), (3.70), (3.71) and (3.28a) that, for any fixed $\ell \in \mathbb{N}$,

$$\begin{aligned} \limsup_{\kappa\to 0_+} \|\mathfrak{t}_\ell(\rho) - \mathfrak{t}_\ell(\rho_\kappa)\|^{\gamma+1}_{L^{\gamma+1}(\Omega_T)} &\leq \frac{\mu^\star}{c_p}\int_{\Omega_T}\left[\overline{\mathfrak{t}_\ell(\rho)\, \underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}} - \overline{\mathfrak{t}_\ell(\rho)}\, \underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}\right] \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t \\ &\leq \frac{\mu^\star}{c_p}\int_{\Omega_T}\left[\mathfrak{t}_\ell(\rho) - \overline{\mathfrak{t}_\ell(\rho)}\right] \underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t \\ &\leq C\, \|\mathfrak{t}_\ell(\rho) - \overline{\mathfrak{t}_\ell(\rho)}\|_{L^2(\Omega_T)} \\ &\leq C\, \limsup_{\kappa\to 0_+} \|\mathfrak{t}_\ell(\rho) - \mathfrak{t}_\ell(\rho_\kappa)\|_{L^2(\Omega_T)} \\ &\leq C\, \limsup_{\kappa\to 0_+} \|\mathfrak{t}_\ell(\rho) - \mathfrak{t}_\ell(\rho_\kappa)\|_{L^{\gamma+1}(\Omega_T)}, \end{aligned} \tag{3.72}$$

where $C \in \mathbb{R}_{>0}$ is independent of ℓ and κ. It is easily deduced from (3.50), (3.56e) and (3.61) that, for all $\ell \in \mathbb{N}$, $\kappa > 0$ and $r \in [1, \gamma+\vartheta)$,

$$\|\rho_\kappa - \mathfrak{t}_\ell(\rho_\kappa)\|_{L^r(\Omega_T)} + \|\rho - \mathfrak{t}_\ell(\rho)\|_{L^r(\Omega_T)} + \|\rho - \overline{\mathfrak{t}_\ell(\rho)}\|_{L^r(\Omega_T)} \leq C\, \ell^{1-\frac{\gamma+\vartheta}{r}}, \tag{3.73}$$

where $C \in \mathbb{R}_{>0}$ is independent of ℓ and κ. It follows from (3.72), (3.73) and (2.9) that

$$\lim_{\ell\to\infty} \limsup_{\kappa\to 0_+} \|\mathfrak{t}_\ell(\rho) - \mathfrak{t}_\ell(\rho_\kappa)\|_{L^{\gamma+1}(\Omega_T)} = 0. \tag{3.74}$$

It immediately follows from (3.73), (3.74), (3.50), (2.9) and (3.56f), on possibly extracting a further subsequence (not indicated) of the subsequence in Lemma 3.6, that, as $\kappa \to 0_+$,

$$\rho_\kappa \to \rho \qquad \text{strongly in } L^s(\Omega_T), \qquad \text{for any } s \in [1, \gamma+\vartheta(\gamma)), \tag{3.75a}$$

$$\rho_\kappa^\gamma \to \rho^\gamma \qquad \text{weakly in } L^{\frac{\gamma+\vartheta}{\gamma}}(\Omega_T), \qquad \text{that is, } \overline{\rho^\gamma} = \rho^\gamma. \tag{3.75b}$$

Finally, we have the following result.

Theorem 3.2. *The triple $(\rho, \underset{\sim}{u}, \widehat{\psi})$, defined as in Lemmas 3.1 and 3.4, is a global weak solution to problem (P) for $d = 2$, $\gamma > 1$ and $\mathfrak{z} = 0$, in the sense that (3.58a,c), with their respective initial conditions, hold and*

$$\int_0^T \left\langle \frac{\partial(\rho \underset{\sim}{u})}{\partial t}, \underset{\sim}{w} \right\rangle_{W_0^{1,r}(\Omega)} \mathrm{d}t + \int_0^T \int_\Omega \left[\underset{\approx}{S}(\underset{\sim}{u}) - \rho \underset{\sim}{u} \otimes \underset{\sim}{u} - c_p \rho^\gamma \underset{\approx}{I} \right] : \underset{\approx}{\nabla}_x \underset{\sim}{w} \, \mathrm{d}\underset{\sim}{x} \, \mathrm{d}t$$

$$= \int_0^T \int_\Omega \left[\rho \underset{\sim}{f} \cdot \underset{\sim}{w} - \underset{\approx}{\tau}_1(M \widehat{\psi}) : \underset{\approx}{\nabla}_x \underset{\sim}{w} \right] \mathrm{d}\underset{\sim}{x} \, \mathrm{d}t \qquad \forall \underset{\sim}{w} \in L^{\frac{\gamma+\vartheta}{\vartheta}}(0, T; \underset{\sim}{W}_0^{1,r}(\Omega)), \tag{3.76}$$

with $(\rho \underset{\sim}{u})(\cdot, 0) = (\rho_0 \underset{\sim}{u}_0)(\cdot)$, $\vartheta(\gamma)$ defined as in (3.38), $r = \max\{\frac{\gamma+\vartheta}{\vartheta}, s_2'\}$ and $s_2 \in (1, \frac{2\gamma}{1+\gamma})$. In addition, the weak solution $(\rho, \underset{\sim}{u}, \widehat{\psi})$ satisfies the energy inequality (3.59).

