Artin group actions on derived categories of threefolds

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Abstract. Motivated by the enhanced gauge symmetry phenomenon of the physics literature and mirror symmetry, this paper constructs an action of an Artin group on the derived category of coherent sheaves of a smooth quasiprojective threefold containing a configuration of ruled surfaces described by a finite type Dynkin diagram. The action extends over deformations of the threefold via a compatible action of the corresponding reflection group on the base of its deformation space. All finite type Dynkin diagrams are realized.

Introduction

The purpose of this work is to construct actions of finite type Artin groups on derived categories of coherent sheaves of complex threefolds. The construction is motivated by a correspondence between Calabi-Yau threefolds containing ruled surfaces and Lie algebras, which arises in type II string theory; see [12], [1], [7] and references in these works, as well as [6] which also considers some of the geometries studied in this paper. I explain the connection further in [21]; it suffices to say here that the threefolds I consider are of the most simple kind for which the physics correspondence works, and the main theorem of this paper says that in these cases the derived category of coherent sheaves of the threefold is acted on by an Artin group which covers the Weyl group of the corresponding Lie algebra.

The main result of this paper can also be viewed in the context of homological mirror symmetry, representing a generalization of a result of Seidel and Thomas [16]. They construct representations of the classical (type A) braid group on derived categories of coherent sheaves of a much larger class of varieties than those considered here. However, their construction is more algebraic in flavour, whereas the Artin group actions in this paper are governed in a very precise geometric way by deformation theory. As explained in [21], the two constructions coincide in dimension two; the braid group actions on derived categories of threefolds obtained in this paper are new even in the type A case.

Autoequivalences of derived categories for threefolds containing ruled surfaces were first constructed by Horja in [9]–[10]. In [20] it was observed that these equivalences are essentially given by classical correspondences (structure sheaves of subschemes in the prod-

uct), and also that they deform to derived equivalences given by flops first found by Bondal and Orlov [2]. The proof of the braid relations uses in an essential way both of these facts.

In certain cases, the Artin group acts faithfully on the derived category. The proof of this statement will be reduced to the injectivity statement of Seidel-Thomas [16] for type A, using a hyperplane section argument.

Structure of the paper. Section 1 deals with Dynkin diagrams, reflection groups and braid groups. In Section 2, I first recall some results about resolutions of Kleinian surface singularities, and then turn to the construction of certain quasiprojective threefolds and their deformations. Section 3 discusses generalities about families of Fourier-Mukai functors. Section 4 contains the main results. Families of Fourier-Mukai functors constructed in Section 4.1 are shown to satisfy braid relations in Section 4.2, where faithfulness is also proved in some cases. The paper is concluded in Section 4.3 by a brief discussion of the projective case.

Conventions. A *smooth family* means a smooth morphism $e : \mathscr{X} \to S$ of smooth varieties over $\mathbb{C}$ with $\mathscr{X}$ quasiprojective over S. The base S will always be very simple in this paper, typically affine space $\mathbb{A}^r$ or an open set thereof. For a brief period in Section 2, r can be infinite, but this will cause no complications. The fiber $e^{-1}(s)$ over $s \in S$ will be denoted by X_s. By definition a *Dynkin diagram* Δ means an irreducible diagram of finite type $A_n \ldots G_2$. Nodes of Δ will be denoted $i, j, \ldots$; for $i \neq j$, $m_{ij} = m_{ji} \in \{2, 3, 4, 6\}$ is the label associated to the pair of nodes (i, j). As usual, the pair (i, j) is said to span an edge if $m_{ij} > 2$. The diagram Δ is simply laced (type ADE) if $m_{ij} \in \{2, 3\}$.

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1. Dynkin diagrams and Artin groups

1.1. The reflection group and the Artin group. Take an arbitrary Dynkin diagram Δ with n nodes. Let $\Sigma_\Delta \subset \mathfrak{h}_{\Delta, \mathbb{R}}$ be the corresponding root system, where $(\mathfrak{h}_{\Delta, \mathbb{R}}, \langle \, , \, \rangle)$ is a Euclidean inner product space of dimension n. Fix sets of simple and positive roots

$$\Sigma_\Delta^0 = \{\lambda_1, \ldots, \lambda_n\} \subset \Sigma_\Delta^+ \subset \Sigma_\Delta.$$

The reflections $r_i : \mathfrak{h}_{\Delta, \mathbb{R}} \to \mathfrak{h}_{\Delta, \mathbb{R}}$ $(1 \leqq i \leqq n)$ defined by the simple roots generate a finite reflection group

$$W_\Delta = \langle r_i \rangle < \mathrm{GL}(\mathfrak{h}_{\Delta, \mathbb{R}}).$$

As an abstract group,

$$W_\Delta \cong \langle r_i : i \in \mathrm{Nodes}(\Delta) \rangle / \langle r_i^2 = 1, (r_i r_j)^{m_{ij}} = 1 \rangle$$

with one relation for every node i and one for every pair of different nodes (i, j) with label m_{ij}. The set of reflections in W_Δ is in one-to-one correspondence with the set Σ_Δ^+. The

group W_Δ also acts on the complex vector space $\mathfrak{h}_\Delta = \mathfrak{h}_{\Delta,\mathbb{R}} \otimes \mathbb{C}$; for a reflection $w \in W_\Delta$, let $\Pi_w \subset \mathfrak{h}_\Delta$ denote the fixed hyperplane of w.

Define the *Artin group* (also called generalized braid group) B_Δ by generators and relations as

$$(1) \qquad B_\Delta = \langle R_i : i \in \mathrm{Nodes}(\Delta) \rangle / \langle \underbrace{R_i R_j \ldots}_{m_{ij}} = \underbrace{R_j R_i \ldots}_{m_{ij}} \rangle$$

with one relation for every pair of different nodes (i, j) of Δ, the *braid relation*. There is a group homomorphism $B_\Delta \to W_\Delta$ sending R_i to r_i.

1.2. Quotiens of Dynkin diagrams. Let Δ be a simply laced Dynkin diagram and A a non-trivial subgroup of its automorphism group $\mathrm{Aut}(\Delta)$, excluding the case $(\Delta, A) = (A_{2n}, \mathbb{Z}/2)$. Then A permutes the set of simple roots in $\mathfrak{h}_\Delta$ which forms a basis; hence A also acts on $\mathfrak{h}_\Delta$. Let $\mathfrak{h}_\Xi = (\mathfrak{h}_\Delta)^A$ and let $\Sigma_\Xi = \Sigma_\Delta \cap \mathfrak{h}_\Xi$ be the set of invariant roots. It is well known that, as the notation suggests, Σ_Ξ is a root system for a Dynkin diagram Ξ, the "quotient" diagram Δ/A. For $\Delta = A_{2n-1}, D_n, E_6$, Ξ is the non-simply laced diagram of type C_n, B_{n-1}, F_4 respectively; in the case $\Delta = D_4$, Ξ is either G_2 or C_3 according to whether A acts transitively on the outer nodes of D_4 or not. The set of nodes of Ξ is in one-to-one correspondence with the set of orbits of nodes of Δ under the action of A; there is a corresponding set of simple and positive roots

$$\Sigma_\Xi^0 = \{\mu_1, \ldots\} \subset \Sigma_\Xi^+ \subset \Sigma_\Xi$$

and a reflection group

$$W_\Xi = \langle \rho_i : i \in \mathrm{Nodes}(\Xi) \rangle$$

acting on $\mathfrak{h}_\Xi$. Finally Ξ also defines an Artin group B_Ξ.

Lemma 1.1. *The group A acts naturally on the Artin group B_Δ and the reflection group W_Δ equivariantly with respect to the map $B_\Delta \to W_\Delta$. The fixed subgroups are isomorphic to B_Ξ and W_Ξ respectively.*

Proof. For $a \in A$, the action is defined on generators of B_Δ by $R_i \mapsto R_{a(i)}$. This action clearly leaves the relations invariant and descends to an action on W_Δ. If $\{k_j\}$ is an A-orbit of nodes of Δ corresponding to a node k of Ξ, then $R_{k_j} \in B_\Delta$ commute by the braid relations and their product $R_k = \prod R_{k_j}$ is invariant under the action. It is an easy check to show that the R_k satisfy the braid relations of the group B_Ξ. By [13], Corollary 4.4, these elements generate the fixed subgroup and they do not satisfy any further relations. $\quad\square$

Remark 1.2. The proof also shows that the action of A on the reflection group W_Δ is simply the conjugation action of $A < \mathrm{GL}(\mathfrak{h}_\Delta)$ on $W_\Delta < \mathrm{GL}(\mathfrak{h}_\Delta)$.

2. Surfaces, threefolds and deformations

2.1. Finite subgroups of SL(2,$\mathbb{C}$). Fix a finite subgroup Γ of $\mathrm{SL}(2, \mathbb{C})$ together with its canonical two-dimensional representation ρ_c. Following McKay, consider the diagram

$\tilde{\Delta}$ consisting of a node for every irrep (irreducible representation) ρ_j of Γ and an edge between irreps ρ_j, ρ_k whenever ρ_j is a direct summand of $\rho_k \otimes \rho_c$. It is well known that this defines a symmetric relation and $\tilde{\Delta}$ is an affine Dynkin diagram of type $\tilde{A}_n, \tilde{D}_n$ or $\tilde{E}_n$ with distinguished affine node corresponding to the trivial rep ρ_0. Let $\Delta = \tilde{\Delta} \backslash \{\rho_0\}$ be the corresponding finite diagram.

Lemma 2.1. *There exists an exact sequence of groups*

(2) $$1 \to C_\Gamma \xrightarrow{\gamma} N_\Gamma \xrightarrow{\delta} \mathrm{Aut}(\Delta) \to 1,$$

where $N_\Gamma = N_{\mathrm{GL}(2,\mathbb{C})}(\Gamma)/\Gamma$ and $C_\Gamma = C_{\mathrm{GL}(2,\mathbb{C})}(\Gamma)/Z_\Gamma$ are the normalizer of Γ in $\mathrm{GL}(2,\mathbb{C})$ modulo Γ and the centralizer of Γ modulo the center Z_Γ of Γ respectively, and $\mathrm{Aut}(\Delta)$ is the automorphism group of the diagram Δ.

Proof. For an irrep $\rho : \Gamma \to \mathrm{GL}(V)$ and an element $g \in N_{\mathrm{GL}(2,\mathbb{C})}(\Gamma)$, define a new irrep ρ^g of Γ by $\rho^g(h) = \rho(g^{-1}hg)$. The isomorphism class of the irrep ρ^g only depends on the class of g in N_Γ. For all $g \in N_{\mathrm{GL}(2,\mathbb{C})}(\Gamma)$, ρ_0^g is isomorphic to ρ_0 and ρ_c^g to ρ_c, so the diagram Δ is mapped to itself by the action of g. So $\rho \mapsto \rho^g$ defines a map $\delta : N_\Gamma \to \mathrm{Aut}(\Delta)$. The proof of the surjectivity of this map as well as the computation of its kernel are easy on a case-by-case basis. $\square$

2.2. The surface Y. For a finite subgroup $\Gamma < \mathrm{SL}(2,\mathbb{C})$, let $g : Y \to \mathbb{C}^2/\Gamma$ be the minimal resolution of the Kleinian quotient singularity with exceptional locus $E = \bigcup_{j=1}^{n} E_j$. The incidence graph of the components of E can be identified with the (simply laced) McKay diagram Δ defined by Γ in 2.1; fix such an identification.

Proposition 2.2. *There is an injection $j : N_\Gamma \hookrightarrow \mathrm{Aut}(Y)$. The composite*

$$N_\Gamma \to \mathrm{Aut}(Y) \to \mathrm{Aut}(\Delta),$$

where the latter is the map given by permuting exceptional divisors, coincides with the map δ of Lemma 2.1. In particular, an element of N_Γ fixes all the exceptional divisors if and only if it is in C_Γ.

Proof. By definition, an element $h \in N_{\mathrm{GL}(2,\mathbb{C})}(\Gamma)$ induces an automorphism of $\mathbb{C}^2$ normalizing the action of Γ; hence this automorphism descends to the quotient $\mathbb{C}^2/\Gamma$ and only depends on the class of h in N_Γ. The resolution f is the unique minimal model of $\mathbb{C}^2/\Gamma$, hence every automorphism of $\mathbb{C}^2/\Gamma$ lifts to a unique automorphism of Y. This defines the injection $j : N_\Gamma \to \mathrm{Aut}(Y)$. The last statement follows from an explicit computation on the resolution. $\square$

Next I collect information about the cohomology and deformations of the surface Y. The first statement is well known.

