Rogers-Ramanujan type identities, knots and UCD

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Outline

- Motivation
- Knots, the GLZ conjectures and our work
- Opportunities at UCD

Partitions

- A *partition* of a natural number n is a non-increasing sequence of positive integers whose sum is n.
- There are 5 partitions of 4, namely

$$4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1.$$

Theorem (Rogers (1894), Ramanujan (1913))

The number of partitions of n such that all parts differ by at least 2 is equal to the number of partitions of n such that all parts are congruent to 1 or 4 modulo 5.

q-series

- The analytic version is

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}$$

where

$$(a)_n = (a;q)_n = \prod_{k=1}^{n} (1 - aq^{k-1})$$

valid for $n \in \mathbb{N} \cup \{\infty\}$.

- Three comments ...

- The RR identities have been generalized in several ways (Andrews, Gordon, Warnaar, Lepowsky, ...).
- On the other hand ...

Theorem (Zagier, 2007)

The q-series

$$\sum_{n \geq 0} \frac{q^{an^2+bn+c}}{(q)_n}$$

is modular if and only if $(a, b, c) = (1, 0, -1/60)$, $(1, 1, 11/60)$, $(1/2, 0, -1/48)$, $(1/2, 1/2, 1/24)$, $(1/2, -1/2, 1/24)$, $(1/4, 0, -1/40)$, $(1/4, 1/2, 1/40)$.

q-multisums

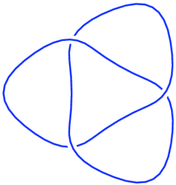
- In 1984, Andrews used the "Bailey machinery" to prove

$$\sum_{n_{k-1}\geq n_{k-2}\geq\cdots\geq n_1\geq 0} \frac{q^{n_{k-1}^2+n_{k-2}^2+\cdots+n_1^2}}{(q)_{n_{k-1}-n_{k-2}}\cdots(q)_{n_2-n_1}(q)_{n_1}} = \frac{(q^k;q^{2k+1})_\infty(q^{k+1};q^{2k+1})_\infty(q^{2k+1};q^{2k+1})_\infty}{(q)_\infty}.$$

- q-multisums also occur in many other areas (combinatorics, statistical mechanics, Lie algebras, group theory, knot theory ...).

Knots

- A *knot* K is an embedding of a circle in $\mathbb{R}^3$. For example, the trefoil knot 3_1 is given by

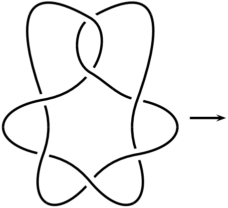


- We consider *alternating* knots and their recent connection to q-series.

GLZ conjectures

- (Garoufalidis and Lê, 2011): alternating knot $K \rightsquigarrow q$-multisum $\Phi_K(q)$.

- For example, for $K = 7_2$,



$$\Phi_{7_2}(q) = (q)_\infty^7 \sum_{a,b,c,d,e,f,g\geq 0} \frac{q^{3a^2+2a+b^2+bc+ac+ad+ae+af+ag+cd}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_{b+c}}$$

$$\times \frac{q^{de+ef+fg+c+d+e+f+g}}{(q)_{a+c}(q)_{a+d}(q)_{a+e}(q)_{a+f}(q)_{a+g}}$$

$$\stackrel{?}{=} h_6 \quad (\text{“false theta function”})$$

GLZ conjectures

- They (along with Zagier) conjectured 43 Rogers-Ramanujan type identities relating $\Phi_K(q)$ to products of h_b where

$$h_b = h_b(q) = \sum_{n \in \mathbb{Z}} \epsilon_b(n) q^{\frac{bn(n+1)}{2} - n}$$

and

$$\epsilon_b(n) = \begin{cases} (-1)^n & \text{if } b \text{ is odd,} \\ 1 & \text{if } b \text{ is even and } n \geq 0, \\ -1 & \text{if } b \text{ is even and } n < 0. \end{cases}$$

- One can show that $h_1(q) = 0$, $h_2(q) = 1$ and $h_3(q) = (q)_\infty$.

