Combinatorics

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LMS Prospects in Mathematics 2014

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What I'll talk about...

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2. combinatorics elsewhere

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3. pure mathematics at Bristol
4. practicalities and prerequisites

Combinatorics at Bristol

Our group:

- Dr Misha Rudnev (m.rudnev@bristol.ac.uk)
- Dr Julia Wolf (julia.wolf@bristol.ac.uk)
- Prof Trevor Wooley (trevor.wooley@bristol.ac.uk)
- several postdocs and PhD students
- weekly seminar and reading group

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The problems we work on are often easy to state but are tackled by a wide range of techniques.

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- Techniques: hands-on counting, probability, discrete Fourier analysis, linear algebra
- Connections: analytic number theory, group theory, ergodic theory, theoretical computer science

The sum-product phenomenon

Suppose A is a finite subset of $\mathbb{R}$. Let

$$A + A := \{a + a' : a, a' \in A\}$$

and

$$A \cdot A := \{a \cdot a' : a, a' \in A\}.$$

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Conjecture (Erdős-Szemerédi, 1983)

Let $A \subseteq \mathbb{R}$ be a finite set. Then for any $\epsilon > 0$,

$$\max(|A + A|, |A \cdot A|) \gg |A|^{2-\epsilon}.$$

Incidence geometry

Given a set P of n points and a set L of m lines in the plane, how many incidences between points and lines can there be?

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Theorem (Szemerédi-Trotter, 1983)

Let P be a finite set of points in $\mathbb{R}^2$, and let L be a finite set of lines. Then the number of incidences between P and L, i.e. the number of pairs $(p, \ell) \in P \times L$ such that $p \in \ell$ is

$$I(P, L) \leq 4|L|^{2/3}|P|^{2/3} + 4|P| + |L|.$$

A sum-product theorem

Theorem (Elekes, 1997)

Let $A \subseteq \mathbb{R}$ *be a finite set. Then* $|A+A|^2|A\cdot A|^2 \gg |A|^5$, *and in particular*

$$\max(|A+A|, |A\cdot A|) \gg |A|^{5/4}.$$

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Deduction: Consider $P = (A+A) \times (A \cdot A)$, together with $L = \{y = a(x-b) : a, b \in A\}$.

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$$|P|^{2/3}|L|^{2/3}+|P|+|L| = |A+A|^{2/3}|A\cdot A|^{2/3}|A|^{4/3}+|A+A||A\cdot A|+|A|^2.$$

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The best known exponent is $4/3$, due to Solymosi.

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For $|A| \gg \sqrt{p}$ this question is completely resolved.

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Theorem (Roche-Newton, Rudnev and Shkredov, 2014)

Let p be a prime and let $A \subseteq \mathbb{F}_p$. Suppose $|A| < p^{5/8}$. Then

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The proof is based on the following incidence theorem of Rudnev.

Theorem (Rudnev, 2014)

Let p be a prime, let P be a set of points and Π be a set of planes in $\mathbb{P}^3\mathbb{F}_p$. Let s and σ be the maximum number of points and planes incident to a single line, respectively. Suppose $|P| \geq |\Pi|$ and $|\Pi|^{3/4}|P|^{-1/4} \leq cp$ for some absolute constant c. Then

$$I(P,\Pi) \ll (|P||\Pi|)^{3/4} + s|\Pi|^{3/2}|P|^{-1/2} + \sigma|P|^{1/2}|\Pi|^{1/2}.$$

Further reading

- Zeev Dvir. *Incidence theorems and their applications*, 2010.
- Oliver Roche-Newton, Misha Rudnev and Ilya Shkredov. *New sum-product type estimates over finite fields*, 2014.
- Terence Tao and Van Vu. *Additive combinatorics*, Cambridge University Press, 2006.

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Other pure research themes at Bristol

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- dynamical systems
- probability
- group theory
- interactions with computer science, quantum information

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