Proposition 2.3. *The second cohomology $H^2(Y, \mathbb{Z})$ of the surface Y is a free $\mathbb{Z}$-module of rank n, with dual $H_c^2(Y, \mathbb{Z})$ which has a natural $\mathbb{Z}$-basis consisting of the classes $\{[E_j]\}$ of the exceptional divisors. Every exceptional divisor $E_j \subset Y$ gives rise to a reflection*

$$\omega \mapsto \omega + ([E_j] \cdot \omega) c_1\big(\mathcal{O}_Y(E_j)\big)$$

on $H^2(Y, \mathbb{C})$. These reflections generate a finite reflection group. $\square$

Remark 2.4. The data in this proposition can be identified with the Dynkin diagram data as follows. Mapping the class $[E_j] \in H_c^2(Y, \mathbb{C})$ to the simple root λ_j gives an isomorphism between the lattice $H_c^2(Y, \mathbb{Z})$ and the root lattice $\mathbb{Z}\langle \lambda_1, \ldots, \lambda_r \rangle$. Dually, $H^2(Y, \mathbb{C}) \cong \mathfrak{h}_\Delta$ and the reflection defined by E_j is the reflection r_j associated to the simple root λ_j. Hence the reflection group in (ii) is isomorphic to the reflection group W_Δ. Note also that the group $\mathrm{Aut}(\Delta)$ acts both on $H^2(Y, \mathbb{C})$ (by acting on a basis) and on $\mathfrak{h}_\Delta$ (as defined in Section 1.2) and these actions are obviously compatible.

Recall the hyperplanes $\Pi_w \subset \mathfrak{h}_\Delta$ which are the fixed loci of reflections of W_Δ; in particular, these include the fixed hyperplanes $\Pi_{r_j} = \{\omega \in \mathfrak{h}_\Delta \mid \langle \omega, \lambda_i \rangle = 0\}$ of r_j.

Proposition 2.5. (i) *The universal deformation space of Y is a smooth family $d : \mathcal{Y} \to Z$ with central fiber $d^{-1}(0) \cong Y$. There is a simultaneous contraction $G : \mathcal{Y} \to \overline{\mathcal{Y}}$ over Z with central fiber $g : Y \to \overline{Y}$. More generally, for any subset $I = \{E_{j_i}\}$ of the exceptional curves, there exists a contraction $G_I : \mathcal{Y} \to \overline{\mathcal{Y}}_I$ over Z contracting curves in I in the central fiber and giving an isomorphism on fibers Y_t to which none of the curves in I deform.*

(ii) *Choosing a generator of the relative canonical bundle $\omega_{\mathcal{Y}/Z}$ over Z gives rise to a period map*

$$\varphi : Z \to H^2(Y, \mathbb{C}) \cong \mathfrak{h}_\Delta$$

which is an isomorphism. Different choices of the generator give rise to the same isomorphism up to multiplication by a nonzero constant on $\mathfrak{h}_\Delta$.

(iii) *The group N_Γ acts naturally on the base Z and compatibly on the total space of the family $\mathcal{Y} \to Z$ by automorphisms. There is an induced action of N_Γ on $\mathfrak{h}_\Delta$ via φ. This action factors via the morphism*

$$(\delta, \det) : N_\Gamma \to \mathrm{Aut}(\Delta) \times \mathbb{C}^*$$

with $\mathbb{C}^*$ acting on $\mathfrak{h}_\Delta$ with weight one.

(iv) *The action of the reflection group W_Δ on $\mathfrak{h}_\Delta$ induces, via the period map φ, a W_Δ-action on Z. This extends compatibly to an action on the total space $\overline{\mathcal{Y}} \to Z$ by automorphisms. More generally, if $w \in W$ is a reflection and I denote the set of nodes corresponding to positive roots mapped to negative roots by w, then there is a diagram*

$$
\begin{array}{ccc}
\overline{\mathcal{Y}}_I & \xrightarrow{\sim} & \overline{\mathcal{Y}}_I \\
\downarrow & & \downarrow \\
Z & \xrightarrow{w} & Z.
\end{array}
$$

There is no W_Δ-action by automorphisms on the total space $\mathcal{Y} \to Z$, but for $w \in W_\Delta$ and $t \in Z$ the fibers $g^{-1}(t), g^{-1}(tw)$ are isomorphic.

(v) *For a point $s \in \mathfrak{h}_\Delta$, the fiber $g^{-1}\varphi^{-1}(s) = Y_s$ contains projective curves if and only if $s \in \bigcup\limits_{w \in \Sigma_\Delta} \Pi_w$; otherwise it is affine. For $w \in \Sigma_\Delta^+$ corresponding to the positive root λ, write $\lambda = \sum\limits_j a_j \lambda_j$. Then $s \in \Pi_w$ if and only if Y_s contains a rational (possibly reducible) curve which is a flat deformation of the effective rational curve $\sum\limits_j a_j E_j$ on Y. If $s \in \Pi_w \setminus \bigcup\limits_{w' \neq w} \Pi_{w'}$ then this curve is smooth and it is the unique projective curve in Y_s.*

Proof. (i) is well known. (ii) is proved for example in [17], Section 4. For (iii), note that by universality, $\mathrm{Aut}(Y)$ and hence its subgroup N_Γ act on the family $\mathcal{Y} \to Z$, and in particular on its base Z. By [18], Proposition 8.6ii, the determinant acts by weight one in the induced action on $\mathfrak{h}_\Delta$. The action of the reflection group W_Δ on Z and its properties stated in (iv) go back to Brieskorn, see e.g. [18]. Finally (v) is spelled out in [11], Theorem 1. □

Remark 2.6. Using (ii), I will identify Z with $\mathfrak{h}_\Delta$ everywhere below. The space $\mathfrak{h}_\Delta$ carries actions of both N_Γ and W_Δ.

2.3. The group cohomology class α and related constructions. Let B be a smooth, not necessarily projective curve over $\mathbb{C}$. Fix a cohomology class

$$\alpha \in H^1(B_{\text{ét}}, N_\Gamma)$$

in the nonabelian group cohomology set $H^1(B_{\text{ét}}, N_\Gamma)$ in the étale topology (compare [14], p. 122). All constructions below depend on this class α; for ease of reading, this dependence is dropped from the notation.

The exact sequence (2) of Lemma 2.1 gives rise to an exact sequence of pointed sets ([14], Proposition 4.5):

$$H^1(B_{\text{ét}}, C_\Gamma) \xrightarrow{\gamma} H^1(B_{\text{ét}}, N_\Gamma) \xrightarrow{\delta} H^1(B_{\text{ét}}, \mathrm{Aut}(\Delta)).$$

Consider the class $\delta(\alpha) \in H^1(B_{\text{ét}}, \mathrm{Aut}(\Delta))$; let A be the minimal subgroup of $\mathrm{Aut}(\Delta)$ such that $\delta(\alpha)$ is in the image of

$$H^1(B_{\text{ét}}, A) \hookrightarrow H^1(B_{\text{ét}}, \mathrm{Aut}(\Delta)).$$

The preimage of $\delta(\alpha)$ in $H^1(B_{\text{ét}}, A)$ defines a finite étale cover $b : \tilde{B} \to B$. Since the cocycle cannot be reduced to a smaller subgroup, $\tilde{B}$ is connected.

For the rest of this paper, I fix the type of the simply laced diagram Δ (equivalently the finite subgroup Γ) and the cohomology class α; hence also A and $\Xi = \Delta/A$ are fixed. By the exact sequence above, A is trivial if and only if α takes values in C_Γ.

Since Γ is a subgroup of $\mathrm{SL}(2, \mathbb{C})$, there is a map $\det : N_\Gamma \to \mathbb{C}^*$. Hence α gives rise to an induced cocycle

$$\det(\alpha) \in H^1(B_{\text{ét}}, \mathbb{C}^*) \cong \mathrm{Pic}(B);$$

I will denote by $\mathscr{M}$ the corresponding line bundle on B. Recall also that Proposition 2.5(iii) defines a map $N_\Gamma \to \mathrm{GL}(\mathfrak{h}_\Delta)$. The image of the cohomology class $\alpha \in H^1(B_{\text{ét}}, N_\Gamma)$ in $H^1(B_{\text{ét}}, \mathrm{GL}(\mathfrak{h}_\Delta)) \cong H^1(B, \mathrm{GL}(\mathfrak{h}_\Delta))$ induces a locally free sheaf $\mathscr{H}$ of rank r on B.

If A is trivial, this construction gives a class in

$$H^1(B, \mathbb{C}^*) \subset H^1(B, \mathrm{GL}(\mathfrak{h}_\Delta)),$$

and in this case $\mathscr{H} \cong \mathscr{M} \otimes \mathfrak{h}_\Delta$. A positive root $\lambda \in \Sigma^+_\Delta \subset \mathfrak{h}_\Delta$ defines a map $\lambda : \mathfrak{h}_\Delta \to \mathbb{C}$ using the inner product, which globalizes to a map of bundles $\lambda : \mathscr{H} \to \mathscr{M}$ over B and hence to a map on sections

$$(3) \qquad\qquad m_\lambda : H^0(B, \mathscr{H}) \to H^0(B, \mathscr{M}).$$

If A is nontrivial, there is an isomorphism $b^*\mathscr{H} \cong b^*\mathscr{M} \otimes \mathfrak{h}_\Delta$ on the étale cover $\tilde{B}$ of B. On the other hand, by Leray

$$H^0(\tilde{B}, b^*\mathscr{H}) \cong H^0(B, b_*b^*\mathscr{H}) \cong H^0(B, \mathscr{H} \otimes b_*\mathcal{O}_{\tilde{B}})$$

and taking A-invariants,

$$H^0(B, \mathscr{H}) \cong H^0(\tilde{B}, b^*\mathscr{H})^A$$

$$\cong H^0(\tilde{B}, b^*\mathscr{M})^A \otimes \mathfrak{h}^A_\Delta \cong H^0(\tilde{B}, b^*\mathscr{M})^A \otimes \mathfrak{h}_\Xi.$$

Hence if $\mu \in \Sigma^+_\Xi$ is a positive root in the root system of Ξ, there is an induced map

$$(4) \qquad\qquad m_\mu : H^0(B, \mathscr{H}) \to H^0(\tilde{B}, b^*\mathscr{M})^A.$$

2.4. The threefold X. Represent the group cohomology element $\alpha \in H^1(B_{\text{ét}}, N_\Gamma)$ by a Čech cocycle $\{\alpha_{ij} \in \Gamma(B_{ij}, N_\Gamma)\}$ with respect to an étale covering $\{B_l\}$ of B. Consider the product $\mathbb{C}^2 \times B_l$, its quotient $\overline{X}_l = \mathbb{C}^2/\Gamma \times B_l$ and resolution $X_l = Y \times B_l \to \overline{X}_l$ over B_l. By étale descent, I can glue the morphisms $X_l \to \overline{X}_l$ over B using the cocyle α with values in $N_\Gamma \subset \mathrm{Aut}(Y)$ (Proposition 2.2) to get a diagram of quasiprojective varieties

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & \overline{X} \\ & \searrow{\scriptstyle\pi} & \downarrow{\scriptstyle\bar{\pi}} \\ & & B \end{array}$$

which up to isomorphism only depends on the cohomology class of α.

Proposition 2.7. (i) *The singular locus of $\overline{X}$ is a section of $\bar{\pi}$ and is a curve of cDV singularities of the type of the diagram Δ. The morphism f is a crepant resolution of singularities. The canonical bundle of X is*

$$\omega_X \cong \pi^*(\omega_B \otimes \mathscr{M}^{-1})$$

and $H^2(X, \mathcal{O}_X) = 0$.

