# The Cohomology of the Moduli Stack of Principal Bundles on a Curve

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# Classification of Principal Bundles

## Background Data

- A smooth projective curve $X$ over the field $k$.
- A (connected) split reductive group $G$ over the field $k$.

## Problem

Classify principal $G$-bundles on $X$ (of "topological type" $\vartheta \in \pi_1(G)$) up to isomorphism!

## "Abstract" answers

- The moduli stack $\mathrm{Bun}_G = \bigsqcup_{\vartheta\in\pi_1(G)} \mathrm{Bun}_G^{\vartheta}$ of principal $G$-bundles parameterizes all principal $G$-bundles on $X$.
- The moduli scheme $\mathscr{M}^{\mathrm{ss}}(\vartheta)$ parameterizes Ramanathan-semistable principal $G$-bundles on $X$ (up to S-equivalence).

# Moduli Spaces of Principal Bundles



## Assumption

$k$ is algebraically closed.

## Idea

Use an appropriate representation $\rho\colon G \longrightarrow \mathrm{GL}(V)$.

## Example

- $G = \mathrm{Sp}_r(k) \overset{\rho}{\subset} \mathrm{SL}_{2r}(k)$.
  $G$-bundle = Vector bundle $E$ (of rank $2r$) with anti-symmetric non-degenerate bilinear form $E \otimes E \longrightarrow \mathscr{O}_X$.
- $G = \mathrm{Aut}(\mathfrak{g}) \overset{\rho}{\subset} \mathrm{GL}(\mathfrak{g})$.
  $G$-bundle = Vector bundle $E$ with fiber $\mathfrak{g}$ with bilinear map $E \otimes E \longrightarrow E$, fiberwise isomorphic to $\mathfrak{g}$.

# Moduli Spaces of Principal Bundles over $\mathbb{C}$

## Ramanathan's Construction (1976, published 1996)

G connected reductive linear algebraic group. Use

$$G \overset{\mathrm{AD}}{\twoheadrightarrow} \mathrm{Aut}^0(\mathrm{Lie}(G)) \subset \mathrm{Aut}(\mathrm{Lie}(G)).$$

Important:

- Rigidity of semisimple Lie algebras.
  (Fails in positive characteristic!)
- A principal $G$-bundle is semistable, if and only if the associated "adjoint" vector bundle is semistable.
  (Needs low height in positive characteristic.)

# Moduli Spaces of Principal Bundles over $\mathbb{C}$

## Other Constructions of the Moduli Spaces

- Faltings (1993): Approach via Theta functions.
- Balaji/Seshadri (2002): $G$ semisimple group, $\rho\colon G \longrightarrow \mathrm{SL}(V)$ a faithful representation.

Faltings and Balaji/Seshadri provide proofs for the semistable reduction theorem (via deformations of Lie algebras and Bruhat-Tits theory, respectively).

- Gómez/Sols (2005): Ramanathan's strategy.
- S. (2002, 2004); Gómez/Langer/S./Sols (2006): $G$ semisimple group, $\rho\colon G \longrightarrow \mathrm{SL}(V)$ a faithful representation.

These recent GIT constructions "automatically" yield projective moduli spaces.

# Moduli Spaces of Principal Bundles in Positive Characteristic



## Balaji/Parameswaran (2003)

$G$ semisimple group, $\rho\colon G \longrightarrow \mathrm{SL}(V)$ a faithful representation of low height and $\mathrm{SL}(V)/G \hookrightarrow W$ an embedding into an $\mathrm{SL}(V)$-module of low separable index.

A principal $G$-bundle is semistable, if and only if the associated vector bundle with fiber $V$ is semistable. (Ilangovan, Mehta, Parameswaran)

The semistable reduction theorem via Bruhat-Tits theory can be generalized.

## Gómez/Langer/S./Sols (2005)

$G$ semisimple group, $\rho\colon G \longrightarrow \mathrm{SL}(V)$ a faithful representation.

The moduli spaces of semistable principal $G$-bundles exist as quasi-projective varieties in any characteristic.

## Heinloth (2006)

Heinloth generalizes Langton's algorithm for the semistable reduction theorem via affine Graßmannians.

The assumptions on the characteristic $p$ are:

| | |
|---|---|
| $B, C, D$: | $p \neq 2$ |
| $G_2$: | $p > 7$ |
| $F_4, E_6$: | $p > 19$ |
| $E_7$: | $p > 31$ |
| $E_8$: | $p > 58$. |

The result would hold in any characteristic, if Behrend's conjecture were true[a].

