Numerical Solution of ODEs – Problem Sheet III

1. Use Matlab or Maple to plot the solution curves to the following initial value problems. Which of the problems would you consider stiff in the sense that the solution exhibits two scales: a fast scale, over a short subinterval where the solution has rapid variation, and a slow scale, over a longer subinterval where the solution changes only mildly?

   a) $y' = -(10^5 \mathrm{e}^{-10^4 x} + 1)(y-1)$, $y(0) = 2$ on the interval $x \in [0,1]$. Note that the solution can be found in closed form:

   $$y(x) = \mathrm{e}^{10(\mathrm{e}^{-10^4 x}-1)}\mathrm{e}^{-x} + 1 \, .$$

   b)

   $$\begin{aligned} y_1' &= -0.5y_1 + 0.501y_2 \, , \qquad y_1(0) = 1.1 \, , \\ y_2' &= 0.501y_1 - 0.5y_2 \, , \qquad y_2(0) = -0.9 \, , \end{aligned}$$

   on the interval $x \in [0, 10^3]$.

   c) $\mathbf{y}' = A(x)\mathbf{y}$, $\mathbf{y}(0) = (1,1,1)^T$, where

   $$A(x) = \begin{pmatrix} -1 + 100\cos 200x & 100(1 - \sin 200x) & 0 \\ -100(1 + \sin 200x) & -(1 + 100\cos 200x) & 0 \\ -1200(\cos 100x + \sin 100x) & 1200(\cos 100x - \sin 100x) & -501 \end{pmatrix} .$$

2. Consider the θ-method

   $$y_{n+1} = y_n + h\left[(1-\theta)f_n + \theta f_{n+1}\right]$$

   for $\theta \in [0,1]$.

   a) Show that the method is A-stable for $\theta \geq 1/2$.

   b) Show that the method is $A(\alpha)$-stable if and only if it is A-stable.

3. Find $X_M > 0$ as large as possible such that the system of differential equations

   $$\begin{aligned} y_1' &= -y_1 + xy_2 \\ y_2' &= x^2(y_1 - y_2) \end{aligned}$$

   is dissipative in the interval $[0, X_M]$. Deduce that any two solutions with respective initial conditions are then contractive on $[0, X_M]$ in the Euclidean norm.

4. Show that the second-order backward differentiation method and its one-leg twin are G-stable with

   $$G = \begin{bmatrix} 5/2 & -1 \\ -1 & 1/2 \end{bmatrix} .$$