(ii) *Assume that $(\Delta, A) \neq (A_{2n}, \mathbb{Z}/2)$ as usual. The exceptional locus of f is a union of irreducible divisors D_j indexed by the nodes of Ξ; every such divisor is a smooth geometrically ruled surface $\pi_j : D_j \to B_j$. If j is a node of Ξ representing an A-orbit of size one then $B_j \cong B$, otherwise $B_j \cong \tilde{B}$. Two divisors D_i, D_j intersect if and only if the corresponding nodes i, j of Ξ are connected by an edge ($m_{ij} > 2$).*

(iii) *For every subset I of the nodes of Ξ, there exists a morphism*

$$f_I : X \to \overline{X}_I$$

which contracts the divisors D_j for $j \in I$ along their rulings and is an isomorphism on $X \backslash \bigcup_{j \in I} D_j$.
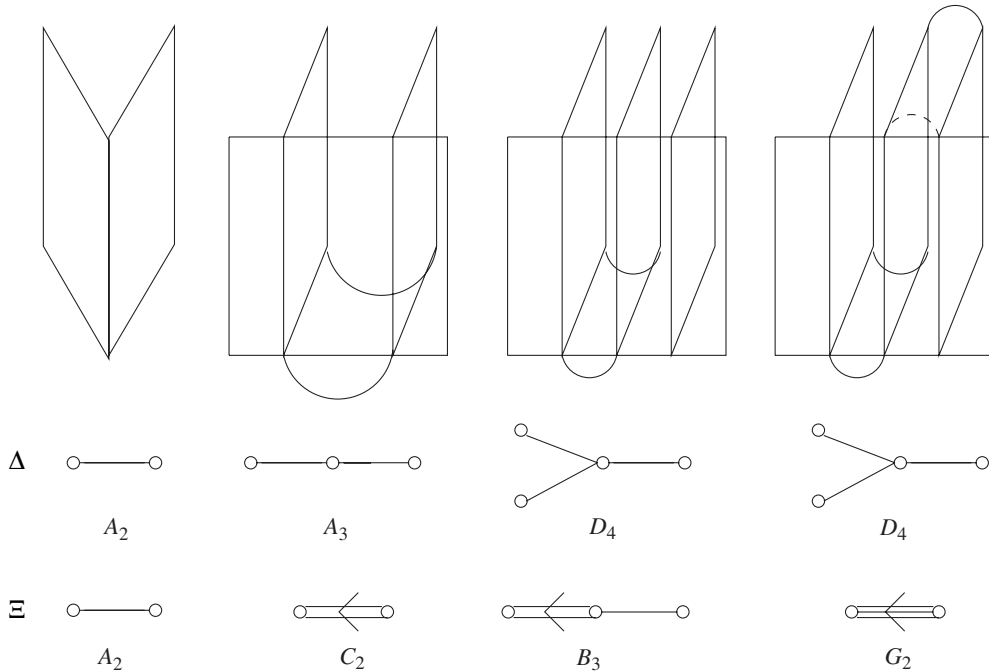

Figure 1.   Dynkin diagrams and configurations of surfaces.

Proof.   The first statement of (i) is clear. The statement that f is a crepant resolution is local and hence follows from the fact that $Y \to \mathbb{C}^2/\Gamma$ is a crepant resolution. To compute the canonical bundle of X, note that with $\pi_j : X_j = Y \times B_j \to B_j$, $\omega_{X_j} = \pi_j^*(\omega_{B_j})$, and these line bundles glue together after a twisting by the inverse of the determinant cocycle. $H^2(X, \mathcal{O}_X) = 0$ follows from the Leray spectral sequence for π and well-known properties of Y.

For (ii), recall that X was glued together from étale open subsets $X_l = Y \times B_l$. If α can be represented by a cocycle with values in C_Γ, in other words if A is trivial, then the glueing process will not permute the exceptional divisors $\{E_j\}$ in Y by Proposition 2.2. Hence $\{E_j \times B_l\}$ will glue for every j to a smooth exceptional divisor D_j ruled over the

curve B, and these surfaces will intersect as dictated by the diagram $\Xi = \Delta$. If A is non-trivial, then the set of exceptional lines $\{E_j\}$ is acted on by monodromy over B, this action being given by the cocyle $\delta(\alpha)$ with values in A. The lines E_i corresponding to nodes fixed by the action of A can still be glued globally over B, leading to exceptional divisors in X ruled over B. However, nontrivial orbits $\{E_{j_i}\}$ of exceptional curves under the A-action are glued together to an irreducible exceptional divisor, where the glueing is governed by the cohomology class $\delta(\alpha)$; hence the corresponding surfaces are ruled over the étale cover $\tilde{B}$ of B. Finally the morphism $f_I : X \to \overline{X}_I$ can be glued over B from the morphism $G_J : Y \to \overline{Y}_J$, contracting the exceptional curves on Y given by the A-invariant set J of nodes of Δ lying over I. $\quad\square$

Remark 2.8. In the special case $(\Delta, A) = (A_{2n}, \mathbb{Z}/2)$ the exceptional divisors D_i are still indexed by nodes of the quotient diagram Δ/A, defined to be the A_n-diagram with a marked node at one end corresponding to the adjacent $\mathrm{Aut}(\Delta)$-orbit of nodes. However, the marked node n of Ξ corresponds to a singular exceptional surface. It is an irreducible non-normal surface $\pi_n : D_n \to B$ whose double locus is a section and whose fiber over any point $b \in B$ is a line pair. The main results of this paper do not apply in this special case; see [21], Remark 4.5 for further discussion.

2.5. A family of deformations of X. Representing the cohomology class α as a Čech cocyle again with respect to an étale covering $\{B_l\}$ of B, there are isomorphisms $\mathscr{H}|_{B_l} \cong \mathfrak{h}_\Delta \times B_l$. Using these isomorphisms, pull back the universal deformation space $\mathscr{Y} \to \mathfrak{h}_\Delta$ of Y from Proposition 2.5 to a family of surfaces $\tilde{\mathscr{X}}_l \to \mathscr{H}|_{B_l}$. Glue these families over B using the identification given by the cocyle with values in $N_\Gamma < \mathrm{Aut}(\mathscr{Y})$ (compare Proposition 2.5(iii)) to get a global family of surfaces $\tilde{\mathscr{X}} \to \mathscr{H}$ over the total space of the vector bundle $\mathscr{H}$. Finally use the tautological map $B \times H^0(B, \mathscr{H}) \to \mathscr{H}$ to pull back $\tilde{\mathscr{X}}$ to a family $\mathscr{X}$ over $B \times H^0(B, \mathscr{H})$. This leads to a diagram

$$
\begin{array}{ccc}
\mathscr{X} & \longrightarrow & \tilde{\mathscr{X}} \\
\downarrow & & \downarrow \\
B \times T & \longrightarrow & \mathscr{H} \\
\downarrow & & \\
T & &
\end{array}
$$

with $T = H^0(B, \mathscr{H})$. The composite of the left-hand vertical maps gives rise to a morphism $e : \mathscr{X} \to T$, which is a smooth family of threefolds by construction. Using the projection to B shows that for $s \in T$, the fiber $X_s = e^{-1}(s)$ admits a smooth map $\pi_s : X_s \to B$. The central fiber $e^{-1}(0)$ corresponds to the zero section of the bundle $\mathscr{H}$; it is obtained by glueing varieties $Y \times B_l$, coming from the central fiber of $\mathscr{Y} \to \mathfrak{h}_\Delta$, over the curve B as dictated by the cohomology class α. Thus $e^{-1}(0) \cong X$. Note that at this point T may well be infinite dimensional, but the meaning of the following statements should be obvious also in this case.

Proposition 2.9. *The family $e : \mathscr{X} \to T$ with central fiber $X \cong e^{-1}(0)$ has the following properties*:

(i) *The Kodaira-Spencer map of the family is injective at $0 \in T$.*

(ii) *For any set I of nodes of Ξ, there is a contraction morphism $F_I : \mathscr{X} \to \bar{\mathscr{X}}_I$ over T with central fiber $f_I : X \to \bar{X}_I$.*

(iii) *The group W_Ξ acts on the base T with the following properties:*

(a) *For i a node of the Dynkin diagram Ξ, the fixed locus $T_i = \mathrm{Fix}(\rho_i)$ of the reflection $\rho_i \in W_\Xi$ acting on T is exactly the locus of points $s \in T$ for which the map $f_{i,s} : X_s \to X_{i,s}$ (the fiber of F_i at $s \in T$) contracts a surface.*

(b) *For a reflection $w \in W_\Xi$, if I is the set of simple roots reflected by w, there exists a diagram*

$$
\begin{array}{ccc}
\mathscr{X} & \dashrightarrow^{\theta_w} & \mathscr{X} \\
\downarrow & & \downarrow \\
\bar{\mathscr{X}}_I & \xrightarrow{\sim} & \bar{\mathscr{X}}_I \\
\downarrow & & \downarrow \\
T & \xrightarrow{w} & T.
\end{array}
$$

The relative birational morphism $\theta_w : \mathscr{X} \dashrightarrow \mathscr{X}$ over T restricts to a birational morphism $\theta_{w,s} : X_s \dashrightarrow X_{w(s)}$ on fibers.

Proof. The Kodaira-Spencer map of the family $e : \mathscr{X} \to T$ at $0 \in T$ is a map

$$\varphi : \Theta_{T,0} \cong H^0(B, \mathscr{H}) \to H^1(X, \Theta_X)$$

where $\Theta_{T,0}$ is the holomorphic tangent space of T at $0 \in T$, and Θ_X is the holomorphic tangent bundle of X. This map sits in a composition

$$(5) \qquad H^0(B, \mathscr{H}) \to H^1(X, \Theta_X) \to H^0(B, R^1\pi_*\Theta_X) \to H^0(B, R^1\pi_*\Theta_{X/B})$$

where the second map comes from the Leray spectral sequence, and the last map comes from the natural map of sheaves $\Theta_X \to \Theta_{X/B}$ on X. On the other hand, it is easy to check from the construction that the Kodaira-Spencer map $\mathfrak{h}_\Delta \xrightarrow{\sim} H^1(Y, \Theta_Y)$ for Y globalizes to an isomorphism between the sheaf $\mathscr{H}$ and the sheaf $R^1\pi_*\Theta_{X/B}$ on B, and the composite $H^0(B, \mathscr{H}) \to H^0(B, R^1\pi_*\Theta_{X/B})$ of the maps in (5) is the induced isomorphism. Hence the Kodaira-Spencer map of $e : \mathscr{X} \to T$ is injective.

For (ii), let J be the subset of exceptional curves in Y corresponding to the set of nodes I of Ξ, and let $G_J : \mathscr{Y} \to \mathscr{Y}_J$ be the corresponding contraction from Proposition 2.5(vi). J is fixed under the monodromy action by A, so I can glue G_J over B according to the cocycle α to get a morphism $F_I : \mathscr{X} \to \bar{\mathscr{X}}_I$ with central fiber f_I.

For (iii), note that $W_\Xi < W_\Delta$ acts on $\mathfrak{h}_\Delta$ and by Remark 1.2 this action commutes with that of A and of course the scalars. Hence it acts on the vector bundle $\mathscr{H}$ over B (trivial action on B) and so on $T = H^0(B, \mathscr{H})$.

To show property (a) of this action, assume first that A is trivial. Fix a section $s \in T$ of $\mathscr{H}$ and a node i of $\Xi = \Delta$. Note that by Proposition 2.5(v), the surface $X_{s,p} = \pi_s^{-1}(p)$ over $p \in B$ contains a deformation of the rational curve $E_i \subset Y$ if and only if $m_{r_i}(s)$ vanishes at $p \in B$, where m_{r_i} is the map in (3). The contraction $f_{i,s}$ contracts a surface if and only if this happens at every point $p \in B$, in other words if $m_{r_i}(s) = 0$. However, this simply says that in every fiber $s(p)$ is on the reflection hyperplane $\Pi_{r_i} \subset \mathfrak{h}_\Delta$ or equivalently that it is fixed by r_i. Hence the fixed locus of r_i acting on T is exactly the locus where $f_{i,s}$ contracts a surface.

If A is nontrivial, the node i of Ξ corresponds to an A-orbit $\{i_j\}$ of nodes of Δ. For $s \in T$, the surface fiber $X_{s,p}$ over $p \in B$ will contain deformations of the exceptional curves E_{i_j} of Y if and only if $m_{\rho_i}(s)$ vanishes at the preimages $b^{-1}(p) \subset \tilde{B}$ of $p \in B$, where m_{ρ_i} is the map in 4. Hence $f_{i,s}$ contracts a surface if and only if $m_{\rho_i}(s) = 0$; equivalently, if in every fiber $s(p) \in \mathscr{H}_p \cong \mathfrak{h}_\Delta$ lies on all reflection hyperplanes $\Pi_{r_{i_j}} \subset \mathfrak{h}_\Delta$. This however happens if and only if the section s is fixed by $\rho_i = \prod r_{i_j} \in W_\Xi \subset W_\Delta$.

The isomorphism $\bar{\mathscr{X}}_I \xrightarrow{\sim} \bar{\mathscr{X}}_I$ fitting into the diagram of (b) for a reflection $w \in W_\Xi$ is the pullback of the diagram of Proposition 2.5(iv) and it naturally induces a birational map between resolutions; the details are left to the reader. $\square$

Remark 2.10. The W_Ξ-action on T will be essential in what follows. I continue calling an element $w \in W_\Xi$ a reflection if it is a reflection on $\mathfrak{h}_\Xi$, that is if it corresponds to a positive root in Σ_Ξ^+. Of course in general the fixed locus of w on T is not a hyperplane.

If the linear system $\mathscr{M}$ on B is small, then the family $e : \mathscr{X} \to T$ can be rather uninteresting (for example, T could be a point). Under an extra assumption, a lot more geometry emerges.

