FEMs for PDEs - Problem Sheet 2

1. Given that α is a non-negative real number, consider the boundary value problem

$$-u'' + u = f(x) \quad \text{for } x \in (0,1), \qquad u(0) = 0, \quad \alpha u(1) + u'(1) = 0.$$

State the weak formulation of the problem. Using continuous piecewise linear basis functions on a uniform mesh of size $h = 1/N$, $N \geq 2$, write down the finite element approximation to this problem and show that this has a unique solution u_h. Expand u_h in terms of the finite element basis functions ϕ_i, where $\phi_i(x) = (1 - |x - x_i|/h)_+$, $i = 1, \ldots, N$, by writing

$$u_h(x) = \sum_{i=1}^{N} U_i \phi_i(x)$$

to obtain a system of linear equations for the vector of unknowns $(U_1, \ldots, U_N)^T$.

Suppose that $\alpha = 0$, $f(x) \equiv 1$ and $h = 1/3$. Solve the resulting system of linear equations and compare the corresponding numerical solution $u_h(x)$ with the exact solution $u(x)$.

2. Consider the elliptic equation

$$-\Delta u = f(x,y) \quad \text{for } (x,y) \in \Omega = (0,1)^2$$

subject to the homogeneous Dirichlet boundary condition $u = 0$ on

$$\Gamma_D = \{(x,y) \in \partial\Omega \,:\, x = 0 \text{ or } y = 0 \text{ or } y = 1\}$$

and non-homogeneous Neumann boundary condition $\frac{\partial u}{\partial x} = 1$ on

$$\Gamma_N = \{(x,y) \in \partial\Omega \,:\, x = 1\}.$$

State the weak formulation of the problem.

Consider a triangulation of Ω which has been obtained from a square mesh of spacing $h = 1/N$, $N \geq 2$, in both co-ordinate directions by subdividing each square into two triangles with the diagonal of negative slope. Using continuous piecewise linear basis functions on this triangulation, state the finite element approximation to the boundary value problem. Rewrite the finite element method as a system of linear algebraic equations and comment on the structure of the matrix.