FEMs for PDEs - Problem Sheet 6

1. Given that K is a simply connected bounded open set in $\mathbb{R}^n$ with piecewise smooth boundary, $\mathcal{P}$ a k-dimensional linear space of functions defined on K and $\mathcal{N} = \{N_1, \ldots, N_k\}$ a basis for $\mathcal{P}'$, the algebraic dual of $\mathcal{P}$, consider the finite element $(K, \mathcal{P}, \mathcal{N})$.

   What does it mean to say that $\Psi = \{\psi_1, \ldots, \psi_k\}$ is a *nodal basis* for $\mathcal{P}$? Supposing that there exists a nodal basis $\Psi = \{\psi_1, \ldots, \psi_k\}$ for $\mathcal{P}$, show that $\mathcal{N}$ uniquely determines $\mathcal{P}$ (namely, if $\psi \in \mathcal{P}$ with $N(\psi) = 0$ for all $N \in \mathcal{N}$ then $\psi = 0$).

   Assuming that v is a function such that $N_i(v)$ is defined for all $N_i \in \mathcal{N}$, $i = 1, \ldots, k$, and Ψ is a nodal basis for $\mathcal{P}$, give a definition of the *local interpolant* $\mathcal{I}_K(v)$ involving the nodal basis Ψ.

   Consider the triangle $\hat{K}$ in the (ξ, η)-plane shown in Figure 1, let $\hat{\mathcal{P}}$ be the set of all polynomials in ξ and η of degree 2, $\hat{\mathcal{N}} = \{\hat{N}_1, \ldots, \hat{N}_6\}$ with $\hat{N}_i(\hat{v}) = \hat{v}(\hat{\mathbf{z}}_i)$ where $\hat{v}$ is a continuous function of ξ and η and $\hat{\mathbf{z}}_1, \ldots, \hat{\mathbf{z}}_6$ are as in the figure. Find a nodal basis for $\hat{\mathcal{P}}$. Given that $\hat{v}$ is a function of ξ and η defined by $\hat{v}(\xi, \eta) = 1 + \xi^4 + \eta^4$, find the local interpolant $\mathcal{I}_{\hat{K}}(\hat{v})$.

2. Suppose that Ω is a convex polygonal domain in the plane and $\mathcal{T} = \{K\}$ is a triangulation of Ω. Consider a triangle K in the triangulation $\mathcal{T}$; suppose that the vertices of K have position vectors labelled $\mathbf{r}_i$, $i = 1, 2, 3$, in the anti-clockwise direction. Further, let $\mathbf{r}_4 = (\mathbf{r}_2 + \mathbf{r}_3)/2$, $\mathbf{r}_5 = (\mathbf{r}_3 + \mathbf{r}_1)/2$, $\mathbf{r}_6 = (\mathbf{r}_1 + \mathbf{r}_2)/2$, and denote the coordinates of $\mathbf{r}_i$ by (x_i, y_i), $i = 1, \ldots, 6$.

   Given that $\mathbf{r} = (x, y)$ is a point in K, write $\mathbf{r} = (1 - \xi - \eta)\mathbf{r}_1 + \xi\mathbf{r}_2 + \eta\mathbf{r}_3$ with $0 \le \xi \le 1$, $0 \le \eta \le 1 - \xi$, and show that any quadratic polynomial $v_h(x, y)$ defined on K may be expressed as

$$v_h(x, y) = v_h(\mathbf{r}(\xi, \eta)) = \sum_{i=1}^{6} V_i \hat{\psi}_i(\xi, \eta),$$

   where $V_i = v_h(x_i, y_i)$, and $\{\hat{\psi}_1, \ldots, \hat{\psi}_6\}$ is a suitable nodal basis for the set of quadratic polynomials in ξ and η on the canonical triangle

$$\hat{K} = \{(\xi, \eta) \,:\, 0 \le \xi \le 1, \quad 0 \le \eta \le 1 - \xi\}.$$

   Show further that

$$\frac{\partial}{\partial \xi} v_h(\mathbf{r}(\xi, \eta)) = (4\xi + 4\eta - 3)V_1 + (4\xi - 1)V_2 + 4\eta(V_4 - V_5) + (4 - 8\xi - 4\eta)V_6,$$
$$\frac{\partial}{\partial \eta} v_h(\mathbf{r}(\xi, \eta)) = (4\eta + 4\xi - 3)V_1 + (4\eta - 1)V_3 + 4\xi(V_4 - V_6) + (4 - 8\eta - 4\xi)V_5.$$

   Deduce that

$$\frac{1}{2}\int_K |\nabla v_h(x, y)|^2 \,\mathrm{d}x\,\mathrm{d}y = [V_1, \ldots, V_6] A^K \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix},$$

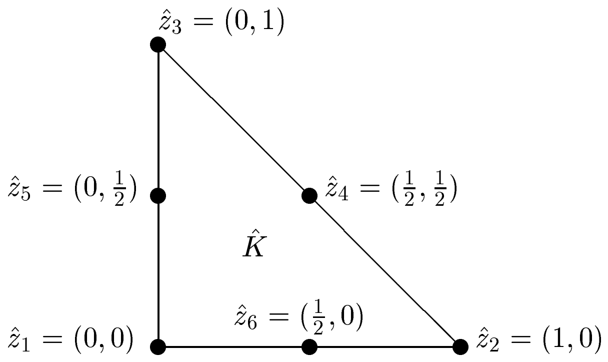

Figure 1: Quadratic Lagrange triangle.

where A^K is the symmetric 6×6 element stiffness matrix given as

$$A^K = \frac{1}{4A_{123}} \begin{bmatrix} a_{11}^K & a_{12}^K & a_{13}^K & a_{14}^K & a_{15}^K & a_{16}^K \\ & a_{22}^K & a_{23}^K & a_{24}^K & a_{25}^K & a_{26}^K \\ & & a_{33}^K & a_{34}^K & a_{35}^K & a_{36}^K \\ & & & a_{44}^K & a_{45}^K & a_{46}^K \\ & & & & a_{55}^K & a_{56}^K \\ & & & & & a_{66}^K \end{bmatrix}$$

and A_{123} is the area of K. (You should only calculate explicitly one of the entries of the matrix, say a_{11}^K, in terms of the position vectors $\mathbf{r}_i$; you don't need to spend time on similar(ly tedious) calculations for the other 20 entries). Identity (2.16) in the lecture notes may be used without proof.

Show further that the global stiffness matrix A can be expressed as

$$A = \sum_{K \in \mathcal{T}} L^K A^K (L^K)^T,$$

where $(L^K)^T$ is the transpose of a certain Boolean matrix L^K of size $N \times 6$, with N denoting the number of internal nodes in the triangulation $\mathcal{T}$.

How would you use this result in the process of assembly of the global stiffness matrix arising in the piecewise quadratic finite element approximation of the homogeneous Dirichlet boundary value problem

$$\begin{aligned} -\Delta u &= f(x,y), \qquad (x,y) \in \Omega, \\ u &= 0 \qquad\qquad\;\; \text{on } \partial\Omega? \end{aligned}$$