Proposition 2.11. *Assume that $\mathscr{M}$ is a moving linear system on B.*

(i) *For a general point $s \in T$, the exceptional locus of $f_s : X_s \to \bar{X}_s$ consists of a finite number of disjoint smooth rational curves, naturally indexed by positive roots $\mu \in \Sigma_\Xi^+$, with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1, -1)$.*

(ii) *For such general $s \in T$ and a reflection $w \in W_\Xi$, the birational map $\theta_{w,s} : X_s \dashrightarrow X_{w(s)}$ is a flop, flopping exactly those $(-1, -1)$-curves which are indexed by positive roots mapped to negative roots by w.*

(iii) *For $w = \rho_i$ with i a node of the Dynkin diagram Ξ, there are two possibilities: either $s \in T_i$ and $\theta_{\rho_i, s} : X_s \dashrightarrow X_{\rho_i(s)}$ is the identity isomorphism, or $s \in T \setminus T_i$ and the birational map $\theta_{\rho_i, s} : X_s \dashrightarrow X_{\rho_i(s)}$ flops a disjoint union of rational curves indexed by the simple root μ_i, with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1, -1)$ or $\mathcal{O}_{\mathbb{P}^1}(0, -2)$.*

Proof. To prove (i), assume first that A is trivial. Let $p \in B$ a closed point and $X_{s,p} = \pi_s^{-1}(p)$ the corresponding surface. According to Proposition 2.5, the surface $X_{s,p}$ is affine if and only if $s(p) \in \mathfrak{h}_\Delta = \mathscr{H}_s$ does not lie on any reflection hyperplane Π_w, and it contains a single smooth rational curve if it lies in a unique such hyperplane. Said invariantly, using the map (3), $X_{s,p}$ is affine if and only if $m_\lambda(s)$ does not vanish at p for any $\lambda \in \Sigma_\Delta^+$, and it contains a unique curve if $m_\lambda(s)$ vanishes at p for a unique $\lambda \in \Sigma_\Delta^+$.

I claim that for general $s \in T$, there is a finite set of $p \in B$ such that $m_\lambda(s)$ vanishes at p for some positive root λ, and at such a p there is a unique $\lambda \in \Sigma_\Delta^+$ with $m_\lambda(s)(p) = 0$. This shows that for such s there is a finite number of disjoint rational curves in X_s, which are indexed by positive roots.

To show the claim, take two sections $t, t' \in H^0(B, \mathcal{M})$ which have a disjoint set of simple zeroes; as $|\mathcal{M}|$ has no base points, this is possible. Choose also $h, h' \in \mathfrak{h}_\Delta$ so that for $\lambda, \lambda' \in \Sigma_\Delta^+$ different positive roots,

(6) $$\langle \lambda, h \rangle \langle \lambda', h' \rangle - \langle \lambda, h' \rangle \langle \lambda', h \rangle \neq 0.$$

Setting $s = h \otimes t + h' \otimes t'$, the section $m_\lambda(s)$ of $\mathcal{M}$ vanishes only at finitely many points of B for any positive root λ. Also, if $m_\lambda(s)$ vanishes at the same point $p \in B$ for two different roots, then by condition (6), I get that $t(p) = t'(p) = 0$ which contradicts the choice of t, t'.

The proof in the general case, when A is nontrivial, is similar. In this case, the claim is that for general $s \in T$ there is a finite number of points $p \in B$ such that $m_\lambda(s) \in H^0(\tilde{B}, b^*\mathcal{M})$ vanishes at a point $q \in \tilde{B}$ lying over $p \in B$ and for a positive root $\mu \in \Sigma_\Xi^+$, and at such points p, vanishing happens for a unique μ. The claim follows by considering

$$b^* t \otimes h + b^* t' \otimes h' \in H^0(\tilde{B}, b^*\mathcal{M})^A \otimes \mathfrak{h}_\Xi \cong H^0(B, \mathcal{H})$$

for sufficiently general $h, h' \in \mathfrak{h}_\Xi$ and $t, t' \in H^0(B, \mathcal{M})$ with no common zeros.

By construction and the discussion above, a small analytic neighbourhood of every rational curve on the general fiber X_s looks like the standard one-dimensional deformation of a (-2)-curve in a surface, the deformation direction being transversal to the hyperplane along which the curve deforms. In other words, locally near the curve, the threefold looks like a small resolution of the ordinary threefold double point $\{xy = z^2 - t^2\}$. It is well known that the normal bundle of such a curve is $\mathcal{O}_{\mathbb{P}^1}(-1, -1)$. This also proves statement (ii): by Proposition 2.9(iii), the birational map $\theta_{w,s}$ factors as $X_s \to \overline{X}_{I,s} \leftarrow X_{w(s)}$, where the first map contracts exactly those $(-1, -1)$-curves which are indexed by positive roots mapped to negative ones by w; locally analytically the two maps give the two small resolutions of the resulting ordinary double points, in other words the flop.

To prove (iii), note that one possibility is $m_{\rho_i}(s) = 0$, hence $s \in T_i$ and the birational map $\theta_{\rho_i,s}$ is the identity. Otherwise $m_{\rho_i}(s)$ vanishes at a finite set of points $\{p_l\} \subset B$, and the map $X_s \to \overline{X}_{i,s}$ contracts a finite number of disjoint rational curves contained in the surfaces X_{s,p_l}. If at p_l the section $m_{\rho_i}(s)$ meets the zero-section of $\mathcal{M}$ transversally, then the singularity on $\overline{X}_{i,s}$ is analytically isomorphic to $\{xy = z^2 - t^2\}$ and the corresponding rational curve has normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1, -1)$. If the intersection of the two sections is tangential of order $n > 1$ then the singularity on $\overline{X}_{i,s}$ is analytically isomorphic to $\{xy = z^2 - t^{2n}\}$ and the normal bundle is $\mathcal{O}_{\mathbb{P}^1}(0, -2)$. In any case, $\theta_{\rho_i,s}$ flops all these curves (compare [15]). $\square$

Remark 2.12.   Points $s \in T$ in the base with f_s having an exceptional locus consisting of a disjoint union of smooth $(-1, -1)$-curves as in (i) will be called *sufficiently general*; on Figure 2, $s \in T$ is a sufficiently general point but $t \in T$ is not. As the proof shows, the locus of sufficiently general points is a non-empty Zariski open subset of T.
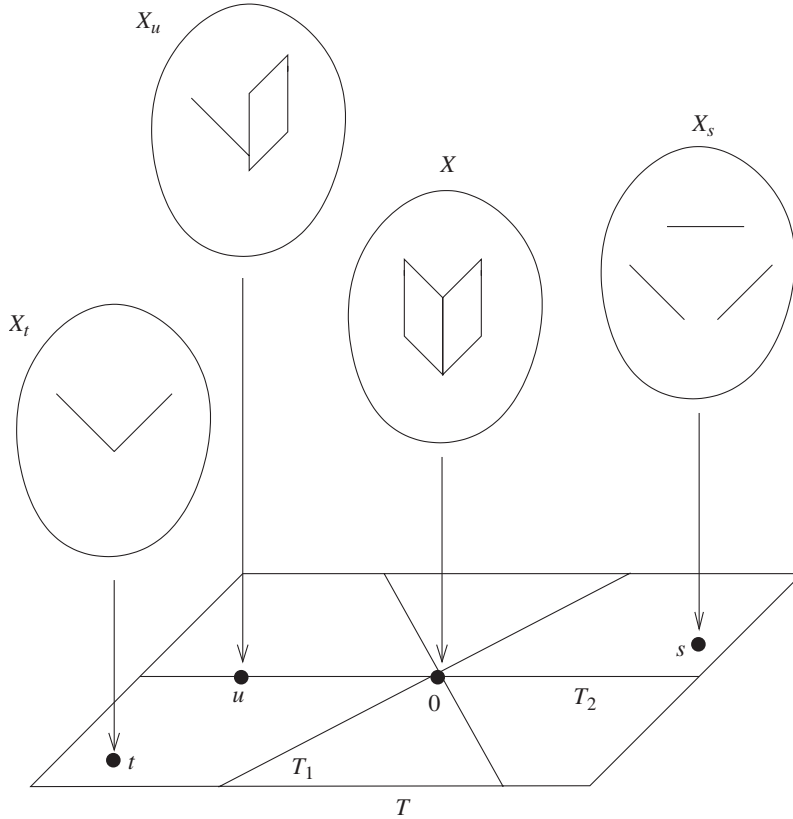
Figure 2.   Possible exceptional loci for type A_2.

Note also that it can perfectly well happen that $\mathcal{M}$ is the trivial line bundle on B and its only sections are the constants. All the statements of the above discussion remain true, with the small proviso that the maps $\theta_{w,s}$ flop an empty set of curves, in other words they are isomorphisms for all s. This phenomenon (in the projective case) is well known in the literature; the nontrivial birational contraction $f : X \to \overline{X}_i$ on the central fiber deforms to an isomorphism $f_s : X_s \xrightarrow{\sim} \overline{X}_{i,s}$, hence the cone of ample divisors jumps in the family. The variety X is Calabi-Yau if and only if $\mathcal{M} \cong \omega_B$ by Proposition 2.7(i), hence such surfaces are elliptic ruled surfaces in Calabi-Yau threefolds. Compare [22]–[23].

3. Derived categories and equivalences in families

3.1. Kernels and Fourier-Mukai functors.   If X is a smooth quasiprojective variety, let $D^b(X)$ denote the bounded derived category of coherent sheaves on X. A *kernel* (derived correspondence) between smooth quasiprojective varieties X_i ($i = 1, 2$) is an object $U \in D^b(X_1 \times X_2)$, whose support is proper over both factors. There is a composition product on kernels given for $U \in D^b(X_1 \times X_2)$ and $V \in D^b(X_2 \times X_3)$ by the standard formula

$$U \circ V = \mathbf{R}p_{13*}\left(p_{23}^*(V) \otimes^{\mathbf{L}} p_{12}^*(U)\right) \in D^b(X_1 \times X_3);$$

here $p_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j$ are the projection maps and the pullbacks are ordi-

nary pullbacks since p_{ij} is flat. A kernel $U \in D^b(X_1 \times X_2)$ is *invertible*, if there is a kernel $V \in D^b(X_2 \times X_1)$ such that the products $U \circ V$ and $V \circ U$ are isomorphic in $D^b(X_i \times X_i)$ to $\mathcal{O}_{\Delta_{X_i}}$, the (complexes consisting of) the structure sheaves of the diagonals. A kernel $U \in D^b(X_1 \times X_2)$ defines a functor

$$\Psi^U : D^b(X_2) \to D^b(X_1)$$

by

$$\Psi^U(-) = \mathbf{R}p_{1*}\big(U \otimes^{\mathbf{L}} p_2^*(-)\big),$$

with $p_i : X_1 \times X_2 \to X_i$ the projections. If U is invertible then Ψ^U is a *Fourier-Mukai functor*, an equivalence of triangulated categories. Let $\mathrm{Auteq}(X)$ denote the group of invertible kernels on X.

3.2. Kernels in families. Suppose that $\pi_i : \mathscr{X}_i \to S_i$, $i = 1, 2$ are smooth families. A *relative kernel for* (π_1, π_2) is a pair (U, φ), where

- $\varphi : S_1 \to S_2$ is an isomorphism, giving rise to the fiber product diagram

$$
\begin{array}{ccc}
\mathscr{X}_1 \times_\varphi \mathscr{X}_2 & \xrightarrow{\;p_2\;} & \mathscr{X}_2 \\
\big\downarrow{\scriptstyle p_1} & {\scriptstyle \pi_{12}} & \big\downarrow{\scriptstyle \varphi^{-1}\circ\pi_2} \\
\mathscr{X}_1 & \xrightarrow{\;\pi_1\;} & S_1
\end{array}
$$

and

- $U \in D^b(\mathscr{X}_1 \times_\varphi \mathscr{X}_2)$, whose derived restriction to the fiber of π_{12} over every $s \in S_1$ is isomorphic to an object in $D^b(X_{1,s} \times X_{2,\varphi(s)})$ with proper support over both factors.

