Research in (Complex) Dynamics

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A typical motivating example in dynamics

$$\frac{d^2y}{dt^2} = \sin y$$

can be turned into an equation of first order by writing $\frac{dy}{dt} = p$ and then the system of equations is

$$\frac{dy}{dt} = p, \qquad \frac{dp}{dt} = \sin y$$

giving

$$p\frac{dp}{dt} = \sin y \frac{dy}{dt}$$

and hence

$$p^2 = \left(\frac{dy}{dt}\right)^2 = -2\cos y + C$$

which can be solved explicitly (with some difficulty) but the important point is that the solutions (y, p) for $2 + C$ near 0 are periodic, one solution through each point in the plane.

If $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $\varphi_t(y_0, p_0) = (p(t), y(t))$ for the solution with $(y(0), p(0)) = (y_0, p_0)$, then

$$\varphi_{t+s} = \varphi_t \circ \varphi_s \quad \forall t,\ s \in \mathbb{R}.$$

Such a group $\{\varphi_t : t \in \mathbb{R}\}$ of homeomorphisms of $\mathbb{R}^2$ is called a flow on $\mathbb{R}^2$.

A motivating example in Mathematical Biology (from Rachel Bearon)

The swimming of an organism is determined by the directions of its two flagella, which are given by three-dimensional unit vectors $\mathbf{p}$ and $\mathbf{n}$ respectively. The ODE's determining the motion are

$$\frac{d\mathbf{p}}{dt} = \Omega \wedge \mathbf{p}, \tag{1}$$

$$\frac{d\mathbf{n}}{dt} = \Omega \wedge \mathbf{n}, \tag{2}$$

where

$$\Omega = G\mathbf{p} \wedge \mathbf{k} + R\mathbf{n} + \frac{1}{2}\omega\mathbf{j}, \tag{3}$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ is the standard orthonormal basis, with $\mathbf{k}$ pointing in the upwards vertical direction, and ω, G and R are positive constants. It is immediate from (1) and (2) that

$$\mathbf{p} \cdot \frac{d\mathbf{n}}{dt} + \mathbf{n} \cdot \frac{\mathbf{dp}}{\mathbf{dt}} = \mathbf{0} \tag{4}$$

Therefore for some constant $\gamma \in [0, \pi]$, along any solution of (1) and (2),

$$\mathbf{p} \cdot \mathbf{n} = \cos\gamma. \tag{5}$$

If $\gamma = 0$ or π then this is a 2-sphere. If $\gamma \in (0, \pi)$ then $(\mathbf{p}, \mathbf{n})$ lie in $\mathbb{RP}^3$.

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It is unknown whether it is possible to have a flow on $\mathbb{RP}^3$ for which every orbit is dense.

Cross-sections and discrete dynamical systems

Some information can be found about this flow by using transversal cross-sections.

A *cross section* to a flow is a submanifold which is transversal to the flow. If one is lucky, one can find transversals which are forward invariant under the flow. This does happen for some parameter values in this example, where the cross section is a topological disc and the return map maps the disc into itself. The return map therefore has a fixed point and the flow has a periodic orbit.

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Complex dynamics is about iteration of holomorphic maps - on the Riemann sphere (rational maps) or on the plane (entire functions, or transcendental meromorphic functions).

Stable dynamical systems

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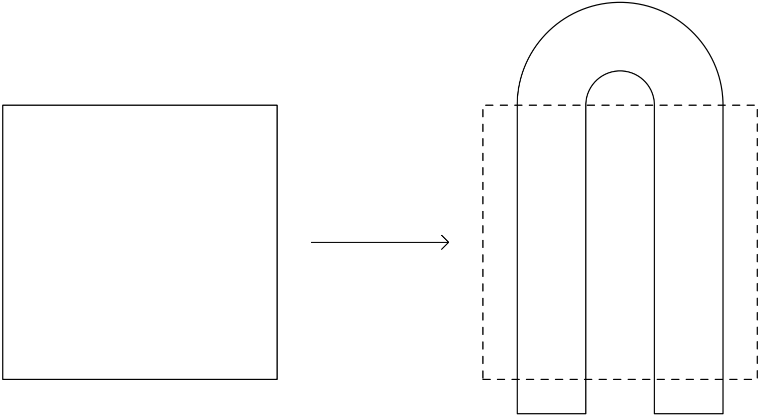

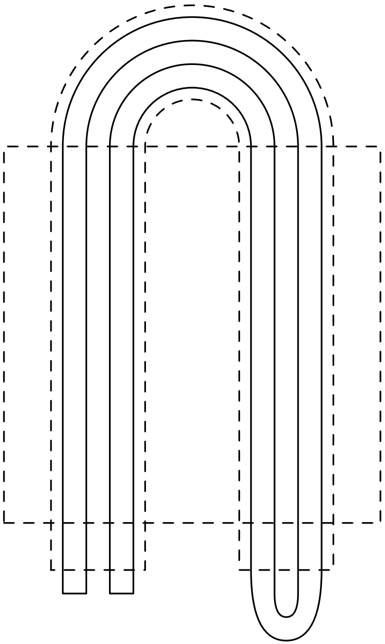

Because we never know exactly what model is the right one in applications, it is very important to know if all nearby maps (or flows) behave in the same way, in the long term. Maps for which nearby maps do have the same behaviour in the long term are called stable

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There was hope, for a while, that stable maps are dense. This turned out not to be true in some classes of maps (C^1 diffeomorphisms but true in some others (quadratic polynomials on the interval, and some polynomials of higher degree, C^r unimodal maps) and is thought to be true in many classes of holomorphic maps but this is still unknown.

Holomorphic maps provide, in many ways, a good microcosm for study of holomorphic dynamics in general. It is easy to provide a variety of examples. This is largely due to the fact that dynamical behaviour of all points is very strongly influenced by that of the critical points. The way in which this happens was largely worked out nearly a hundred years ago by Fatou and Julia.

A good example of this is provided by the Mandelbrot set for quadratic polynomials. In many cases, conjectured to be dense, the orbit of the critical point completely determines the topology of the Julia set and the dynamics of the polynomial on the complex plane. There are a number of programmes available illustrating this. See for example http://math.bu.edu/DYSYS/applets/Quadr.html

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All positions are for 3.5 years, with an annual stipend (initially) of around 14K. The closing date for applications is 15 February and we will be interviewing shortly afterwards. Skype interview is likely to be an option.