GLZ conjectures

K	$\Phi_K(q)$	$\Phi_{-K}(q)$	K	$\Phi_K(q)$	$\Phi_{-K}(q)$
3_1	h_3	1	7_5	$h_3 h_4$	h_4
4_1	h_3	h_3	7_6	$h_3 h_4$	h_3^2
5_1	h_5	1	7_7	h_3^3	h_3^2
5_2	h_4	h_3	8_1	h_7	h_3
6_1	h_5	h_3	8_2	$h_3 h_6$	h_3
6_2	$h_3 h_4$	h_3	8_3	h_5	h_5
6_3	h_3^2	h_3^2	8_4	h_3	$h_4 h_5$
7_1	h_7	1	8_5	?	h_3
7_2	h_6	h_3	$K_p,\ p > 0$	h_{2p}	h_3
7_3	h_5	h_4	$K_p,\ p < 0$	$h_{2\|p\|+1}$	h_3
7_4	h_4^2	h_3	$T(2,p),\ p > 0$	h_{2p+1}	1

Theorem (–, Keilthy, 2014)

All of the above identities are true.

Algebra and Number Theory at UCD

- **Permanent members:**

  Vincent Astier (Quadratic forms, Model theory),
  Eimear Byrne (Coding theory),
  Kevin Hutchinson (K-theory, Homology),
  Gary McGuire (Elliptic curves, coding and cryptography),
  – (Number theory, combinatorics),
  Helena Smigoc (Matrix theory),
  Masha Vlasenko (Number theory, Mirror symmetry).

- **Weekly seminar:** in 2013-2014, 20 talks including international visitors (Essen, Berkeley, Tokyo, Max Planck, Paris, Baton Rouge, Bordeaux).

- **International Conferences/Workshops:**

  Workshop on Coding and Cryptography (2007, 2008, 2009, 2010, 2011),
  Workshop on Elliptic Curves and Cryptography (2007)
  Finite Fields 9 (2009),
  The Lewisfest (2009),
  IEEE Information Theory Workshop (2010),
  Prospects in q-series and modular forms (2010),
  Representations and Finite Fields (2013),
  Automorphic Forms Workshop (2013).

Opportunities at UCD

- "Research Demonstratorships" (4 years, 15K per year, 2K for travel funds, 50% fees covered), deadline is April 1, 2015. See

  `http://www.ucd.ie/mathsciences/research`

  or

  `http://www.ucd.ie/mathsciences/graduatestudents`

- "IRC Postgraduate Scholarship Scheme" (4 years, 16K per year, 2,250 for travel funds (per year), all fees paid), deadline is February 11, 2015. See

  `http://research.ie/funding/postgraduate-funding`

- Questions? Contact

  `simon.williams@ucd.ie`

Thank you!

Comments

- There is no (known) conjectural evaluation for the 8_5 knot. In this case, we have

$$\Phi_{8_5}(q) := (q)_\infty^8 \sum_{\substack{a,b,c,d,e,f,g,h\geq 0\\ a+f\geq b}} (-1)^{b+f} \frac{q^{2a+3a^2-\frac{b}{2}-2ab+\frac{3b^2}{2}+c+ac+d+ad}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_h}$$

$$\times \frac{q^{cd+e+ae+de+\frac{3f}{2}+4af-4bf+ef+\frac{5f^2}{2}+g+ag-bg+eg+fg+h+ah-bh+fh+gh}}{(q)_{a+c}(q)_{a+d}(q)_{a+e}(q)_{a-b+f}(q)_{a-b+e+f}(q)_{a-b+f+g}(q)_{a-b+f+h}}$$

$$= (q)_\infty^2 \sum_{a,b\geq 0} \frac{q^{a^2+a+b^2+b}(q)_{a+b}}{(q)_a^2(q)_b^2} \quad \rightsquigarrow \quad ?$$

- (Zagier's question) Is the list of 43 conjectural identities complete?

- Generalizations? RR-type identities for HOMFLY polynomials or superpolynomials?