ROBUST REPROJECTION METHODS FOR THE RESOLUTION OF THE GIBBS PHENOMENON *

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To David Gottlieb in honor of his 60th birthday for providing much wisdom, guidance, and encouragement over the years.

Abstract. The classical Gibbs' phenomenon exhibited by global Fourier projections and interpolants can be resolved in smooth regions by reprojecting in a truncated Gegenbauer series, achieving high resolution recovery of the function *up to* the point of discontinuity, [16]. Unfortunately, due to the poor conditioning of the Gegenbauer polynomials, the method suffers both from numerical round-off error and the Runge phenomenon. In some cases the method fails to converge, [3] and [12]. Following the work in [17], a more general framework for reprojection methods is introduced here. From this insight we propose an additional requirement on the reprojection basis which ameliorates the limitations of the Gegenbauer reconstruction. The new robust Gibbs complementary basis yields a reliable exponentially accurate resolution of the Gibbs phenomenon up to the discontinuities.

Key words. Fourier series, robust Gibbs complementary, round-off error, weight functions, Freud polynomials, Gegenbauer post-processing

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1. Introduction.

Spectral methods using either orthogonal polynomial or Fourier series expansions yield highly accurate approximations for smooth (and periodic in the Fourier case) functions. It is in part for this reason that they have become popular for such applications as partial differential equations as well as signal and image processing. However, when functions are only piecewise smooth (and/or non-periodic in the Fourier case), the accuracy of spectral methods is reduced to first order away from discontinuities, and spurious $\mathcal{O}(1)$ oscillations form as the jump discontinuities are approached. This behavior is the well known Gibbs' phenomenon, and its removal has been the subject of many investigations. *Of course it would be preferable to appropriate a piecewise smooth function using spectrally accurate bases over regions of smoothness, and in doing so avoid the Gibbs' phenomenon. However, in many applications (e.g. tomography) the only information available is the set of global (pseudo-)spectral coefficients, which comes from a region that includes discontinuities.*

Some techniques which resolve the Gibbs' phenomenon *up to* the discontinuity include: subtracting off discontinuities to increase smoothness [7], re-expansions in terms of singular Padé approximations [6], analytic continuation methods [5], inverse methods [21], and Gegenbauer post-processing developed in [18] and expanded in a series of papers (see [16] for references). These different techniques offer different advantages, and no one method is inherently superior. However, so far only the Gegenbauer post-processing method avoids solving a (frequently ill-conditioned) linear system, and as as result is often computationally more efficient.

The general theory for reprojection methods such as Gegenbauer post-processing can be found in [17], where the requirements for a reprojection basis are given as

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follows:

1. For a function analytic on the interval $[-1, 1]$, the function's expansion in the orthogonal reprojection basis is exponentially convergent.
2. The projection of the high modes in the original basis on the low modes in the new basis is exponentially small.

These two requirements define the new projection basis as a *Gibbs complement*, which will be discussed further in §2 and §3.

The first requirement is easily motivated and accomplished. Clearly orthogonal polynomials, such as Chebyshev or Legendre polynomials, are well suited for approximating analytic functions in an interval (via linear transformation to $[-1, 1]$), as they yield exponential convergence for analytic functions. The second requirement measures the error due to having limited information about the original function, i.e., its truncated series approximation. Although easily understood, it is this requirement that causes added complications. As was discovered originally in [18], under certain conditions the Gegenbauer polynomials satisfy this second requirement, rendering it a suitable basis for reprojection. However, as indicated in the more general theory given in [17], the Gegenbauer polynomials serve only as an example of a Gibbs complementary basis. The purpose of this paper is to study other possible Gibbs complements that offer advantages over the traditional Gegenbauer reconstruction method. In particular we seek to address the most notable difficulties in the Gegenbauer reconstruction method, specifically its extrapalatory nature which causes round-off error and the generalized Runge phenomenon, as termed in [3]. This is especially problematic when the proximity of an off-axis singularity is less than 1. Although some compensating techniques have been successfully applied, [12] and [13], the consequences are that for some functions spectral accuracy is compromised, and in the most severe cases, the method will fail to converge when implemented numerically.

Hence in this paper we reject the use of the Gegenbauer polynomials as a reprojection basis and develop an alternative Gibbs complementary basis that is less susceptible both to round-off error as well as to the Runge phenomenon. Moreover, we introduce a generalization of the theory developed in [17] which makes the selection of a Gibbs' complement basis more transparent, and as a result allows for an improved understanding as to how to achieve the desirable properties of a reprojection basis. This insight suggests the following additional requirement for a Gibbs complement:

3. As the order of the original projection N increases, the weight function of the reprojection basis converges to a weight function whose associated orthogonal polynomial family satisfies the first requirement of a Gibbs complement.

