FEMs for PDEs - Problem Sheet 4

1. Suppose that Ω is a bounded polygonal domain in $\mathbf{R}^2$ with boundary $\partial\Omega$. Assume that $p \in C^1(\bar{\Omega})$ and $q \in C(\bar{\Omega})$ are two functions, defined and positive on $\bar{\Omega}$, and let $f \in L_2(\Omega)$. Consider the partial differential equation

$$-\nabla \cdot (p(x,y)\nabla u) + q(x,y)u = f(x,y) \qquad \text{in } \Omega,$$

subject to the homogeneous Dirichlet boundary condition $u = 0$ on $\partial\Omega$.

Suppose further that V_h is a finite-dimensional subspace of $H_0^1(\Omega)$ (consisting, say, of continuous piecewise polynomial functions defined on a certain subdivision of Ω). Given that u is the weak solution in $H_0^1(\Omega)$ to the boundary value problem and $u_h \in V_h$ is its finite element approximation, show that the following, so called, *Galerkin orthogonality property* holds:

$$a(u - u_h, v_h) = 0 \qquad \text{for all } v_h \in V_h,$$

with $a(\cdot,\cdot)$ a suitable bilinear functional on $H_0^1(\Omega) \times H_0^1(\Omega)$ that you should define. Show further that there exists a positive constant c_0 such that

$$\|u - u_h\|_{H^1(\Omega)}^2 \leq \frac{1}{c_0} a(u - u_h, u - u_h).$$

Apply the Galerkin orthogonality property to deduce that

$$\|u - u_h\|_{H^1(\Omega)}^2 \leq \frac{1}{c_0} a(u - u_h, u - v_h) \qquad \text{for all } v_h \in V_h.$$

Verify that there exists a positive constant c_1 such that

$$\|u - u_h\|_{H^1(\Omega)} \leq \frac{c_1}{c_0} \min_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)}.$$

How would these results be affected (if at all) if you replaced the boundary condition $u = 0$ on $\partial\Omega$ by $u = 0$ on Γ_1 and $\frac{\partial u}{\partial \nu} = 0$ on Γ_2 where Γ_i, $i = 1, 2$, are nonempty unions of edges contained in $\partial\Omega$, $\Gamma_1 \cup \Gamma_2 = \partial\Omega$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, and ν denotes the unit outward normal vector to Γ_2? Think of a physical situation that would be modelled by this mixed Dirichlet-Neumann boundary value problem.

2. Consider the two-point boundary value problem

$$-(p(x)u')' + q(x)u = f(x), \quad x \in (0,1), \qquad u(0) = 0, \quad u(1) = 0,$$

where $p(x) \geq \tilde{c} > 0$, $q(x) \geq 0$ for $x \in [0,1]$, $p \in C^1[0,1]$, $q \in C[0,1]$ and $f \in L_2(0,1)$. Given that u_h denotes the continuous piecewise linear finite element approximation to u on a uniform subdivision of $[0,1]$ into subintervals of length $h = 1/N$, $N \geq 2$, show that

$$\|u - u_h\|_{H^1(0,1)} \leq C_1 h \|u''\|_{L_2(0,1)},$$

where C_1 is a positive constant that you should specify. Show further that there is a positive constant C_2 such that

$$\|u - u_h\|_{H^1(0,1)} \leq C_2 h \|f\|_{L_2(0,1)}.$$

Calculate the right-hand sides in these inequalities in the case when $p(x) \equiv 1$, $q(x) \equiv 0$, $f(x) \equiv 1$ for $x \in [0,1]$, and $h = 10^{-3}$.