If $\pi_i : \mathscr{X}_i \to S_i$, $i = 1, 2, 3$ are smooth families and $(U, \varphi), (V, \psi)$ relative kernels for $(\pi_1, \pi_2), (\pi_2, \pi_3)$, then there is a composition on kernels defined as follows. Let $\chi = \psi \circ \varphi : S_1 \to S_3$, and let

$$(7) \qquad \pi_{123} : \tilde{\mathscr{X}} = (\mathscr{X}_1 \times_\varphi \mathscr{X}_2) \times_\chi \mathscr{X}_3 \to S_1$$

be the twice-fiber product. There are maps $p_{12} : \tilde{\mathscr{X}} \to \mathscr{X}_1 \times_\varphi \mathscr{X}_2$ and $p_{13} : \tilde{\mathscr{X}} \to \mathscr{X}_1 \times_\chi \mathscr{X}_3$ which commute with the maps to S_1. On the other hand, it is easy to check that $(\mathscr{X}_1 \times_\varphi \mathscr{X}_2) \times_\chi \mathscr{X}_3 \cong \mathscr{X}_1 \times_\varphi (\mathscr{X}_2 \times_\psi \mathscr{X}_3)$ hence there is a map $p_{23} : \tilde{\mathscr{X}} \to \mathscr{X}_2 \times_\psi \mathscr{X}_3$ which satisfies $\pi_{23} \circ p_{23} = \varphi \circ \pi_{123}$. Hence finally I can put

$$W = \mathbf{R}p_{13*}\big(p_{23}^*(V) \otimes^{\mathbf{L}} p_{12}^*(U)\big)$$

and set

$$(V, \psi) \circ (U, \varphi) = (W, \chi).$$

It is easy to see that (W, χ) is a relative kernel for (π_1, π_3).

For reference I record the composition of three kernels, leaving the obvious generalization to the reader. Let $\pi_i : \mathscr{X}_i \to S_i$ be families for $i = 1, \ldots, 4$, and $(U, \varphi), (V, \psi), (T, \eta)$ relative kernels for $(\pi_1, \pi_2), (\pi_2, \pi_3)$ and (π_3, π_4). Let

$$\hat{\mathscr{X}} = \mathscr{X}_1 \times_\varphi \left(\mathscr{X}_2 \times_\psi \left(\mathscr{X}_3 \times_\eta \mathscr{X}_4 \right) \right)$$

with maps $s_{12} : \hat{\mathscr{X}} \to \mathscr{X}_1 \times_\varphi \mathscr{X}_2$, $s_{23} : \hat{\mathscr{X}} \to \mathscr{X}_2 \times_\psi \mathscr{X}_3$, $s_{34} : \hat{\mathscr{X}} \to \mathscr{X}_3 \times_\eta \mathscr{X}_4$ and

$$s_{14} : \hat{\mathscr{X}} \to \mathscr{X}_1 \times_{\eta \circ \psi \circ \varphi} \mathscr{X}_4.$$

Then

Lemma 3.1.

$$(T, \eta) \circ (V, \psi) \circ (U, \varphi) \cong (\tilde{U}, \eta \circ \psi \circ \varphi),$$

where

$$\tilde{U} = \mathbf{R}s_{14*} \left(s_{34}^*(T) \otimes^{\mathbf{L}} s_{23}^*(V) \otimes^{\mathbf{L}} s_{12}^*(U) \right). \quad \square$$

A relative kernel (U, φ) for (π_1, π_2) is called *invertible*, if there is a relative kernel (V, φ^{-1}) for (π_2, π_1), with the property that the compositions $(U, \varphi) \circ (V, \varphi^{-1})$ and $(V, \varphi^{-1}) \circ (U, \varphi)$ are isomorphic to the relative kernels $(\mathcal{O}_{\Delta_{\mathscr{X}_i}}, \mathrm{id})$.

Let $U_s \in D^b(X_{1,s} \times X_{2,\varphi(s)})$ denote the derived restriction of the relative kernel U to fibers of π_{12}.

Proposition 3.2. *If a relative kernel $U \in D^b(\mathscr{X}_1 \times_\varphi \mathscr{X}_2)$ for (π_1, π_2) is invertible then the restricted kernel $U_s \in D^b(X_{1,s} \times X_{2,\varphi(s)})$ is invertible for every $s \in S_1$. Conversely if the restricted kernel is invertible for every $s \in S_1$, then every $s \in S_1$ has a neighbourhood $s \in T \subset S_1$ such that the relative kernel restricted to the families $\pi_1^{-1}(T) \to T$ and $\pi_2^{-1}(\varphi(T)) \to \varphi(T)$ is invertible.*

Proof. If $x_{i,s} : X_{i,s} \hookrightarrow \mathscr{X}_i$ denotes the inclusion, then

$$\mathbf{L}x_{i,s}^* \mathcal{O}_{\Delta_{\mathscr{X}_i}} \cong \mathcal{O}_{\Delta_{X_{i,s}}}$$

as the maps π_i are flat. Hence if U is invertible with inverse V, then U_s is invertible with inverse V_s.

Conversely, take a relative kernel $U \in D^b(\mathscr{X}_1 \times_\varphi \mathscr{X}_2)$ and suppose that the restrictions are all invertible. Let

$$V = \mathbf{R}\mathscr{H}\mathrm{om}(U, \mathcal{O}_{\mathscr{X}_1 \times_\varphi \mathscr{X}_2}) \otimes p_1^* \omega_{\mathscr{X}_1/S_1}[n] \in D^b(\mathscr{X}_2 \times_\varphi \mathscr{X}_1)$$

where n is the dimension of the fibers $X_{i,s}$. Then by standard adjunctions the functor $\Psi^{(V, \varphi^{-1})}$ is right adjoint to the functor $\Psi^{(U, \varphi)}$. Hence Ψ^{V_s} is right adjoint to Ψ^{U_s} on the fibers. However, adjoints are unique, so V_s is the inverse of the kernel U_s. In other words,

$$\mathbf{L}z_{1,s}^*(U \circ V) \cong U_s \circ V_s \cong \mathcal{O}_{\Delta_{X_{1,s}}} \in D^b(X_{1,s} \times X_{1,s}),$$

where $z_{1,s} : X_{1,s} \times X_{1,s} \to \mathscr{X}_1 \times_S \mathscr{X}_1$ is the inclusion. By [3], Lemma 4.3, this implies that $U \circ V$ is a sheaf on $\mathscr{X}_1 \times_S \mathscr{X}_1$, flat over S. Take $s \in S_1$, then the natural map

$$H^0(\mathscr{X}_1 \times_S \mathscr{X}_1, U \circ V) \to H^0(X_{1,s} \times X_{1,s}, U_s \circ V_s)$$

is surjective, hence there is a map of sheaves

$$\mathcal{O}_{\mathscr{X}_1 \times_S \mathscr{X}_1} \to U \circ V$$

which is surjective at $s \in S_1$. So this map is surjective over a neighbourhood T of $s \in S_1$ and restricted to that neighbourhood, $U \circ V$ is a structure sheaf of a subscheme, fiberwise isomorphic to the diagonal $\Delta_{X_{1,s}} \subset X_{1,s} \times X_{1,s}$. By Lemma 3.5 below, $U \circ V$ restricted over T is isomorphic to the structure sheaf of the relative diagonal in the fiber product $\mathscr{X}_1 \times_S \mathscr{X}_1$. To conclude, repeat the argument with $V \circ U$ and take the intersection of the resulting open sets. $\quad \square$

Let now $\pi : \mathscr{X} \to S$ be a fixed smooth family, and consider relative kernels (U, φ) where $\varphi : S \to S$ is an automorphism of the base. Let $M(\mathscr{X}/S)$ be the set of such pairs up to isomorphism. $M(\mathscr{X}/S)$ has a monoid multiplication given by composition with a two-sided unit $(\mathrm{id}_S, \mathcal{O}_{\Delta_{\mathscr{X}}})$. Let $\mathrm{Auteq}(\mathscr{X}/S)$ be the group of invertible elements of the monoid $M(\mathscr{X}/S)$, the *group of relative equivalences* of the family $\mathscr{X} \to S$.

3.3. Three lemmas.   I record some auxiliary results on sheaves and kernels.

Lemma 3.3.   *Suppose that $\pi : \mathscr{X} \to S$ is a smooth family,*

$$(U, \varphi), (V, \psi) \in M(\mathscr{X}, S)$$

are relative kernels with composite (W, χ) and assume further that the sheaf V is isomorphic to the structure sheaf $i_* \mathcal{O}_Y$ of a subscheme $i : \mathscr{Y} \hookrightarrow \mathscr{X} \times_\psi \mathscr{X}$. Then every map $\mathcal{O}_{\mathscr{X} \times_\varphi \mathscr{X}} \to U$ in $D^b(\mathscr{X} \times_\varphi \mathscr{X})$ induces a map $\mathcal{O}_{\mathscr{X} \times_\chi \mathscr{X}} \to W$ in $D^b(\mathscr{X} \times_\chi \mathscr{X})$.

Proof.   Since V is the structure sheaf of a subscheme, $p_{23}^*(V) \cong j_* \mathcal{O}_{\tilde{\mathscr{Y}}}$ is also the structure sheaf of the subscheme $j : \tilde{\mathscr{Y}} \hookrightarrow \tilde{\mathscr{X}}$, where $\tilde{\mathscr{X}}$ is the fiber product of (7). But by the projection formula

$$p_{23}^*(V) \otimes^{\mathbf{L}} p_{12}^*(U) \cong j_* \mathcal{O}_{\tilde{\mathscr{Y}}} \otimes^{\mathbf{L}} p_{12}^*(U) \cong j_* \big(\mathbf{L}j^* p_{12}^*(U) \big).$$

On the other hand, by functoriality, a map $\mathcal{O}_{\mathscr{X} \times_\varphi \mathscr{X}} \to U$ induces a map $\mathcal{O}_{\tilde{\mathscr{X}}} \to p_{12}^*(U)$, hence a map $\mathcal{O}_{\tilde{\mathscr{Y}}} \to \mathbf{L}j^* p_{12}^*(U)$ and hence a map $j_*(\mathcal{O}_{\tilde{\mathscr{Y}}}) \to j_* \big(\mathbf{L}j^* p_{12}^*(U) \big)$. Composing this with the natural map $\mathcal{O}_{\tilde{\mathscr{X}}} \to j_*(\mathcal{O}_{\tilde{\mathscr{Y}}})$ I get a map $\mathcal{O}_{\tilde{\mathscr{X}}} \to p_{23}^*(V) \otimes^{\mathbf{L}} p_{12}^*(U)$. But $\mathcal{O}_{\tilde{\mathscr{X}}} \cong p_{13}^*(\mathcal{O}_{\mathscr{X} \times_\chi \mathscr{X}})$, so by adjunction I finally get a map

$$\mathcal{O}_{\mathscr{X} \times_\chi \mathscr{X}} \to \mathbf{R}p_{13*} \big(p_{23}^*(V) \otimes^{\mathbf{L}} p_{12}^*(U) \big) = W. \quad \square$$

Lemma 3.4.   *Let $f : X \to Y$ be a projective morphism of quasiprojective varieties. Let $i : Z \hookrightarrow X$ be a reduced subscheme of X and $j : T \hookrightarrow Y$ its reduced image under f. Assume that all fibers of $f|_Z : Z \to T$ are projective spaces. Then $\mathbf{R}f_*(i_* \mathcal{O}_Z) \cong j_* \mathcal{O}_T$.*

Proof. As i and j are closed immersions, the Grothendieck spectral sequences for $\mathbf{R}(f \circ i)_*$ and $\mathbf{R}(j \circ f|_Z)_*$ degenerate, hence

$$\mathbf{R}f_*(i_*\mathcal{O}_Z) \cong \mathbf{R}(f \circ i)_*(\mathcal{O}_Z) \cong \mathbf{R}(j \circ f|_Z)_*(\mathcal{O}_Z) \cong j_*(\mathbf{R}f|_{Z_*}\mathcal{O}_Z).$$

On the other hand, $f|_{Z_*}\mathcal{O}_Z \cong \mathcal{O}_T$ since fibers of $f|_Z$ are connected, and $R^i f|_{Z_*}\mathcal{O}_Z = 0$ for $i > 0$ by the Theorem on Formal Functions and the fact that the higher cohomologies of $\mathcal{O}$ on projective spaces vanish. This proves the statement. $\square$

Lemma 3.5. *Let $e : \mathcal{X} \to S$ be a smooth family. Assume that U_1, U_2 are sheaves on $\mathcal{X}$, flat over S, with surjective maps $g_i : \mathcal{O}_X \to U_i$ for $i = 1, 2$ (i.e. the U_i are structure sheaves of subschemes). Suppose further that for some dense open $S^0 \subset S$, if $s \in S^0$ then there is an isomorphism $U_{1,s} \cong U_{2,s}$ which is compatible with the maps $g_{i,s} : \mathcal{O}_{X_s} \to U_{i,s}$. Then $U_1 \cong U_2$ as sheaves on $\mathcal{X}$, compatibly with the maps g_i.*

Proof. For $i = 1, 2$ the structure sheaves U_i of subschemes of $\mathcal{X}$ give rise to morphisms

$$\varphi_i : S \to \mathrm{Hilb}(\mathcal{X}/S)$$

over S, where $\mathrm{Hilb}(\mathcal{X}/S) \to S$ represents the Hilbert functor of the quasiprojective morphism $\mathcal{X} \to S$ (this is constructed using a projective completion $\mathcal{X} \hookrightarrow \bar{\mathcal{X}} \to S$ along the fibers), such that the morphisms g_i are pullbacks of a universal surjection $\mathcal{O}_{\mathcal{X} \times_S \mathrm{Hilb}(\mathcal{X}/S)} \to \mathcal{U}$. By the condition on restrictions, $\varphi_1|_{S^0} = \varphi_2|_{S^0}$. But the Hilbert scheme is separated, so the maps φ_1 and φ_2 coincide. Hence $U_1 \cong U_2$ compatibly with the maps g_i, since they are pullbacks of $\mathcal{U}$ along the same map. $\square$

4. Artin group actions on derived categories

4.1. Relative equivalences for threefolds containing ruled surfaces. Recall the family of threefolds $e : \mathcal{X} \to T$ constructed in Section 2.4 together with the action of the reflection group W_Ξ on the base T. Note that T can a priori be infinite dimensional. For any finite dimensional W_Ξ-invariant vector subspace $Q \subset T$, I can consider the restricted family $e_Q : \mathcal{X}_Q \to Q$ which I will simply denote by $e : \mathcal{X} \to Q$. The central fiber of this family is still $e^{-1}(0) \cong X$. Restrict all contractions and the W_Ξ-action to Q.

