FEMs for PDEs - Problem Sheet 3

1. Given that Ω is a bounded open set in $\mathbb{R}^n$, consider the elliptic boundary value problem (labelled (P)):

$$\begin{aligned} -\Delta u + c(x)u &= f(x), \quad x \in \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned}$$

with $c(x) \geq 0$, $x = (x_1, \ldots, x_n) \in \bar{\Omega}$ and assume that $c \in L_\infty(\Omega)$, $f \in L_2(\Omega)$.

   a) Show that there exists a quadratic functional $J : H_0^1(\Omega) \to \mathbb{R}$ of the form

   $$J(v) = \frac{1}{2}a(v, v) - l(v), \qquad v \in H_0^1(\Omega),$$

   where $a(\cdot, \cdot)$ is a symmetric bilinear functional on $H_0^1(\Omega) \times H_0^1(\Omega)$ with the property $a(v, v) > 0$ for all $v \in H_0^1(\Omega)$, and $l(\cdot)$ is a linear functional on $H_0^1(\Omega)$, such that the weak solution u to (P) satisfies

   $$J(u) \leq J(v) \qquad \forall v \in H_0^1(\Omega).$$

   Does J have other minimisers in $H_0^1(\Omega)$? (Justify your answer!)

   b) Show that if u minimises $J(\cdot)$ over $H_0^1(\Omega)$ then u is the weak solution to problem (P).

   c) Assume that $n = 1$, $\Omega = (0, 1)$, $c(x) \equiv 1$ and $f \in L_2(0, 1)$. Suppose further that V_h is a finite element subspace of $H_0^1(0, 1)$ consisting of continuous piecewise linear functions on a uniform subdivision of $[0, 1]$ into subintervals of length $h = 1/N$. Show that there exists a unique $u_h \in V_h$ such that $J(u_h) \leq J(v_h)$ for all $v_h \in V_h$. Show further that u_h can be found by solving a suitable system of linear algebraic equations with a tri-diagonal matrix A (the global stiffness matrix) whose entries you should calculate.

2. Suppose that Ω is a bounded polygonal domain in $\mathbb{R}^2$ whose closure $\bar{\Omega}$ has been subdivided into M closed triangles K so that any pair of triangles intersect along a complete edge, at a vertex or not at all. Consider a triangle K whose vertices P_i, $i = 1, 2, 3$, have position vectors $\mathbf{r}_i$, $i = 1, 2, 3$, and a linear function v_h defined on K such that $v_h(P_i) = V_i$, $i = 1, 2, 3$.

   Show that

   $$\int_K |\nabla v_h(x, y)|^2 \, \mathrm{d}x \, \mathrm{d}y = [V_1, V_2, V_3] A^k \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix},$$

   where $k \in \{1, \ldots, M\}$ is the number of the triangle K in the global numbering, A^k is the symmetric 3×3 element stiffness matrix given as

   $$A^k = \frac{1}{4A_{123}} \begin{bmatrix} |\mathbf{r}_2 - \mathbf{r}_3|^2 & (\mathbf{r}_2 - \mathbf{r}_3) \cdot (\mathbf{r}_3 - \mathbf{r}_1) & (\mathbf{r}_2 - \mathbf{r}_3) \cdot (\mathbf{r}_1 - \mathbf{r}_2) \\ & |\mathbf{r}_3 - \mathbf{r}_1|^2 & (\mathbf{r}_3 - \mathbf{r}_1) \cdot (\mathbf{r}_1 - \mathbf{r}_2) \\ \text{symm.} & & |\mathbf{r}_1 - \mathbf{r}_2|^2 \end{bmatrix}$$

and A_{123} is the area of K. Show further that the *full* stiffness matrix A can be expressed as

$$A = \sum_{k=1}^{M} L^k A^k (L^k)^T,$$

where $(L^k)^T$ is the transpose of a certain Boolean matrix L^k.

How would you use this result in the process of assembly of the *global* stiffness matrix arising in the piecewise linear finite element approximation of the homogeneous Dirichlet boundary value problem

$$\begin{aligned} -\Delta u &= f(x,y), \qquad && (x,y) \in \Omega, \\ u &= 0 && \text{on } \partial\Omega? \end{aligned}$$

E.S.