FEMs for PDEs - Problem Sheet 5

1. Let $\Omega = (0,1)^2$. Consider the finite element space $V_h \subset H^1(\Omega)$ consisting of all continuous piecewise linear functions on a triangulation of Ω obtained from a uniform square mesh of size $h = 1/N$, $N \geq 2$, by subdividing each square into two triangles with the diagonal of negative slope. Given that $u \in H^2(\Omega)$ let $\mathcal{I}_h u$ denote its continuous piecewise linear interpolant from V_h. You may take it for granted that

$$\|u - \mathcal{I}_h u\|_{H^1(\Omega)} \leq K_1 h |u|_{H^2(\Omega)},$$

where K_1 is a positive constant, independent of u, u_h and h. [If you are really ambitious, you may try to prove this, but it is not compulsory; be warned: it is hard work!]

Now consider the elliptic boundary value problem

$$-\Delta u + u = f(x,y) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where $f \in L_2(\Omega)$.

Given that V_h is the finite element space introduced above and u_h denotes the finite element approximation to u from V_h, show that

$$\|u - u_h\|_{H^1(\Omega)} \leq K_2 h |u|_{H^2(\Omega)}$$

where K_2 is a positive constant, independent of u, u_h and h.

Show further that

$$\|u - u_h\|_{L_2(\Omega)} \leq K_3 h^2 |u|_{H^2(\Omega)}$$

where K_3 is a positive constant, independent of u, u_h and h.

2. Consider the boundary value problem from Exercise 1. Show that the finite element method based on the space V_h defined above can be restated as a seven-point difference scheme for the vector U of nodal values of the finite element solution u_h, subject to a certain homogeneous boundary condition. Comment on the sparsity structure of the matrix of the associated system of linear algebraic equations.