FEMs for PDEs - Problem Sheet 1

1. Draw the graph of the function ϕ defined by $\phi(x) = (1-|x|)_+$ for $x \in [-2,2]$. Is it true that $\phi \in C[-2,2] \cap C^1(-2,2)$?

   Calculate the first (weak) derivative $\phi' = D\phi$ of ϕ on the interval $[-2,2]$. Verify that ϕ, $\phi' \in L_p(-2,2)$ for all $p \in [1,\infty]$. Hence deduce that $\phi \in W_p^1(-2,2)$ for all $p \in [1,\infty]$.

2. Suppose that $u(x) = x^\alpha$, $x \in [0,1]$, where α is a fixed real number, $0 < \alpha < 1$. Show that $u \in C^\infty(0,1)$, but $u \notin W_p^1(0,1)$ for $p \geq (1-\alpha)^{-1}$.

   Let $\Omega = \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 < \frac{1}{4}\}$ and consider the function w defined on Ω by $w(x,y) = \log|\log\sqrt{x^2+y^2}|$. Show that $w \in W_2^1(\Omega)(= H^1(\Omega))$ but $w \notin C(\Omega)$.

3. Given that (a,b) is an open interval of the real line, let $H_{E_0}^1(a,b) = \{v \in H^1(a,b) : v(a) = 0\}$.

   a) By writing
   $$v(x) = \int_a^x v'(\xi)\,\mathrm{d}\xi, \qquad a \leq x \leq b,$$
   for $v \in H_{E_0}^1(a,b)$, show that the following (Poincaré-Friedrichs) inequality holds for each $v \in H_{E_0}^1(a,b)$:
   $$\|v\|_{L_2(a,b)}^2 \leq \frac{1}{2}(b-a)^2 |v|_{H^1(a,b)}^2.$$

   b) By writing
   $$[v(x)]^2 = \int_a^x \frac{d}{d\xi}[v(\xi)]^2\,\mathrm{d}\xi = 2\int_a^x v(\xi)v'(\xi)\,\mathrm{d}\xi, \qquad a \leq x \leq b,$$
   for $v \in H_{E_0}^1(a,b)$, show that the following (Agmon's) inequality holds for each $v \in H_{E_0}^1(a,b)$:
   $$\max_{x\in[a,b]} |v(x)|^2 \leq 2\|v\|_{L_2(a,b)}|v|_{H^1(a,b)}.$$

4. Given that $f \in L_2(0,1)$, state the weak formulation of each of the following boundary value problems:

   a) $-u'' + u = f(x)$ for $x \in (0,1)$, $u(0) = 0$, $u(1) = 0$;

   b) $-u'' + u = f(x)$ for $x \in (0,1)$, $u(0) = 0$, $u'(1) = 0$;

   c) $-u'' + u = f(x)$ for $x \in (0,1)$, $u(0) = 0$, $u(1) + u'(1) = 0$.

   Apply the Lax-Milgram lemma to show that each of the three weak formulations has a (corresponding) unique weak solution[1].

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[1] Hint: You may wish to use the inequality from part b) of Question 3 when attempting to prove that the bilinear form, associated with the boundary value problem in part c), is bounded on $H_{E_0}^1(0,1)$.