FEMs for PDEs - Problem Sheet 7

1. Consider the two-point boundary value problem

$$-u'' + u = f(x), \quad x \in (0,1), \qquad u(0) = 0, \ \ u(1) = 0,$$

with $f \in C^2[0,1]$. State the piecewise linear finite element approximation to this problem on a non-uniform mesh $0 = x_0 < x_1 < \ldots < x_N = 1$, $N \geq 2$, with $h_i = x_i - x_{i-1}$, assuming that, for a continuous piecewise linear function $v_h(x)$,

$$\int_0^1 f(x) v_h(x)\,\mathrm{d}x$$

has been approximated by applying the trapezium rule on each subinterval $[x_{i-1}, x_i]$. Verify that the following *a posteriori* bound holds for the error between $u(x)$ and its finite element approximation $u_h(x)$:

$$\begin{aligned} \|u - u_h\|_{L_2(0,1)} \ \leq \ & K_0 \left(\sum_{i=1}^{N} h_i^4 \|R(u_h)\|^2_{L_2(x_{i-1},x_i)} \right)^{1/2} \\ & + K_1 \max_{1 \leq i \leq N} h_i^2 \left(\|f''\|^2_{C[x_{i-1},x_i]} + 4\|f'\|^2_{C[x_{i-1},x_i]} \right)^{1/2}, \end{aligned}$$

where $R(u_h) = f(x) - (-u_h''(x) + u_h(x))$ for $x \in (x_{i-1}, x_i)$, $i = 1, \ldots, N$, and K_0, K_1 are constants which you should specify.

How would you use this bound to compute u to within a specified tolerance TOL?

2. Let $u(x,t)$ denote the solution to the initial-boundary-value problem

$$\begin{aligned} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad & 0 < x < 1, \quad 0 < t \leq T, \\ u(0,t) = 0, \quad u(1,t) = 0, \qquad & 0 \leq t \leq T, \\ u(x,0) = u_0(x), \qquad & 0 < x < 1, \end{aligned}$$

where $T > 0$ and $u_0 \in L_2(0,1)$. Construct a finite element method for the numerical solution of this problem, based on the backward Euler scheme with time step $\Delta t = T/M$, $M \geq 2$, and a piecewise linear approximation in x on a uniform subdivision of the interval $[0,1]$ of spacing $h = 1/N$, $N \geq 2$, denoting by u_h^m the finite element approximation to $u(\cdot, t^m)$ where $t^m = m\Delta t$, $0 \leq m \leq M$.

Show that, for $0 \leq m \leq M-1$,

$$\|u_h^{m+1}\|^2_{L_2(0,1)} + 2\Delta t |u_h^{m+1}|^2_{H^1(0,1)} \leq \|u_h^m\|^2_{L_2(0,1)}.$$

Deduce that the method is unconditionally stable in the L_2 norm in the sense that, for any Δt, independent of the choice of h,

$$\|u_h^m\|_{L_2(0,1)} \leq (1 + 4\Delta t)^{-m/2} \|u_h^0\|_{L_2(0,1)}, \quad 1 \leq m \leq M. \qquad (\star)$$

Show further that

$$|u_h^m|_{H^1(0,1)} \leq \frac{\alpha}{\sqrt{4\Delta t}} (1 + 4\Delta t)^{-m/2} \|u_h^0\|_{L_2(0,1)}, \quad 1 \leq m \leq M, \qquad (\star\star)$$

where $\alpha > 0$ is certain constant (independent of h and Δt, but possibly dependent on T) that you should specify. [The Poincare-Friedrichs inequality, $\|v\|^2_{L_2(0,1)} \leq \frac{1}{2}|v|^2_{H^1(0,1)}$ for $v \in H^1_0(0,1)$, may be used without proof.]

Observe that for Δt small the right-hand side in ($\star\star$) is much larger than in ($\star$). The aim of the final part of this exercise is to prove that this behaviour is *not* an artifact of a crude analysis, and the bound ($\star\star$) is sharp. For this purpose, choose $u^0_h(x)$ as a suitable constant multiple of the finite element basis function $\phi_{i_0}(x) = (1 - |(x - i_0 h)/h|)_+$ where i_0 is a fixed integer, $1 \leq i_0 < N$, so that $\|u^0_h\|_{L_2(0,1)} = 1$; then show that, for $\Delta t = \frac{1}{6}h^2$,

$$|u^1_h|_{H^1(0,1)} \geq \frac{\beta}{\sqrt{4\Delta t}} \geq \frac{\beta}{\sqrt{4\Delta t}}(1 + 4\Delta t)^{-1/2}\|u^0_h\|_{L_2(0,1)},$$

where β is a positive constant, $\beta < \alpha$.