Numerical Solution of Differential Equations:
Problem Sheet 3 (of 4)

1. We consider the system of scalar ODEs

$$y' = v, \quad v' = f(y), \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function.

(a) Let F be a primitive function of f. Show that $H(v, y) = v^2/2 - F(y)$ is a Hamiltonian of (1) and verify that it is indeed a first integral.

(b) Let $\boldsymbol{z} = \begin{pmatrix} y \\ v \end{pmatrix}$ and $\boldsymbol{g}(\boldsymbol{z}) = \begin{pmatrix} v \\ f(y) \end{pmatrix}$, and let $\boldsymbol{\Psi}$ be the discrete evolution operator of the implicit midpoint rule associated with (1). Show that

$$\boldsymbol{D}_{\boldsymbol{z}_0}(\boldsymbol{\Psi}(0, \boldsymbol{z}_0, h, \boldsymbol{g})) = \frac{1}{1 - \frac{h^2}{4} f'(*)} \begin{pmatrix} 1 + \frac{h^2}{4} f'(*) & h \\ h f'(*) & 1 + \frac{h^2}{4} f'(*) \end{pmatrix},$$

where $f'(*) := f'\left(\frac{y_0 + y_1}{2}\right)$.

(c) Hence deduce that the implicit midpoint rule is symplectic.

Suppose that we have discrete data $\{U_j\}$ defined on an infinite grid $x_j = j\Delta x$, $j = 0, \pm 1, \pm 2, \ldots$. Let δ and μ be the discrete differentiation and smoothing operators defined by

$$(\delta U)_j = (U_{j+1} - U_{j-1})/(2\Delta x), \qquad (\mu U)_j = (U_{j+1} + U_{j-1})/2.$$

2. Determine the functions δU, δV, μU, μV for $U = (\ldots, 1, -1, 1, -1, 1, -1, 1, \ldots)$ and $V = (\ldots, 1, 0, -1, 0, 1, 0, -1, 0, \ldots)$.

3. Determine what effect δ and μ have on the function U defined by $U_j = \mathrm{e}^{\imath k x_j}$, $j = 0, \pm 1, \pm 2, \ldots$, where k is a real constant (the wave number).

4. The semidiscrete Fourier transform of a function U defined on the infinite grid $x_j = j\Delta x$, $j = 0, \pm 1, \pm 2, \ldots$, is the function $k \mapsto \hat{U}(k)$, $k \in [-\pi/\Delta x, \pi/\Delta x]$, defined by

$$\hat{U}(k) = \Delta x \sum_{j=-\infty}^{\infty} \mathrm{e}^{-\imath k x_j} U_j \,.$$

[The reason for the restriction on k is that the wave numbers $|k| > \pi/\Delta x$ are not resolvable on a grid of spacing Δx; this is the phenomenon of *aliasing*.]

Show that the inverse of the semidiscrete Fourier transform is given by the formula

$$U_j = \frac{1}{2\pi} \int_{-\pi/\Delta x}^{\pi/\Delta x} \mathrm{e}^{\imath k j \Delta x} \hat{U}(k) \, \mathrm{d}k \,.$$

Describe the relationship between $\hat{U}(k)$, and $\widehat{\delta U}(k)$ and $\widehat{\mu U}(k)$. [Note that this is a restatement of Question 3.]

P.T.O.

The ratios $\widehat{\delta U}/\hat{U}$ and $\widehat{\mu U}/\hat{U}$ are referred to as *Fourier multipliers*. Sketch the graphs of these Fourier multipliers as functions of $k \in [-\pi/\Delta x, \pi/\Delta x]$.

One would think that applying μ repeatedly to U should lead to a function that is much smoother than U. Explain this effect by considering a sketch of the multiplier function $\widehat{\mu^m U}/\hat{U}$ for $m \gg 1$. Your analysis should reveal that taking successive powers of μ is not a perfect smoothing procedure. Explain.

5. The $\ell_2(-\infty,\infty)$ norm of U and the $L_2(-\pi/\Delta x,\, \pi/\Delta x)$ norm of $\hat{U}$ are defined, respectively, by

$$\|U\|_{\ell_2} = \left(\Delta x \sum_{j=-\infty}^{\infty} |U_j|^2\right)^{1/2}, \qquad \|\hat{U}\|_{L_2} = \left(\int_{-\pi/\Delta x}^{\pi/\Delta x} |\hat{U}(k)|^2 \,\mathrm{d}k\right)^{1/2}.$$

Prove *Parseval's identity*:

$$\|U\|_{\ell_2} = \frac{1}{\sqrt{2\pi}} \|\hat{U}\|_{L_2}.$$

6. In the lectures we considered the simplest finite difference approximation of the heat equation $u_t = u_{xx}$, given by

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{(\Delta x)^2}, \qquad j = \ldots, -2, -1, 0, 1, 2, \ldots\,; \quad n = 0, 1, 2, \ldots.$$

What would the analogous difference approximation be based on values of U at just every other point in the x direction, i.e., U_{j+2}^n, U_j^n and U_{j-2}^n? Now suppose that you create a new difference approximation from these two schemes by adding 1/2 of the first difference approximation to 1/2 of the second difference approximation. Using Fourier analysis, explore how large Δt can be in relation to Δx if this last scheme is to be stable in the norm of $\ell_2 = \ell_2(-\infty,\infty)$.