For every node i of the diagram Ξ there is a contraction $F_i : \mathcal{X} \to \mathcal{X}_i$ and a map $\rho_i : Q \to Q$ fitting into a diagram

which is just the diagram from Proposition 2.11(iii) re-drawn and completed to a fiber product on the top. Let

$$U_i = \mathcal{O}_{\mathcal{X} \times_{\bar{\mathcal{X}}_i, \rho_i} \mathcal{X}} \in D^b(\mathcal{X} \times_{\rho_i} \mathcal{X})$$

be the structure sheaf of this fiber product.

Theorem 4.1. *There is a W_Ξ-invariant open subset $0 \in S \subset Q$, such that the restriction of each pair (U_i, ρ_i) to the pullback family $e_S : \mathcal{X}_S \to S$ is in the group of relative equivalences* $\mathrm{Auteq}(\mathcal{X}_S/S)$.

The proof of this theorem relies on the following fact, which will be very important also later.

Proposition 4.2. *The morphism $e_i : \mathcal{X} \times_{\bar{\mathcal{X}}_i, \rho_i} \mathcal{X} \to Q$ is flat and has reduced fibers.*

Proof. Using the notation and constructions of Section 2.5, $\mathcal{X} \to Q$ factors through a morphism $\mathcal{X} \to B \times Q$. Hence e_i factors through $\mathcal{X} \times_{\bar{\mathcal{X}}_i, \rho_i} \mathcal{X} \to B \times Q$ which is the pullback of a morphism $\tilde{\tilde{\mathcal{X}}} \times_{\bar{\mathcal{X}}_i, \rho_i} \tilde{\mathcal{X}} \to \mathcal{H}$ along the natural map $B \times Q \to \mathcal{H}$. It is clearly enough to prove that this latter morphism is flat with reduced fibers. However, this statement is (étale) local over the base, hence it suffices to prove it over the étale open set $\mathfrak{h}_\Delta \times B_l$ of $\mathcal{H}$; recall that $\{B_l\}$ is an étale open covering of the curve B. Over $\mathfrak{h}_\Delta \times B_l$, everything is a pullback along the map $\mathfrak{h}_\Delta \times B_l \to \mathfrak{h}_\Delta$, so finally it is enough to show that the morphism

$$d_I : \mathcal{Y} \times_{\bar{\mathcal{Y}}_I, r_I} \mathcal{Y} \to \mathfrak{h}_\Delta$$

has the stated properties. Here I is the A-orbit of nodes of Δ corresponding to the node i of Ξ, and r_I is the corresponding A-fixed element of W_Δ.

Since the morphism d_I is surjective with smooth target $\mathfrak{h}_\Delta$, by [8], 15.2.3 and Remark (v), using also [8], 14.4.2, it is flat once its fibers are reduced and its domain $\mathcal{Y} \times_{\bar{\mathcal{Y}}_I, r_I} \mathcal{Y}$ irreducible and equidimensional over $\mathfrak{h}_\Delta$. Equidimensionality is clear, so the issue is to prove that the fibers are reduced and the domain irreducible.

Assume first that $I = i$ is a single node of Δ. Then the central fiber $\overline{Y}_I$ of $\bar{\mathcal{Y}}_I \to \mathfrak{h}_\Delta$ is a surface with an ordinary surface double point, and the total family is a deformation family of this surface, where the double point survives on a codimension one subspace $\Pi \subset \mathfrak{h}_\Delta$. The map $r_i : \mathfrak{h}_\Delta \to \mathfrak{h}_\Delta$ is the reflection in Π. Finally the family $\mathcal{Y} \to \mathfrak{h}_\Delta$ is a simultaneous resolution of $\bar{\mathcal{Y}}_I \to \mathfrak{h}_\Delta$, constructed simply by blowing up the singular locus. Hence near the singularity, up to a local analytic change in coordinates I can simply write

$$
\begin{array}{ccc}
\bar{\mathcal{Y}}_i \cong \{xy = z^2 - t_1^2\} \subset \mathbb{A}^{n+3}_{x,y,z,t_1,\ldots,t_n} & & \\
\downarrow & & \downarrow \\
\mathfrak{h}_\Delta & = & \mathbb{A}^n_{t_1,\ldots,t_n}
\end{array}
$$

with singular locus $\mathrm{Sing}(\bar{\mathcal{Y}}_i) = \{x = y = z = t_1 = 0\}$ mapping to $\Pi = \{t_1 = 0\} \subset \mathfrak{h}_\Delta$, the fixed locus of $r_i : t_1 \mapsto -t_1$. The resolution $\mathcal{Y}$ can be constructed explicitly as the graph of the rational map $\bar{\mathcal{Y}}_i \dashrightarrow \mathbb{P}^1$ defined by $(x, y, z, t_1, \ldots, t_n) \mapsto (x : (z - t_1))$. Using the affine variable $s = x/(z - t_1)$, one affine piece of this graph is

$$\mathcal{Y}^{(1)} \cong \{ys = z + t_1\} \subset \mathbb{A}^{n+3}_{y,z,s,t_1,\ldots,t_n}.$$

Hence the fiber product has an affine open piece

$$(\mathcal{Y} \times_{\bar{\mathcal{Y}}_i, r_i} \mathcal{Y})^{(1,1)} = \left\{ \begin{array}{c} ys_1 = z + t_1 \\ ys_2 = z - t_1 \\ s_2(z + t_1) = s_1(z - t_1) \end{array} \right\} \subset \mathbb{A}^{n+4}_{y, z, s_1, s_2, t_1, \ldots, t_n}$$

which is isomorphic to the hypersurface

$$(8) \qquad \{y(s_1 - s_2) = 2t_1\} \subset \mathbb{A}^{n+3}_{y, s_1, s_2, t_1, \ldots, t_n}.$$

The map to $\mathfrak{h}_\Delta$ is still given by projection to the t_i coordinates. The equation in (8), together with similar equations for the other affine pieces, show that $\mathcal{Y} \times_{\bar{\mathcal{Y}}_i, r_i} \mathcal{Y}$ is irreducible, and the map to $\mathfrak{h}_\Delta$ has reduced fibers. This concludes the proof for the case when $i = I$ is a single node of Δ. The other cases reduce to this, since locally the morphism $Y \to \bar{Y}_I$ contracts a union of disjoint rational curves to ordinary double points. $\square$

Proof of Theorem 4.1. By Proposition 3.2, it is enough to show that the fiberwise restricted kernels

$$U_{i,s} = \mathbf{L}y_s^* U_i \in D^b(X_s \times X_{\rho_i(s)})$$

are invertible, where $y_s : X_s \times X_{\rho_i(s)} \hookrightarrow \mathcal{X} \times_{\rho_i} \mathcal{X}$ is fiber inclusion. By Proposition 4.2, U_i is a flat family of structure sheaves over Q, and hence the derived restriction $\mathbf{L}y_s^* U_i$ is isomorphic to the ordinary restriction $y_s^* U_i$, which in turn is isomorphic to the structure sheaf $\mathcal{O}_{X_s \times_{\bar{X}_s} X_{\rho_i(s)}}$. The statement that this sheaf defines an invertible kernel is already contained in the literature. There are two cases.

Suppose first that $s \in Q \cap T_i$. Then by Proposition 2.9(iii), s is a fixed point of ρ_i, hence

$$U_{i,s} \cong \mathcal{O}_{X_s \times_{\bar{X}_{i,s}} X_s} \in D^b(X_s \times X_s).$$

On the other hand, the contraction $f_{i,s} : X_s \to \bar{X}_{i,s}$ contracts a single ruled surface $D_{i,s}$ inside X_s to a smooth curve $B_{i,s}$ (which is either B or $\tilde{B}$). There is an exact sequence of sheaves on $X_s \times X_s$

$$0 \to U_{i,s} \to \mathcal{O}_{\Delta_{X_s}} \oplus \mathcal{O}_{D_{i,s} \times_{B_{i,s}} D_{i,s}} \to \mathcal{O}_{\Delta_{D_{i,s}}} \to 0,$$

where Δ_{X_s} and $\Delta_{D_{i,s}}$ are respective diagonals in $X_s \times X_s$. Hence the kernel $U_{i,s}$ is isomorphic to the kernel

$$(9) \qquad \mathrm{Cone}\{\mathcal{O}_{\Delta_{X_s}} \oplus \mathcal{O}_{D_{i,s} \times_{B_{i,s}} D_{i,s}} \to \mathcal{O}_{\Delta_{D_{i,s}}}\} \in D^b(X_s \times X_s).$$

This kernel was introduced in [9], (4.31) and its invertibility proved in [10], Theorem 2.9 and Remark 2.12.

Next suppose that $s \in Q \cap (T \setminus T_i)$. Then by Proposition 2.11(iii), the birational map

$$\theta_{\rho_i, s} : X_s \dashrightarrow X_{\rho_i(s)}$$

is a flop of a disjoint union of smooth rational curves with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1,-1)$ or $\mathcal{O}_{\mathbb{P}^1}(0,-2)$. The kernel U_s is the structure sheaf of the graph of this flop. This kernel was shown to be invertible in [2], Theorem 3.6 and Remark. $\square$

Remark 4.3. I sketch an alternative proof of the invertibility of the kernel $U_{i,s}$, which avoids a case division and also throws some light on the origin of this kernel. The claim is that $U_{i,s}$ is the universal perverse coherent point sheaf on $X_s \times X_{r_i(s)}$ with respect to the contraction $f_{i,s} : X_s \to \overline{X}_{i,s}$, and hence it is invertible. In particular, the variety $X_{r_i(s)}$ is the fine moduli space of perverse point sheaves on X_s for the contraction $f_{i,s}$. Here I am using the terminology of [4]; the essential point is that $f_{i,s}$ has fibers of dimension at most one, so Bridgeland's theory applies. The proof of the claim is not very difficult given the machinery of [4].

For purposes of brevity I will denote the family over the open set $S \subset Q$ also by $\mathcal{X} \to S$, and restrict all contractions, the W_Ξ-action and the relative kernels (U_i, ρ_i) to this family without further notice. The properties spelled out in Propositions 2.9–2.11 continue to hold; the latter of course under the assumption that S contains a sufficiently general point of T.

4.2. The main results. The first main result of the paper is that the derived category of the threefold X and that of its deformation space carry an action of an Artin group.

Theorem 4.4. *Let $\mathcal{X} \to S$ be a finite-dimensional deformation space of the threefold X satisfying the conclusion of Theorem* 4.1. *Then there are homomorphisms*

$$(10) \qquad\qquad B_\Xi \to \operatorname{Auteq}(\mathcal{X}/S)$$

and

$$(11) \qquad\qquad B_\Xi \to \operatorname{Auteq}(X).$$

Proof. Define the map (10) by mapping the generators ρ_i of the Artin group B_Ξ to the element (U_i, ρ_i) of Theorem 4.1. Since W_Ξ fixes $0 \in S$, I can define (11) by restricting these kernels to the central fiber. The point is to prove that the braid relations of (1) defining B_Ξ are satisfied for these kernels. Since (derived) restriction commutes with kernel composition in smooth families, it is enough to show that (10) is a group homomorphism.

