Numerical Solution of ODEs – Problem Sheet IV

1. Develop a shooting method for the numerical solution of the boundary value problem

$$-y'' + ye^y = 1\ , \qquad y(0) = y(1) = 0\ ,$$

on the interval $[0,1]$ using:

   a) The method of bisection;

   b) The Newton-Raphson method.

   Explain how your algorithm can be extended to the case of multiple shooting.

2. Construct a three-point finite difference scheme for the numerical solution of the boundary value problem

$$-y'' + x^2 y = 0\ , \qquad y(0) = 1\ , \qquad y'(1) = 0\ ,$$

on the interval $[0,1]$. Show that the resulting system of equations can be written so that its matrix is tridiagonal. Apply the Thomas algorithm to the linear system when the spacing between the mesh points is $h = 1/10$.

3. Assume that the boundary value problem

$$y'' + f(x,y) = 0\ , \quad 0 < x < 1\ , \qquad y(0) = y(1) = 0\ ,$$

has a unique solution with a continuous fourth derivative on the interval $[0,1]$. Suppose further that a unique approximate solution can be computed satisfying the finite difference scheme

$$h^{-2}\delta^2 y_n + f(x_n, y_n) = 0\ , \quad 1 \le n \le N-1\ , \qquad y_0 = y_N = 0\ ,$$

where $\delta^2 y_n \equiv y_{n+1} - 2y_n + y_{n-1}$, $x_n = nh$ $(0 \le n \le N)$, and $Nh = 1$ for some integer $N \ge 2$.

   a) Show that the truncation error of the finite difference scheme is given by

$$T_n = \frac{1}{12} h^2 y^{(4)}(\xi_n)$$

   for some $\xi_n \in (x_{n-1}, x_{n+1})$.
   Show further that the global error $e_n = y(x_n) - y_n$ satisfies

$$h^{-2}\delta^2 e_n + p_n e_n = T_n\ , \quad 1 \le n \le N-1\ , \qquad e_0 = e_N = 0\ ,$$

   where $p_n = f_y(x_n, \eta_n)$ for some η_n between $y(x_n)$ and y_n, and it is assumed that $f_y(x,y)$ is a continuous function of x and y for $x \in [0,1]$ and $y \in \mathbf{R}$.

   b) Suppose now that $f_y(x,y) \le 0$ for all $x \in [0,1]$ and $y \in \mathbf{R}$. Let $|T_n| \le M$, $1 \le n \le N-1$. Show that $w_n = \frac{1}{2}h^2 M n(N-n)$ satisfies $h^{-2}\delta^2 w_n = -M$, that

$$h^{-2}\delta^2 w_n + p_n w_n \le -M\ , \quad 1 \le n \le N-1\ ,$$

   and that, if $v_n = w_n + e_n$ or $v_n = w_n - e_n$, then v_n satisfies

$$h^{-2}\delta^2 v_n + p_n v_n \le 0\ , \quad 1 \le n \le N-1\ , \qquad v_0 = v_N = 0\ .$$

c) Assume that v_n has a negative minimum for some value of n between 1 and $N-1$ and show that this leads to a contradiction. Deduce that

$$v_n \geq 0 \, , \qquad 1 \leq n \leq N-1 \, ,$$

that

$$|e_n| \leq w_n \, , \qquad 1 \leq n \leq N-1 \, ,$$

and that

$$|e_n| \leq \frac{1}{96} h^2 \max_{0 \leq x \leq 1} |y^{(4)}(x)| \, .$$

4. Consider the boundary value problem

$$-y'' + y' + y = x^2 \, , \qquad y(0) = 0 \, , \qquad y(1) = 0 \, .$$

Develop a collocation method for the numerical solution of this problem on the interval $[0,1]$ using the the collocation points

$$x_j = \frac{j}{N+1} \, , \qquad j = 1, \ldots, N \, .$$

and the basis functions $\psi_i(x) = \sin(i\pi x)$, $i = 1, \ldots, N$. Solve the resulting system of linear equations for increasing values of N and compare the numerical solution with the exact solution to the problem.