We refer to a reprojection basis that satisfies requirements 1 through 3 as a **robust Gibbs complement**. The fundamental difference between the Gegenbauer polynomial basis and a robust Gibbs complement is seen in the limit as the original basis projection order N goes to infinity. As discussed in [3], the Gegenbauer projection approaches the power series expansion, which is guaranteed to converge for $x \in [a, b]$ only when the underlying function is analytic in the complex domain disk $\{z : |z - (b + a)/2| \leq (b - a)/2\}$. In contrast, it will be shown in §3 that the convergence properties of a robust Gibbs complementary basis expansion approaches that of the limiting basis in the third requirement, and by definition this expansion converges exponentially for any function analytic on the real interval $[a, b]$, [16]. This fundamental difference indicates that robust Gibbs complements yield the desired exponential convergence for any piecewise analytic function, whereas the Gegenbauer reconstruction method is only convergent for a subset of piecewise analytic functions, even as the limit is approached.

The paper is organized as follows: In §2 we review the Gegenbauer reconstruction method and determine the causes of the aforementioned difficulties. This discussion will motivate the properties of a new (family of) reprojection bases, the robust Gibbs complements, which we discuss in §3 and design in §3.2. For simplicity of presentation, we limit our discussion to the most widely used case where the original basis is the complex exponentials, i.e., the truncated Fourier series or trigonometric interpolant, and note that the techniques described here should generalize easily to other truncated global series approximations. In §4 several numerical examples are illustrated, including cases where the off-axis singularity proximity to $[-1,1]$ is less than 1. Our results are summarized in §5, and future research proposed.

2. The Gegenbauer Reconstruction Method. In order to motivate the rest of our paper, we describe the Gegenbauer reconstruction method and note its strengths and weaknesses.

Let $f(x) \in L^2[-1,1]$ be a piecewise analytic function, and let $[a,b] \subset [-1,1]$ be one of the analytic sub-intervals. We wish to approximate $f(x)$ in $[a,b]$ from either its truncated Fourier series

$$S_N f(x) := \sum_{|k|\le N} \hat{f}_k e^{i\pi kx}, \qquad \text{with} \quad \hat{f}_k := \frac{1}{2}\int_{-1}^{1} f(x) e^{-i\pi kx}, \tag{2.1}$$

or trigonometric interpolant[1]

$$I_N f(x) := {\sum_{|k|\le N}}' \tilde{f}_k e^{i\pi kx}, \qquad \text{with} \quad \tilde{f}_k := \frac{1}{2N}\sum_{\nu=-N}^{N-1} f\left(\frac{\nu}{N}\right) e^{-i\pi k\nu/N}. \tag{2.2}$$

It is well known that $S_N f(x)$ and $I_N f(x)$ are poor approximations of $f(x)$ in the smooth sub-interval $[a,b]$ with spurious Gibbs oscillations prevalent near the boundaries of the interval and order $\mathcal{O}(\frac{1}{N})$ accuracy in the interior of the interval. However, as was shown in [18] and subsequent papers (see [16] for references), it is possible to reconstruct $f(x)$ with exponential accuracy in the maximum norm over the region of smoothness, $[a,b]$, by reprojecting either (2.1) or (2.2) using the Gegenbauer polynomials defined below.

Definition 2.1. *The Gegenbauer polynomials $C_n^\lambda(x)$ for $\lambda \ge 0$ are the polynomials of degree n with normalization $C_n^\lambda(1) = \Gamma(n+2\lambda)/n!\Gamma(2\lambda)$ that are orthogonal with respect to the weighted $L^2[-1,1]$ inner product*

$$\int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} C_k^\lambda(x) C_n^\lambda(x) dx = 0, \qquad k \ne n. \tag{2.3}$$

The weighted norm of $C_n^\lambda(x)$ is given by

$$\int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} C_n^\lambda(x) C_n^\lambda(x) dx = \sqrt{\pi} C_n^\lambda(1) \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)(n+\lambda)} =: h_n^\lambda, \tag{2.4}$$

and we designate the normalized Gegenbauer polynomials as

$$\Phi_l^\lambda(x) = \frac{1}{\sqrt{h_l^\lambda}} C_l^\lambda(x). \tag{2.5}$$

[1] The $'$ on the summation in equation (2.2) is the standard notation that the first and last elements in the sum should be divide by 2.