Proof. The results (3.58a,c) and (3.59) have already been proved in Lemma 3.7. Equation (3.76) was established in Lemma 3.7 with ρ^γ replaced by $\overline{\rho^\gamma}$; see (3.58b). The desired result (3.76) then follows immediately from (3.58b) and (3.75b). □

4. Conclusions

We have proved the existence of global-in-time weak solutions to a general class of models that arise from the kinetic theory of dilute solutions of nonhomogeneous polymeric liquids, where the polymer molecules are idealized as bead-spring chains with finitely extensible nonlinear elastic (FENE) type spring potentials. The class of models under consideration involved the unsteady, compressible, isentropic, isothermal Navier–Stokes system, with constant shear and bulk viscosity coefficients, in a bounded domain Ω in $\mathbb{R}^d$, $d = 2$, for the density ρ, the velocity $\underset{\sim}{u}$ and the pressure p of the fluid, with an equation of state of the form $p(\rho) = c_p \rho^\gamma$, where c_p is a positive constant and $\gamma > 1$. The right-hand side of the Navier–Stokes momentum equation included an elastic extra-stress tensor, defined as the classical Kramers expression. The elastic extra-stress tensor stems from the random movement of the polymer chains and involves the associated probability density function that satisfies a Fokker–Planck-type parabolic equation, a crucial feature of which is the presence of a centre-of-mass diffusion term. Our analysis required no structural assumptions on the drag term in the Fokker–Planck equation; in particular, the drag term need not be corotational. With a nonnegative initial density $\rho_0 \in L^\infty(\Omega)$ for the continuity equation; a square-integrable initial velocity datum $\underset{\sim}{u}_0$ for the Navier–Stokes momentum equation; and a nonnegative initial probability density function ψ_0 for the Fokker–Planck equation, which has finite relative entropy with respect to the Maxwellian M associated with the spring potential in the model, we proved, *via* a limiting procedure on a pressure-regularization parameter, the existence of a global-in-time bounded-energy weak solution $t \mapsto (\rho(t), \underset{\sim}{u}(t), \psi(t))$ to the coupled Navier–Stokes–Fokker–Planck system, satisfying the initial condition $(\rho(0), \underset{\sim}{u}(0), \psi(0)) = (\rho_0, \underset{\sim}{u}_0, \psi_0)$. This shows that the quadratic interaction term, which the classical Kramers expression was supplemented by in our proof of the existence of global-in-time weak solutions to the compressible Navier–Stokes–Fokker–Planck system for $d \in \{2, 3\}$ in [9], can be omitted in the case of $d = 2$.

Our analysis applies, without alterations, to some other familiar *monotone* equations of state, such as the (Kirkwood-modified) Tait equation of state (cf. Remark 1.1 in [9]). As a starting point for the extension of this work to nonmonotone equations of state we refer the reader to the work of Feireisl [13] in the context of the compressible Navier–Stokes equations; see also Section 7.12.3 in [20], which explains how, in the case of the compressible Navier–Stokes equations at least, the existence proof for monotone equations of state can be extended, with minor modifications, to the case of nonmonotone equations of state. For the limiting case of a barotropic compressible Navier–Stokes system with a linear pressure law ($\gamma = 1$), in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2$, Plotnikov and Weigant [21] have recently shown the existence of finite-energy global weak solutions, the main additional technical difficulty in this case being the lack of gain in the integrability of the pressure. Their proof was based on properties of the Radon transform of the (spatially localized) density, and its connections with the $H^{-\frac{1}{2}}(\Omega)$ norm. Whether a similar result can be shown to hold, with $d = 2$ and $\gamma = 1$, for the coupled Navier–Stokes–Fokker–Planck system considered herein remains unclear.

The reason why in the case of $d = 3$ we are unable to admit $\mathfrak{z} = 0$ is because the bounds (3.11b) in Lemma 3.2 heavily rely on d being equal to 2; with $\mathfrak{z} = 0$ and $d = 3$ we cannot prove a strong enough uniform bound on ϱ_κ to be able to establish the analogue of Lemma 3.3, that is the bound

$$\left\| M \frac{\partial \widehat{\psi}_\kappa}{\partial t} \right\|_{L^r(0,T;H^s(\Omega\times D)')} \leq C, \tag{4.1}$$

with $r \geq 1$, $s > 0$, and a positive constant C, independent of κ. Consequently the proof of Lemma 3.4 fails, and we are then unable to pass to the simultaneous limits $\kappa = \mathfrak{z} \to 0$, as we were able to do in the case of $d = 2$, where, thanks to Lemma 3.3, the bound (4.1) holds with $r = \frac{6}{5}$ and $s > K + 2$. In [9] we proved the existence of global-in-time weak solutions to problem (P) for $d = 2, 3$, provided that $\gamma > \frac{3}{2}$ in (1.3) and $\mathfrak{z} > 0$ in (1.9). Here, in the case of $d = 2$, we have extended these results to $\gamma > 1$ and $\mathfrak{z} = 0$. It is unclear whether for $d = 3$ and $\gamma > \frac{3}{2}$ the existence of global-in-time weak solutions can also be shown to hold when $\mathfrak{z} = 0$. This difficulty is specific to the *compressible* Navier–Stokes–Fokker–Planck system and does not arise in the case of the *incompressible* Navier–Stokes–Fokker–Planck model, where the value of $\mathfrak{z}$ plays no role whatsoever and any value of $\mathfrak{z} \in \mathbb{R}$ is admissible in the proof of existence of a global weak solution for both $d = 2$ and $d = 3$.

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