Take a pair of nodes (i, j) of the Dynkin diagram Ξ. Set

$$(V_l, \varphi_l) = \underbrace{(U_i, \rho_i) \circ (U_j, \rho_j) \circ \cdots}_{m_{ij}}$$

for the left hand side of the braid relation of (1) for the pair of nodes i, j and similarly (V_r, φ_r) for the right hand side. One part is easy: the automorphisms ρ_i and ρ_j of the base S satisfy the relations of the Coxeter group W_Ξ, and consequently also the braid relation; hence $\varphi_l = \varphi_r$ which I will denote simply by φ.

Next note that for $k = i, j$, the sheaf U_k is the structure sheaf of a subscheme of

$\mathscr{X} \times_{\rho_k} \mathscr{X}$, in other words there is a surjective morphism $\mathcal{O}_{\mathscr{X} \times_{\rho_k} \mathscr{X}} \to U_k$. By a repeated use of Lemma 3.3, this implies that there are induced arrows

$$(12) \qquad \begin{array}{ccc} & \mathcal{O}_{\mathscr{X} \times_\varphi \mathscr{X}} & \\ & \swarrow \qquad \searrow & \\ V_l & & V_r \end{array}$$

in $D^b(\mathscr{X} \times_\varphi \mathscr{X})$.

To continue, assume that $m_{ij} = 3$; this will only simplify notation, the other cases being identical. Let $\rho_{ij} = \rho_j \circ \rho_i$. Let also

$$\tilde{\mathscr{X}} = \mathscr{X} \times_{\rho_i} \left(\mathscr{X} \times_{\rho_j} \left(\mathscr{X} \times_{\rho_i} \mathscr{X} \right) \right)$$

with maps $p_{12} : \tilde{\mathscr{X}} \to \mathscr{X} \times_{\rho_i} \mathscr{X}$ etc. For $s \in S$, let

$$x_s : \tilde{X}_s = X_s \times X_{\rho_i(s)} \times X_{\rho_{ij}(s)} \times X_{\varphi(s)} \hookrightarrow \tilde{\mathscr{X}}$$

and

$$y_s : X_s \times X_{\varphi(s)} \hookrightarrow \mathcal{O}_{\mathscr{X} \times_\varphi \mathscr{X}}$$

be inclusion maps of fibers, with projection maps $p_{14s} : \tilde{X}_s \to X_s \times X_{\varphi(s)}$ etc. The derived restriction of diagram (12) is a diagram

$$(13) \qquad \begin{array}{ccc} & \mathcal{O}_{X_s \times X_{\varphi(s)}} & \\ & \swarrow \qquad \searrow & \\ V_{l,s} & & V_{r,s} \end{array}$$

in $D^b(X_s \times X_{\varphi(s)})$. Now compute:

$$(14) \qquad \begin{aligned} V_{l,s} &\cong \mathbf{L} y_s^* \left(\mathbf{R} p_{14*} \left(p_{34}^*(U_i) \otimes^{\mathbf{L}} p_{23}^*(U_j) \otimes^{\mathbf{L}} p_{12}^*(U_i) \right) \right) \\ &\cong \mathbf{R} p_{14s*} \left(\mathbf{L} x_s^* \left(p_{34}^*(U_i) \otimes^{\mathbf{L}} p_{23}^*(U_j) \otimes^{\mathbf{L}} p_{12}^*(U_i) \right) \right) \\ &\cong \mathbf{R} p_{14s*} \left(\mathcal{O}_{X_s \times X_{\rho_i(s)} \times X_{\rho_{ij}(s)} \times_{\bar{X}_{i,\rho_{ij}(s)}} X_{\varphi(s)}} \otimes^{\mathbf{L}} \mathcal{O}_{X_s \times X_{\rho_i(s)} \times_{\bar{X}_{j,\rho_i(s)}} X_{\rho_{ij}(s)} \times X_{\varphi(s)}} \right. \\ &\qquad \left. \otimes^{\mathbf{L}} \mathcal{O}_{X_s \times_{\bar{X}_{j,s}} X_{\rho_i(s)} \times X_{\rho_{ij}(s)} \times X_{\varphi(s)}} \right). \end{aligned}$$

The first isomorphism uses Lemma 3.1, the second follows from a slight generalization of [2], Lemma 1.3 to the quasiprojective case, and the last uses the flatness result Proposition 4.2. There is a similar computation for $V_{r,s}$.

I now distinguish two cases. First assume that $s \in S$ is sufficiently general. It is easy to see that the three subschemes of $\tilde{X}_s$ appearing in the last expression of (14) are transversal, so the (derived) tensor product of their intersections is isomorphic in $D^b(\tilde{X}_s)$ to the structure sheaf of their intersection

$$\tilde{C}_{l,s} = X_s \times_{\bar{X}_{i,s}} X_{\rho_i(s)} \times_{\bar{X}_{j,\rho_i(s)}} X_{\rho_{ij}(s)} \times_{\bar{X}_{i,\rho_{ij}(s)}} X_{\varphi(s)}$$

in $\tilde{X}_s$. To understand this intersection, consider the diagram

(15)
$$
\begin{array}{ccccccc}
X_s & \dashrightarrow & X_{\rho_i(s)} & \dashrightarrow & X_{\rho_{ij}(s)} & \dashrightarrow & X_{\varphi(s)} \\
& \searrow \quad \swarrow & & \searrow \quad \swarrow & & \searrow \quad \swarrow & \\
& \overline{X}_{i,s} & & \overline{X}_{j,\rho_i(s)} & & \overline{X}_{i,\rho_{ij}(s)}.
\end{array}
$$

The support of $\tilde{C}_{l,s}$ in $\tilde{X}_s$ is the set of quadruples $(p_1,\ldots,p_4) \in \tilde{X}_s$ such that p_i, p_{i+1} have the same images under appropriate arrows in the diagram (15).

Since s is sufficiently general, the variety X_s contains a disjoint union of $(-1,-1)$ curves indexed by positive roots. The chain of birational maps in (15) flops, consecutively, the disjoint rational curves on X_s indexed by the positive roots $\mu_i, \mu_i + \mu_j, \mu_j \in \Sigma_\Xi^+$. In other words, the composition of the three flops, the map $X_s \dashrightarrow X_{\varphi(s)}$, is the flop of the disjoint set of all these curves. Hence a quadruple $(p_1,\ldots,p_4)$ is determined by the pair (p_1,p_4), and in particular p_{14s} restricted to the reduced subscheme $\tilde{C}_{l,s}$ of $\tilde{X}_s$ is an isomorphism onto its image in $X_s \times X_{\varphi(s)}$.

On the other hand, the set of positive roots $\{\mu_i, \mu_i + \mu_j, \mu_j\}$ is exactly the set of all positive roots of the sub-root system of type A_2 of the root system of Ξ spanned by the nodes (i,j) (remember $m_{ij} = 3$). The reflection $\varphi \in W_\Xi$ maps exactly these roots to negative roots. So the map $X_s \dashrightarrow X_{\varphi(s)}$ factors as $X_s \to \overline{X}_{ij,s} \leftarrow X_{\varphi(s)}$ in the notation of Proposition 2.11(iii)(b). Hence the image of $\tilde{C}_{l,s}$ under p_{14s} is the reduced subscheme

$$
C_{l,s} = X_s \times_{\overline{X}_{ij,s}} X_{\varphi(s)} \hookrightarrow X_s \times X_{\varphi(s)}.
$$

So finally

$$
V_{l,s} \cong p_{14s*} \mathcal{O}_{\tilde{C}_{l,s}} \cong \mathcal{O}_{C_{l,s}}
$$

and the argument also shows that the map on the left hand side of diagram (13) is just the natural surjection $\mathcal{O}_{X_s \times X_{\varphi(s)}} \to \mathcal{O}_{C_{l,s}}$.

Repeating this argument also for $V_{r,s}$, I obtain that in this case the diagram (13) is isomorphic to the diagram

(16)
$$
\begin{array}{ccc}
& \mathcal{O}_{X_s \times X_{\varphi(s)}} & \\
& \swarrow \qquad \searrow & \\
\mathcal{O}_{X_s \times_{\overline{X}_{ij,s}} X_{\varphi(s)}} & \xrightarrow{\;\sim\;} & \mathcal{O}_{X_s \times_{\overline{X}_{ij,s}} X_{\varphi(s)}}
\end{array}
$$

of sheaves on $X_s \times X_{\varphi(s)}$, with the vertical arrows being surjective.

The other case is when $s \in S$ is not sufficiently general. Transversality of the subschemes in (14) still holds, hence the tensor product in the last expression of (14) is isomophic to the structure sheaf of a subscheme $\tilde{C}_{l,s}$ of $\tilde{X}_s$, a correspondence subscheme with respect to the diagram (15). It is not longer true that the horizontal birational maps in (15) modify X_s on a set of disjoint loci, but in any case p_{14s} restricted to $\tilde{C}_{l,s}$ factors as

$$
p_{14s}|_{\tilde{C}_{l,s}} : \tilde{C}_{l,s} \twoheadrightarrow C_{l,s} \hookrightarrow X_s \times X_{\varphi(s)}.
$$

The fibers of $p_{14s*}|_{\tilde{C}_{l,s}}$ are quadruples $(p_1, \ldots, p_4)$ mapping to a given pair $(p_1, p_4) \in X_s \times X_{\varphi(s)}$ and p_i, p_{i+1} mapping to the same image under the appropriate maps in (15). There are several possible configurations, depending on s and the exceptional loci; for example if $s \in T_i \cap T_j$ then one possibility is that p_1, $p_2 = p_3$ and p_4 all lie in the same fiber of $f_{i,s}$. However, it is easy to check that in all cases when the reduced fiber of $p_{14s*}|_{\tilde{C}_{l,s}}$ is not a point, it is isomorphic to a fiber of either $f_{i,s}$ or $f_{j,s}$, in other words to a projective line. Hence by Lemma 3.4,

$$V_{l,s} \cong \mathbf{R}p_{14s*}\mathcal{O}_{\tilde{C}_{l,s}} \cong \mathcal{O}_{C_{l,s}} \in D^b(X_s \times X_{\varphi(s)}).$$

Using the same reasoning also for $V_{r,s}$, the diagram (13) is isomorphic to a diagram of sheaves on $X_s \times X_{\varphi(s)}$

(17)
$$
\begin{array}{ccc}
 & \mathcal{O}_{X_s \times X_{\varphi(s)}} & \\
 & \swarrow \qquad \searrow & \\
\mathcal{O}_{C_{l,s}} & & \mathcal{O}_{C_{r,s}}
\end{array}
$$

with surjective arrows.

Diagrams (16) and (17) imply by [3], Lemma 4.3 that V_l and V_r are sheaves on $\mathscr{X} \times_\varphi \mathscr{X}$, flat over S. Moreover, since pullback is right exact, the arrows in diagram (12) are necessarily surjective maps of sheaves; in other words, V_l and V_r are structure sheaves of subschemes of $\mathscr{X} \times_\varphi \mathscr{X}$. If I further assume that

$(\star)$ $\mathscr{M}$ is a moving linear system on B, and the finite-dimensional family $e : \mathscr{X} \to S$ contains sufficiently general deformations of X,

then sufficiently general points form an open dense subset of S. Hence Lemma 3.5, together with (16), allows me to conclude that (12) can be extended to a diagram

$$
\begin{array}{ccc}
 & \mathcal{O}_{\mathscr{X} \times_\varphi \mathscr{X}} & \\
 & \swarrow \qquad \searrow & \\
V_l & \xrightarrow{\ \sim\ } & V_r.
\end{array}
$$

So composition of the relative kernels (U_i, ρ_i) and (U_j, ρ_j) in the two different ways gives isomorphic relative kernels; hence, assuming $(\star)$, the braid relation holds up to isomorphism.

To remove assumption $(\star)$, let $B = \bigcup B_\beta$ be a finite decomposition into quasiprojective (e.g. affine) curves so that the restriction $\mathscr{M}_\beta = \mathscr{M}|_{B_\beta}$ moves on B_β. For every β, let $\pi_\beta : X_\beta \to B_\beta$ be the restriction of π over B_β, and let $e_\beta : \mathscr{X}_\beta \to H^0(B_\beta, \mathscr{M}_\beta)$ be the family of deformations of X_β constructed in Proposition 2.9. There is a W_Ξ-equivariant natural injection

$$n_\beta : H^0(B, \mathscr{M}) \hookrightarrow H^0(B_\beta, \mathscr{M}_\beta).$$

Let S_β be a W_Ξ-invariant finite-dimensional subspace of $H^0(B_\beta, \mathscr{M}_\beta)$ containing both the

image of S under the injection n_β and a sufficiently general point of $H^0(B_\beta, \mathcal{M}_\beta)$. There is an induced family $e_\beta : \mathcal{X}_\beta \to S_\beta$ with central fiber X_β.