Before detailing the convergence properties of Gegenbauer series we categorize analytic functions in terms of their extension into the complex plane. Let $f(x)$ be an analytic function for $x \in [-1, 1]$. Then there exists some constant $0 \leq r_0 < 1$ such that the function has a unique analytic extension onto the complex plane for the elliptical region (see e.g. [16])

$$D_\rho := \{z : 2z = re^{i\theta} + r^{-1}e^{-i\theta}, \quad 0 \leq \theta \leq 2\pi,\ r_0 \leq r \leq 1\}. \tag{2.6}$$

The truncated Gegenbauer series expansion of $f(x)$ is defined as

$$G_m^\lambda(f)(x) := \sum_{l=0}^{m} \hat{f}_G^\lambda(l)\Phi_l^\lambda(x), \tag{2.7}$$

where

$$\hat{f}_G^\lambda(l) := \int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}}\Phi_l^\lambda(x)f(x)dx. \tag{2.8}$$

The series converges at the exponential rate

$$\max_{x\in[-1,1]} \left|f(x) - G_m^\lambda(f)(x)\right| \leq Const \cdot m\sqrt{\frac{m+\lambda}{m}}\sqrt{\frac{(m+2\lambda)^{m+2\lambda}}{m^m(2\lambda)^{2\lambda}}}r^m, \tag{2.9}$$

where r is as defined in (2.6). This establishes that the Gegenbauer polynomials satisfy the first Gibbs complement requirement listed in §1.

To apply the local Gegenbauer reprojection of $S_N f(x)$ (2.1) in the region of smoothness $[a, b]$, we make the linear transformation of $x \in [a, b]$ to $\xi \in [-1, 1]$,

$$\xi = -1 + 2\frac{x-a}{b-a}, \tag{2.10}$$

and apply the reprojection $G_m^\lambda(S_N f)(\xi(x))$. We note that a similar reprojection $G_m^\lambda(I_N f)(\xi(x))$ can be formed when the trigonometric interpolant (2.2) is given, and is discussed in §3.1.

The error for a Gegenbauer reconstruction can be decomposed into two parts,

$$f(x) - G_m^\lambda(S_N f)(\xi(x)) \equiv f(x) - G_m^\lambda(f)(\xi(x)) + G_m^\lambda(f - S_N f)(\xi(x)). \tag{2.11}$$

The first component has already been shown in (2.9) to decay exponentially in m for any fixed value of λ. The second term can be bounded by

$$\|G_m^\lambda(f - S_N f)\|_{L^\infty(a,b)} \leq C_{m,\lambda} \cdot \left(\frac{2(\lambda-1)}{e\pi N}\right)^{\lambda-1}\left(1 + \frac{m}{2\lambda+1}\right)^{2\lambda}\left(1 + \frac{2\lambda}{m-1}\right)^{m-\frac{1}{2}} \tag{2.12}$$

where $C_{m,\lambda} := Const \cdot (m+\lambda)\sqrt{(2\lambda-1)(\lambda-1)}$ grows slowly, [18]. For λ fixed, this second error component only decays approximately at the rate $(\frac{m^2}{N})^\lambda$. Hence the Gegenbauer reconstruction can achieve at most a fixed order of accuracy for λ fixed. Exponential decay of (2.12) in N requires that *both* the Gegenbauer projection degree m and weight order λ be selected proportional to N, [18]. Although the second requirement for the Gibbs complement is now satisfied, such a restriction severely inhibits the convergence rate in (2.9). Specifically, it was shown in [16] that to ensure the exponential convergence of (2.9) with $\lambda = \gamma m \sim N$, it was sufficient to select γ small enough such that

$$\frac{(1+2\gamma)^{\frac{1+2\gamma}{2}}}{(2\gamma)^{\gamma}} r < 1,$$

where r is as defined in (2.6). Recently it was demonstrated in [3, 13] that it is in fact necessary to have γ sufficiently small, otherwise the Gegenbauer series will diverge for part of the interval $[a, b]$. Although techniques have been developed in [12] and [13] to properly select the function dependent Gegenbauer parameters m and λ, the problem is more fundamental. For the Gegenbauer polynomials to satisfy both requirements of a Gibbs complement it is necessary to link the parameters $\lambda = \gamma m \sim N$. As a result, the Gegenbauer series converges to the power series expansion as N approaches infinity, [3]. This is a direct consequence of the Gegenbauer weight

$$w_G^{\lambda}(\xi) := (1-\xi^2)^{\lambda-\frac{1}{2}} \tag{2.13}$$

becoming increasingly concentrated at the origin, which causes the projection to become more extrapalatory. Consequently, for Gegenbauer reconstruction with $\lambda = \gamma m \sim N$, in the limit as $N \uparrow \infty$, convergence can only be guaranteed for functions which are analytic in the complex domain disk $\{z : |z - (b+a)/2| \leq (b-a)/2\}$.