[a] Recent work of Heinloth establishes Behrend's conjecture in more cases and shows that is not true in full generality

# Pseudo $G$-Bundles



## Principal Bundles as Decorated Vector Bundles

$G$ semisimple; $\rho\colon G \longrightarrow \mathrm{SL}(V)$ a faithful representation. Then:

Principal $G$-bundle

"="

Vector bundle $E$ with fiber $V$ + Section

$$\sigma\colon X \longrightarrow \underline{\mathrm{Isom}}(V \otimes \mathscr{O}_X, E)/G$$

## The Crucial Commutative Diagram

$$\begin{array}{ccc}
\underline{\mathrm{Isom}}(V \otimes \mathscr{O}_X, E) & \overset{\subset}{\longrightarrow} & \underline{\mathrm{Hom}}(V \otimes \mathscr{O}_X, E) \\
\downarrow & & \downarrow \\
\underline{\mathrm{Isom}}(V \otimes \mathscr{O}_X, E)/G & \overset{\subset}{\longrightarrow} & \underline{\mathrm{Hom}}(V \otimes \mathscr{O}_X, E)/\!\!/G
\end{array}$$

# Pseudo $G$-Bundles

## Conclusion

Principal $G$-bundle

$$\rightsquigarrow$$

$$E + \text{Homomorphism } \tau\colon \underline{\mathrm{Sym}}^\star(V \otimes E^\vee)^G \longrightarrow \mathscr{O}_X$$

## Definition

A **pseudo $G$-bundle** is a pair $(E, \tau)$ which consists of a vector bundle $E$ of rank $\dim(V)$ with $\det(E) \cong \mathscr{O}_X$ and a homomorphism $\tau\colon \underline{\mathrm{Sym}}^\star(V \otimes E^\vee)^G \longrightarrow \mathscr{O}_X$ of $\mathscr{O}_X$-algebras.

## Lemma (Recognizing Principal $G$-Bundles)

Let $(E, \tau)$ be a pseudo $G$-bundle with section $\sigma\colon X \longrightarrow \underline{\mathrm{Hom}}(V \otimes \mathscr{O}_X, E)/\!\!/G$. Then, $(E, \tau)$ is a principal $G$-bundle, if and only if there exists a point $x \in X$ with $\sigma(x) \in \underline{\mathrm{Isom}}(V \otimes \mathscr{O}_X, E)/G$.

# Moduli Spaces of Pseudo $G$-Bundles



### Semistability

The theory of decorated vector bundles yields a notion of **$\delta$-(semi)stability** for pseudo $G$-bundles which depends on $\delta \in \mathbb{Q}_{>0}$.

### Theorem (S. 2002 [$k = \mathbb{C}$]; Gómez/Langer/S./Sols 2005 [Char($k$) $\geq 0$])

Fix $\delta \in \mathbb{Q}_{>0}$. There exists a projective moduli space $\mathscr{M}^{\delta\text{-ss}}$ for $\delta$-semistable pseudo $G$-bundles.
For $\delta \gg 0$, the variety $\mathscr{M}^{\delta\text{-ss}}$ contains $\mathscr{M}(\vartheta)^{\text{ss}}$ as an open subscheme.

# Semistable Reduction



## Unstable Decorations

If, in the pseudo $G$-bundle $(E,\tau)$,

$$\sigma(X) \subset \Big(\underline{\mathrm{Hom}}(V \otimes \mathscr{O}_X, E)/\!\!/ G\Big) \setminus \Big(\underline{\mathrm{Isom}}(V \otimes \mathscr{O}_X, E)/G\Big),$$

then

$$\sigma(\eta) \in \mathrm{Hom}(V_K, E_K)/\!\!/ G$$

is unstable for the action of the group $\mathrm{SL}(E_K)$,
$\eta \in X$ the generic point; $K := k(X)$.

# Semistable Reduction



## The Semistable Reduction Argument

If there is a one parameter subgroup $\lambda\colon \mathbb{G}_m(K) \longrightarrow \mathrm{SL}(E_K)$ with

$$\mu\big(\sigma(\eta),\lambda\big) < 0,$$

then $(E,\tau)$ is $\delta$-unstable for all $\delta > 0$.

## The Difference Between Zero and Positive Characteristic

- In characteristic zero, the Hilbert-Mumford criterion holds also over non-algebraically closed fields. (Kempf's theory of the instability flag.)
- In positive characteristic, it may fail.
  (The semistable reduction argument may be saved, e.g., when we assume that $\rho$ has low separable index).

# Parabolic Principal Bundles
# Joint Work with Jochen Heinloth



## Background Data

- A tuple $\underline{x} = (x_1, \dots, x_b)$ of distinct points on $X$.
- A tuple $\underline{P} = (P_1, \dots, P_b)$ of parabolic subgroups of $G$.