By construction, the family $e_\beta : \mathcal{X}_\beta \to S_\beta$ satisfies assumption $(\star)$ for each β. On the other hand, the natural injection $n_\beta|_S : S \hookrightarrow S_\beta$ is W_Ξ-equivariant by construction, so if I restrict to families $e_{\beta,S} : \mathcal{X}_{\beta,S} \to S$ then the above discussion applies to these families. In particular, the restriction of diagram (12) to $\mathcal{X}_{\beta,S} \times_\varphi \mathcal{X}_{\beta,S}$ can be extended to a diagram of sheaves

$$
(18) \qquad
\begin{array}{ccc}
 & \mathcal{O}_{\mathcal{X}_{\beta,S} \times_\varphi \mathcal{X}_{\beta,S}} & \\
\swarrow & & \searrow \\
V_{l,\beta,S} & \xrightarrow{\ \sim\ } & V_{r,\beta,S}
\end{array}
$$

with the vertical maps being surjective. The horizontal isomorphisms in (18) are compatible with surjections from a fixed sheaf, so they can be glued to an isomorphism

$$
(19) \qquad
\begin{array}{ccc}
 & \mathcal{O}_{\mathcal{X} \times_\varphi \mathcal{X}} & \\
\swarrow & & \searrow \\
V_l & \xrightarrow{\ \sim\ } & V_r
\end{array}
$$

of sheaves on $\mathcal{X} \times_\varphi \mathcal{X}$ extending (12). Hence the braid relation holds between (U_i, ρ_i) and (U_j, ρ_j) with no extra assumption.

The proofs in the cases $m_{ij} = 4, 6$ are, up to writing out longer expressions, identical. The proof for $m_{ij} = 2$ is in fact easier, since the exceptional loci of $f_{i,s}$ and $f_{j,s}$ are disjoint for all s, hence there is no need for a case distinction and assumption $(\star)$. These cases correspond to sub-digrams of Ξ type B_2, G_2 and $A_1 \times A_1$ respectively. The proof of Theorem 4.4 is complete. $\square$

The next result shows that in certain cases, the Artin group action on the derived category of X is faithful.

Theorem 4.5. *Assume that the diagram Ξ describing the configuration of exceptional surfaces in X is of type A_n or C_n. Then the map $B_\Xi \to \mathrm{Auteq}(X)$ is injective.*

Proof. Take any point $b \in B$ and let $j_b : Y_b \to X$ be the fiber of $\pi : X \to B$ over b. Rational exceptional curves $E_{j,b}$ in the surface Y_b are indexed by nodes j of the simply laced Dynkin diagram Δ lying over Ξ.

Let $k_b : Y_b \times Y_b \to X \times X$; for nodes i of Ξ, let $U_{i,b} = \mathbf{L}k_b^*(U_{i,0})$ be the restriction to $Y_b \times Y_b$ of the kernel $U_{i,0}$ (note $X = X_0$ for $0 \in S$). By standard arguments, there is a commutative diagram of functors

$$
\begin{array}{ccc}
D^b(X) & \xrightarrow{\ \mathbf{L}k_b^*\ } & D^b(Y_b) \\
\downarrow{\scriptstyle \Psi^{U_{i,0}}} & & \downarrow{\scriptstyle \Psi^{U_{i,b}}} \\
D^b(X) & \xrightarrow{\ \mathbf{L}k_b^*\ } & D^b(Y_b)
\end{array}
$$

and maps

$$B_\Xi \to \mathrm{Auteq}(X) \to \mathrm{Auteq}(Y_b)$$

where the second arrow is restriction to the fiber over b.

Suppose now that Ξ is of type A_n. Then the following holds (for a proof, see below):

Lemma 4.6. *The kernel $U_{i,b} \in D^b(Y_b \times Y_b)$ is isomorphic to the kernel defining the (inverse) twist functor $T'_{\mathscr{E}_i}$ of [16] for the sheaf $\mathscr{E}_i = \mathscr{O}_{E_{i,b}}(-1)$, where $E_{i,b} \subset Y_b$ is the exceptional rational curve corresponding to the node i of Δ.*

The map $B_\Xi \to \mathrm{Auteq}(Y_b)$ defined by mapping the Artin group generators to the twist functors $T_{\mathscr{E}_i}$ is injective by [16], Theorem 2.18. Hence the map $B_\Xi \to \mathrm{Auteq}(X)$ must be injective as well.

If Ξ is of type C_n, then it has two kinds of nodes: one representing a single node of the simply laced diagram Δ, and the other representing an orbit $\{i_1, i_2\}$ of nodes. For the first type of node, Lemma 4.6 continues to hold; for the second, it gets replaced by

Lemma 4.7. *The kernel $U_{i,b} \in D^b(Y_b \times Y_b)$ is the composite of the commuting kernels defining the (inverse) twist functors $T'_{\mathscr{E}_{i_1}}$ and $T'_{\mathscr{E}_{i_2}}$.*

Hence, recalling the proof of Lemma 1.1, in this case there is a commutative diagram

$$
\begin{array}{ccc}
B_\Xi & \longrightarrow & \mathrm{Auteq}(X) \\
\downarrow & & \downarrow \\
B_\Delta & \longrightarrow & \mathrm{Auteq}(Y_b).
\end{array}
$$

The bottom horizontal arrow is injective by [16], Theorem 2.18 again; the left hand vertical arrow is injective by Lemma 1.1. Hence the composite is injective; so $B_\Xi \to \mathrm{Auteq}(X)$ must be injective as well. $\square$

Proof of Lemmas 4.6 and 4.7. Suppose first that A is trivial. Since all sheaves appearing in (9) are flat with respect to the projection $X \times X \to B$, the kernel $U_{i,b}$ on Y is isomorphic to the kernel

$$\mathrm{Cone}\{\mathscr{O}_{\Delta_{Y_b}} \oplus \mathscr{O}_{E_{i,b} \times E_{i,b}} \to \mathscr{O}_{\Delta_{E_{i,b}}}\} \in D^b(Y_b \times Y_b)$$

where recall $E_{i,b} \subset Y_b$ is one of the exceptional rational curves. Let $y : E_{i,b} \hookrightarrow Y_b$ denote the inclusion of the rational curve in the surface, and let $x : E_{i,b} \times E_{i,b} \hookrightarrow Y_b \times Y_b$ be the induced inclusion. Then up to isomorphism in $D^b(Y_b \times Y_b)$,

$$U_{i,b} \cong \mathrm{Cone}\{\mathscr{O}_{\Delta_{Y_b}} \to (x_*\mathscr{O}_{E_{i,b} \times E_{i,b}} \to x_*\mathscr{O}_{\Delta_{E_{i,b}}})\}$$

$$\cong \mathrm{Cone}\{\mathscr{O}_{\Delta_{Y_b}} \to x_*\big(\mathscr{O}_{E_{i,b} \times E_{i,b}}(-1,-1)\big)\}$$

$$\cong \mathrm{Cone}\{\mathscr{O}_{\Delta_{Y_b}} \to x_*\big(q_2^*\big(\mathbf{R}\mathscr{H}om_{\mathscr{O}_{E_{i,b}}}\big(\mathscr{O}_{E_{i,b}}(-1), \omega_{E_{i,b}}\big)\big) \otimes^{\mathbf{L}} q_1^*\big(\mathscr{O}_{E_{i,b}}(-1)\big)\big)\}$$

where $q_1, q_2 : E_{i,b} \times E_{i,b} \to E_{i,b}$ are projections. It is easiest to conclude now using results of [10]. Comparing the above expression with [10], (2.7) and (2.23) shows that $U_{i,b}$ is isomorphic to the invertible kernel on Y_b defined by the relatively spherical sheaf $\mathcal{O}_{E_{i,b}}(-1)$ with respect to the diagram

$$E_{i,b} \overset{y}{\hookrightarrow} Y_b$$
$$\downarrow$$
$$\star$$

a kernel which by [10], Example 4.1 is isomorphic to the kernel which gives rise to the (inverse) twist functor of [16] defined by the spherical sheaf $y_* \mathcal{O}_{E_{i,b}}(-1)$. This proves Lemma 4.6. Lemma 4.7 follows also on noting that in that case the contraction f_i restricts to Y_b as the contraction of two disjoint exceptional curves. $\quad \square$

4.3. Projective examples. Let $\overline{X}$ be a projective threefold with a curve of singularities

$$B = \operatorname{Sing}(X) \hookrightarrow \overline{X},$$

such that along the curve, $\overline{X}$ has compound du Val singularities of uniform ADE type. The iterated blowup of the singular locus $f : X \to \overline{X}$ is a resolution of singularities, cf. [15], and the exceptional locus consists of a set of geometrically ruled surfaces $\{\pi_j : D_j \to B_j\}$ intersecting in one of the configurations $\Xi = \Delta/A$ described in Section 2.4.

Theorem 4.8. *Assume as usual that $(\Delta, A) \neq (A_{2n}, \mathbb{Z}/2)$. The derived category $D^b(X)$ carries an action of the Artin group B_Ξ. In case Ξ is of type A_n or C_n, this action is faithful.*

Proof. For j a node of the diagram Ξ, define a kernel U_j on X by

$$U_j = \operatorname{Cone}\{\mathcal{O}_{\Delta_X} \oplus \mathcal{O}_{D_j \times_{B_j} D_j} \to \mathcal{O}_{\Delta_{D_j}} \operatorname{Cone}\} \in D^b(X \times X);$$

this is just the kernel of [9], (4.31), proved to be invertible in [10]. The point is that this definition makes sense whether or not there is a contraction morphism on X contracting D_j alone. Define the map

$$B_\Xi \to \operatorname{Auteq}(X)$$

by mapping the generator R_j of B_Ξ to the kernel U_j on X. The issue is again to prove the braid relations. As before, take a pair of nodes (i, j) of Ξ and let $V_l, V_r \in D^b(X \times X)$ be the composite kernels on the two sides of the braid relation for the pair (i, j). Note that the interesting part of the computation of all these kernels takes place in an étale neighbourhood of the exceptional set; away from such a neighbourhood, V_l and V_r are obviously isomorphic to the structure sheaf of the diagonal. There is an étale open covering of a neighbourhood of $B \subset \overline{X}$, such that on the inverse image of this covering on X the restrictions of V_l and V_r are isomorphic by the proof of Theorem 4.4, these isomorphisms being compatible on intersections. Hence by descent, there is an isomorphism $V_l \cong V_r$ on $X \times X$, and so the braid relations hold up to isomorphism.

To prove faithfulness, argue as in the proof of Theorem 4.5: take a quasiprojective surface $\overline{Y}_s \subset \overline{X}$ intersecting the singular locus $B \subset \overline{X}$ transversally at $p \in B$ and in no other points, let $Y_s \subset X$ be its resolution, restrict the kernels U_i to Y_s using Lemmas 4.6 and 4.7 and appeal to the faithfulness result of [16]. $\square$

Examples of varieties $\overline{X}$ with a curve of singularities of uniform type A_n can be found among hypersurfaces or complete intersections in weighted projectice spaces; compare for example [12]. The resolution X is then embedded in a (partial) resolution of the ambient space, typically with n distinct divisors over the relevant singular locus; hence the configuration in X is still of type A_n. Such varieties can be found and in low codimension classified using the graded ring method pioneered by Reid; see the (from the present point of view not very interesting) A_1 case in [19] and the general case in [5]. Examples of type $(A_n, \mathbb{Z}/2)$ can be constructed as quotients; see [21], Examples 4.3 for an explicit example. In favourable cases, the local deformations described in Proposition 2.11 are realized as actual projective deformations. In such cases, the action of the Artin group on $D^b(X)$ can be extended to an action by relative equivalences over its local universal family, in an analogous way to the statement of Theorem 4.4. I leave it to the reader to formulate the precise statement.

Remark 4.9.   The action of the Artin group B_Ξ on the derived category of the three-fold X gives rise to actions on even and odd cohomology, using the Chern class map. In the case when X has trivial canonical bundle, $H^{2,1}(X) \cong H^1(X, \Theta_X)$ is a direct summand of odd cohomology, and it is preserved by the action. Hence the braid group acts on the tangent space to the deformation space, and it is easy to see that this action factors through the reflection group W_Ξ. The action on even cohomology can in turn be restricted to the Picard group to get an action of W_Ξ there. Some of these actions were known before; e.g. [23] discusses the case of elliptic ruled surfaces, whereas [12] has a symmetric group action in the case of type A. The action of the Artin group on the derived category shows the common origin of all these actions.

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