Numerical Solution of Differential Equations: Problem Sheet 2 (of 4)

1. Consider the Runge–Kutta method $y_{n+1} = y_n + h(\alpha\, k_1 + \beta\, k_2)$ where $k_1 = f(x_n, y_n)$ and $k_2 = f(x_n + \gamma h, y_n + \gamma h k_1)$, and where α, β, γ are real parameters.

   (a) Show that there is a choice of these parameters such that the order of the method is 2.

   (b) Suppose that a second-order method of the above form is applied to the initial value problem $y' = -\lambda y$, $y(0) = 1$, where λ is a positive real number. Show that the sequence $(y_n)_{n\geq 0}$ is bounded if and only if $h \leq \frac{2}{\lambda}$.
   Show further that, for such λ,

$$|y(x_n) - y_n| \leq \frac{1}{6}\lambda^3 h^2 x_n, \quad n \geq 0.$$

2. a) What does it mean to say that a linear multistep method is *zero-stable*? Formulate an equivalent characterization of zero-stability of a linear multistep method in terms of the roots of its first characteristic polynomial.

   b) Define the consistency error of a linear multistep method.

   c) Show that there is a value of the parameter b such that the linear multistep method defined by the formula $y_{n+3}+(2b-3)(y_{n+2}-y_{n+1})-y_n = hb(f_{n+2}+f_{n+1})$ is fourth-order accurate. Show further that the method is *not* zero–stable for this value of b.

3. A linear multistep method $\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f(x_{n+j}, y_{n+j})$, $n \geq 0$, for the numerical solution of the initial-value problem $y' = f(x, y)$, $y(x_0) = y_0$, on the mesh $\{x_j : x_j = x_0 + jh\}$ of uniform spacing $h > 0$ is said to be *absolutely stable* for a certain h if, when applied to the model problem $y' = \lambda y$, $y(0) = 1$, with $\lambda < 0$, on the interval $x \in [0, \infty)$, the sequence $(|y_n|)_{n\geq k}$ decays exponentially fast; i.e., $|y_n| \leq \mathrm{C}\,\mathrm{e}^{-\mu n}$, $n \geq k$, for some positive constants C and μ.

   a) Show that a linear multistep method is absolutely stable for $h > 0$ if, and only if, all roots z of its *stability polynomial* $\pi(z; \bar{h}) = \rho(z) - \bar{h}\sigma(z)$, where ρ and σ are the first and second characteristic polynomial of the linear multistep method respectively and $\bar{h} = \lambda h$, belong to the open unit disk $D = \{z : |z| < 1\}$ in the complex plane.

   b) For each of the following methods find the range of $h > 0$ for which it is absolutely stable (when applied to $y' = \lambda y$, $y(0) = 1$, $\lambda < 0$, $x \in [0, \infty)$):

      b1) $y_{n+1} - y_n = hf(x_n, y_n)$;

      b2) $y_{n+1} - y_n = hf(x_{n+1}, y_{n+1})$;

      b3) $y_{n+2} - y_n = \frac{1}{3}h\left(f(x_{n+2}, y_{n+2}) + 4f(x_{n+1}, y_{n+1}) + f(x_n, y_n)\right)$.

4. Which of the following would you regard a stiff initial-value problem?

   a) $y' = -(10^5 \mathrm{e}^{-10^4 x} + 1)(y - 1)$, $y(0) = 2$, on the interval $x \in [0, 1]$. Note that the solution can be found in closed form:

$$y(x) = \mathrm{e}^{10(\mathrm{e}^{-10^4 x}-1)}\mathrm{e}^{-x} + 1.$$

b)

$$\begin{aligned} y_1' &= -0.5y_1 + 0.501y_2, \qquad & y_1(0) = 1.1, \\ y_2' &= 0.501y_1 - 0.5y_2, \qquad & y_2(0) = -0.9, \end{aligned}$$

on the interval $x \in [0, 1]$.

5. Consider the θ–method

$$y_{n+1} = y_n + h\left[(1-\theta)f_n + \theta f_{n+1}\right]$$

for $\theta \in [0, 1]$.

a) Show that the method is A–stable for $\theta \in [1/2, 1]$.

b) A method is said to be $A(\alpha)$–stable, $\alpha \in (0, \pi/2)$, if its region of absolute stability (as a set in the complex plane), contains the infinite wedge $\{\bar{h} : \pi - \alpha < \arg(\bar{h}) < \pi + \alpha\}$. Find all $\theta \in [0, 1]$ such that the θ–method is $A(\alpha)$–stable for some $\alpha \in (0, \pi/2)$.

Note: In the next question you will find it helpful to exploit the following result, known as *Schur's criterion*. Consider the polynomial $\phi(z) = c_k z^k + \cdots + c_1 z + c_0$, $c_k \neq 0$, $c_0 \neq 0$, with complex coefficients. The polynomial ϕ is said to be a *Schur polynomial* if each of its roots z_j satisfies $|z_j| < 1$, $j = 1, \ldots, k$. Given the polynomial $\phi(z)$, as above, consider the polynomial

$$\hat{\phi}(z) = \bar{c}_0 z^k + \bar{c}_1 z^{k-1} + \ldots + \bar{c}_{k-1} z + \bar{c}_k \ ,$$

where $\bar{c}_j$ denotes the complex conjugate of c_j, $j = 1, \ldots, k$. Further, let us define

$$\phi_1(z) = \frac{1}{z}\left[\hat{\phi}(0)\phi(z) - \phi(0)\hat{\phi}(z)\right] \ .$$

Clearly ϕ_1 has degree $\leq k-1$. The polynomial ϕ is a Schur polynomial if, and only if, $|\hat{\phi}(0)| > |\phi(0)|$ and ϕ_1 is a Schur polynomial.

6. Show that the second-order backward differentiation method

$$3y_{n+2} - 4y_{n+1} + y_n = 2hf(x_{n+2}, y_{n+2})$$

is A-stable.