## Definition

A **quasi-parabolic principal $G$-bundle (of type $(\underline{x}, \underline{P})$)** is a tuple $(\mathscr{P}, \underline{s})$ with $\mathscr{P}$ a principal $G$-bundle and $\underline{s} = (s_1, \dots, s_b)$ with $s_i \in \mathscr{P}_{|\{x_i\}}/P_i$, $i = 1, \dots, b$.

## Stability Parameter

The stability parameter is a tuple $\underline{a} = (a_1, \dots, a_b)$ with

$$a_i \in X^\star(P_i)^\vee_{\mathbb{Q},+} \quad = \quad \text{Set of conjugacy classes of } \mathbb{Q}\text{-1PSGS } \lambda \text{ of } P_i, \text{ s.t. } P(\lambda) = P_i, \quad i = 1, \dots, b.$$

# Parabolic Principal Bundles

## Further Notions

- **$\underline{a}$-(semi)stability** of quasi-parabolic principal bundles.
- **admissible stability parameter**: grants existence of the Harder-Narasimhan reduction; needed in the construction of moduli spaces.
- **stability parameter of coprime type**: grants that the notions of "$\underline{a}$-stability" and "$\underline{a}$-semistability" agree.

## Theorem (Heinloth/S. 2006)

Fix the type $(\underline{x}, \underline{P})$ and the stability parameter $\underline{a}$. Then, the moduli space $\mathscr{M}(\underline{x}, \underline{P})^{\underline{a}\text{-ss}}$ of $\underline{a}$-semistable quasi-parabolic principal bundles exists as a quasi-projective variety[a].
It is projective under the hypotheses of Heinloth's semistable reduction theorem.

[a] Over $\mathbb{C}$, see Bhosle/Ramanathan and Teleman/Woodward

# The Cohomology Ring of the Moduli Stack of Principal Bundles



## Assumption

- The ground field is $k = \mathbb{C}$ or a finite field $k = \mathbb{F}_q$, $q = p^n$;
- $G$ is semisimple.

## The Canonical Ring

The Künneth components of the characteristic classes of the universal principal $G$-bundle on $\mathrm{Bun}_G^\vartheta \times X$ generate a free subalgebra $\mathrm{Can}^\star$ of $H^\star(\mathrm{Bun}_G^\vartheta, \overline{\mathbb{Q}}_l)$.
It is called the **ring of canonical classes**.

## Theorem (Atiyah/Bott 1983)

If $k = \mathbb{C}$, then

$$\mathrm{Can}^\star = H^\star\big(\mathrm{Bun}_G^\vartheta, \overline{\mathbb{Q}}_l\big).$$

# The Cohomology Ring of the Moduli Stack of Principal Bundles



## The Case $\mathrm{GL}_n$

- Bifet, Ghione, Letizia (1994)
- Heinloth (1998; Diploma thesis advised by Harder)

## Cohomology from Purity

Assume that the cohomology of $\mathrm{Bun}_G$ is pure. Then, $\mathrm{Can}^\star = H^\star(\mathrm{Bun}_G, \overline{\mathbb{Q}}_l)$.

## Ingredients of the Proof

- Behrend's trace formula
- The Tamagawa number of $G$ equals $\#\pi_1(G)$. (Harder 1974, Ono 1965).

# The Cohomology Ring of the Moduli Stack of Principal Bundles



## Making Unstable Principal Bundles Stable

- Let $U \subset \mathrm{Bun}_G$ be an open substack of finite type
- $\Longrightarrow \exists\ \underline{x} = (x_1, \ldots, x_b)$, $\underline{P} = (B, \ldots, B)$, $\underline{a}$ of coprime type, such that image $\mathrm{Bun}^{\underline{a}\text{-s}}_{G,\underline{x},\underline{P}} \longrightarrow \mathrm{Bun}_G$ contains $U$, and
- $\mathrm{Bun}_{G,\underline{x},\underline{P}} \setminus \mathrm{Bun}^{\underline{a}\text{-s}}_{G,\underline{x},\underline{P}}$ has large codimension.

## Application of the Moduli Spaces of Parabolic Bundles

Existence of proper moduli spaces
$\Longrightarrow$ purity of the cohomology of $\mathrm{Bun}^{\underline{a}\text{-s}}_{G,\underline{x},\underline{P}}$
$\Longrightarrow$ purity of the cohomology of $\mathrm{Bun}_G$

## Base Change

Since the moduli spaces also exist over a base $\mathrm{Spec}(R)$, the result over finite fields implies the result of $\mathbb{C}$ and vice versa.