Numerical Solution of ODEs – Problem Sheet I

1. Suppose that m is a fixed positive integer. Show that the initial value problem

$$y' = y^{2m/(2m+1)} , \qquad y(0) = 0 ,$$

has infinitely many continuously differentiable solutions. Why does this not contradict Picard's theorem?

2. Show that the explicit Euler method fails to approximate the solution $y(x) = (4x/5)^{5/4}$ of the initial value problem $y' = y^{1/5}$, $y(0) = 0$. Justify your answer. Consider the same problem with the implicit Euler method.

3. Consider the initial value problem

$$y' = \log\log(4 + y^2) , \qquad x \in [0, 1] , \qquad y(0) = 1 ,$$

and the sequence $\{y_n\}_{n=0}^{N}$, $N \geq 1$, generated by the explicit Euler method

$$y_{n+1} = y_n + h \log\log(4 + y_n^2) , \qquad n = 0, \dots, N-1 , \qquad y_0 = 1 ,$$

using the mesh points $x_n = nh$, $n = 0, \dots, N$, with spacing $h = 1/N$. Here log denotes the logarithm with base e.

   a) Let T_n denote the truncation error of Euler's method at $x = x_n$ for this initial value problem. Show that $|T_n| \leq h/(4\mathrm{e})$.

   b) Verify that

$$|y(x_{n+1}) - y_{n+1}| \leq (1 + hL)|y(x_n) - y_n| + h|T_n| , \qquad n = 0, \dots, N-1 ,$$

   where $L = 1/(2\log 4)$.

   c) Find a positive integer N_0, as small as possible, such that

$$\max_{0 \leq n \leq N} |y(x_n) - y_n| \leq 10^{-4}$$

   whenever $N \geq N_0$.

4. Consider the one-step method

$$y_{n+1} = y_n + \alpha h f(x_n, y_n) + \beta h f(x_n + \gamma h, y_n + \gamma h f(x_n, y_n)) ,$$

where α, β and γ are real parameters. Show that the method is consistent if and only if $\alpha + \beta = 1$. Show also that the order of the method cannot exceed 2.

Suppose that a second-order method of the above form is applied to the initial value problem $y' = -\lambda y$, $y(0) = 1$, where λ is a positive real number. Show that the sequence $(y_n)_{n \geq 0}$ is bounded if and only if $h \leq \frac{2}{\lambda}$. Show further that, for such λ,

$$|y(x_n) - y_n| \leq \frac{1}{6}\lambda^3 h^2 x_n , \quad n \geq 0 .$$