Numerical Solution of Differential Equations: Problem Sheet 4 (of 4)

1. Consider the implicit Euler scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + b\,\frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\Delta x} = a\,\frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{(\Delta x)^2}, \qquad j = 0, \pm 1, \pm 2, \ldots, \qquad n \geq 0,$$

$$U_j^0 = u_0(x_j), \qquad j = 0, \pm 1, \pm 2, \ldots,$$

for the numerical solution of the initial-value problem

$$\frac{\partial u}{\partial t} + b\,\frac{\partial u}{\partial x} = a\,\frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \quad t > 0,$$

$$u(x, 0) = u_0(x), \qquad -\infty < x < \infty,$$

where $a > 0$ and b are fixed real numbers. Show that the scheme is unconditionally stable in the ℓ_2 norm.

Show further that the consistency error $|T_j^n| \leq C(\Delta t + (\Delta x)^2)$ for all $n \geq 0$ and $j = 0, \pm 1, \pm 2, \ldots$, where C is a constant independent of Δt and Δx, provided that $\partial^2 u/\partial t^2$, $\partial^3 u/\partial x^3$ and $\partial^4 u/\partial x^4$ exist and are bounded functions of x and t, $(x, t) \in (-\infty, \infty) \times [0, \infty)$.

2. Consider the θ-method for the numerical solution of the initial-value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \quad t > 0,$$

$$u(x, 0) = u_0(x), \qquad -\infty < x < \infty.$$

Suppose that the parameter θ has been chosen according to the formula

$$\theta = \frac{1}{2} + \frac{(\Delta x)^2}{12\Delta t}.$$

Show that the resulting scheme is unconditionally stable in the ℓ_2 norm and has a consistency error which is $\mathcal{O}((\Delta t)^2 + (\Delta x)^2)$, provided that derivatives of u of sufficiently high order exist and are bounded functions of x and t, $(x, t) \in (-\infty, \infty) \times [0, \infty)$.

3. The diffusion equation $u_t = u_{xx}$, $-\infty < x < \infty$, subject to the initial condition $u(x, 0) = u_0(x)$, $-\infty < x < \infty$, is approximated by the finite difference scheme (*Crandall's scheme*):

$$U_j^{n+1} - \frac{1}{2}(\nu - \zeta)(U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}) = U_j^n + \frac{1}{2}(\nu + \zeta)(U_{j+1}^n - 2U_j^n + U_{j-1}^n)$$

with $U_j^0 = u_0(x_j)$, where $\Delta t > 0$, $\Delta x > 0$, $\nu = \Delta t/(\Delta x)^2$ and ζ is a fixed constant. Show that if ν is a fixed real number, then the consistency error, T_j^n, satisfies

$$T_j^n = \begin{cases} \mathcal{O}((\Delta x)^2) & \text{if } \zeta \neq 1/6, \\ \mathcal{O}((\Delta x)^4) & \text{if } \zeta = 1/6. \end{cases}$$

4. Letting $\nu = \Delta t/(\Delta x)^2$, $\Delta x = 1/J$, $J \geq 2$, $\Delta t = T/N$, $N \geq 1$, $T > 0$, consider the θ–scheme

$$U_j^{n+1} - U_j^n = \nu[\theta(U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}) + (1 - \theta)(U_{j+1}^n - 2U_j^n + U_{j-1}^n)],$$

where $j = 0, 1, \ldots, J-1$, $0 \le n \le N-1$, with $0 \le \theta \le 1$,

$$U_0^n = 0, \qquad U_J^n = 0, \qquad 0 \le n \le N-1,$$

and

$$U_j^0 = u_0(x_j), \qquad 1 \le j \le J-1,$$

for the numerical solution of the initial-boundary-value problem $u_t = u_{xx}$, $0 < x < 1$, $0 < t \le T$, subject to homogeneous Dirichlet boundary conditions at $x = 0$ and $x = 1$, and the initial condition $u(x, 0) = u_0(x)$, $0 < x < 1$.

Show that if $2\nu(1-\theta) \le 1$, then the θ-scheme satisfies the following maximum principle:

$$U_{\min}^n \le U_j^n \le U_{\max}^n,$$

where

$$U_{\min}^n = \min\{U_0^m,\ 0 \le m \le n\,; U_j^0,\ 0 \le j \le J\,;\ U_J^m,\ 0 \le m \le n\},$$

and

$$U_{\max}^n = \max\{U_0^m,\ 0 \le m \le n\,; U_j^0,\ 0 \le j \le J\,;\ U_J^m,\ 0 \le m \le n\}.$$

5. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous, monotonic nonincreasing function. Consider the initial-boundary-value problem

$$\begin{aligned} u_t - u_{xx} &= f(u) && \text{for } x \in (0,1) \text{ and } t \in (0,T], \\ u(0,t) = 0, \quad u(1,t) &= 0 && \text{for } t \in (0,T], \\ u(x,0) &= u_0(x) && \text{for } x \in [0,1], \end{aligned}$$

where $T > 0$ and $u_0 \in C([0,1])$ is a given function satisfying the compatibility conditions $u_0(0) = 0$, $u_0(1) = 0$.

(a) Show that if there exists a real-valued function u such that $u_t, u_{xx} \in C([0,1] \times [0,T])$, which solves this initial boundary-value problem, then u is unique.

(b) Construct an implicit finite difference scheme for the numerical solution of this problem on a uniform spatial mesh of mesh size $\Delta x = 1/N$ in the x-direction and $\Delta t = T/M$ in the t-direction, where $N \ge 2$ and $M \ge 1$.

(c) Brouwer's fixed point theorem asserts that: *Every continuous function from a closed ball of a Euclidean space into itself has a fixed point.* Show, using Brouwer's fixed point theorem, that the finite difference scheme has a solution

$$U^m = (0, U_1^m, \ldots, U_{N-1}^m, 0)^{\mathrm{T}} \in \mathbb{R}^{N+1}$$

at each time level m, $m \in \{1, \ldots, M\}$. Show further, by mimicking your proof of part (a), that for each $m \in \{1, \ldots, M\}$ the solution U^m is unique.

6. [Optional: See the Lecture Notes for the definition of the ADI scheme.] Consider the heat equation $u_t = u_{xx} + u_{yy} + u$ for $(x,y) \in \mathbb{R}^2$, and $t \in (0,T]$, subject to the initial condition $u(x,y,0) = u_0(x,y)$.

Formulate an ADI scheme, based on the Crank–Nicolson method, for this initial-value problem, on a uniform spatial mesh with mesh-sizes Δx and Δy in the x and y co-ordinate directions, respectively.

Use Fourier analysis to show that your ADI scheme is unconditionally von Neumann stable.