To summarize the previous discussion, the susceptibility of the Gegenbauer reconstruction method to both round-off error and the Runge phenomenon can dramatically impair its convergence properties. Not only is accuracy reduced, but in some cases the method fails to converge at all. These weaknesses are attributed to the weight function for the Gegenbauer reconstruction, (2.13), having the requirement that $\lambda \sim N$. Figure 2.1(a) displays how this weight function becomes more localized to the origin as λ increases. The resulting projection becomes more extrapalatory, causing the generalized Runge phenomenon discussed in [3]. In fact, it was proven in [3] that as the Gegenbauer weight parameter λ increases, the truncated Gegenbauer expansion approaches the power series approximation which is *purely* extrapalatory. An additional consequence, exhibited in Figure 2.1(b), is that the amplitude of the Gegenbauer polynomials grow rapidly, particularly as the boundaries $x = \pm 1$ are approached. Hence the corresponding Gegenbauer coefficients (2.8) must decrease to values smaller than machine epsilon. The combination of large amplitude polynomials and extremely small coefficients leads to substantial round-off errors for even "moderate" values of m and λ, [12].

Since clearly the problems with the Gegenbauer reconstruction method are due to the localization of the weight function to the origin and subsequent growth of the Gegenbauer polynomials, it is desirable to develop new reprojection bases. In addition to the first two requirements for Gibbs complements listed in §1, to alleviate the Runge phenomenon any new reprojection basis must approach a basis that is exponentially convergent for any function analytic on $[-1, 1]$, i.e. the third requirement yielding a robust Gibbs complement on page 2. Such bases will have corresponding weight functions that are not arbitrarily localized at the center, and consequently generate polynomials that are well conditioned in terms of their amplitudes.

Although we do not know how to obtain the optimal weight function, or even if there is one, we observe that a natural choice is one that uniformly weights as much of the region of smoothness as possible. For example, we seek a weight $w(\xi)$ such that

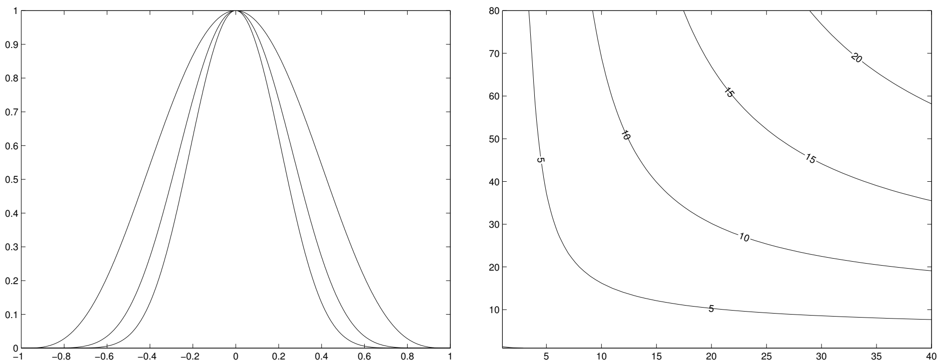

FIG. 2.1. *(a) The Gegenbauer weight* $(1-\xi^2)^{\lambda-1/2}$ *becoming increasingly narrow for increasing* $\lambda = 4, 8,$ *and* 12. *(b) The amplitude of the Gegenbauer polynomials shown as contour lines of* $\log_{10} \max |\Phi_l^\lambda(\xi)|$ *for* $\xi \in [-1,1]$. *The horizontal axis represents the weight order* λ, *and the vertical axis represents the polynomial order* l. *Note the rapid increase in the polynomial amplitude for both parameters.*

$w(\xi) = 1$ for a large portion of the interval and then smoothly decays to be nearly zero at $\xi = \pm 1$. This will be quantified in §3, where we introduce a generalization of the error for a general reprojection as discussed in [17]. We then construct a family of weights that yield both finite order and root exponential convergence rates. The family of weights culminates with the construction of an analytic weight, the Freud weight. We show that the corresponding Freud polynomial bases are a robust Gibbs complement yielding true exponentially convergent reconstructions.

It is important to note that the Gibbs complementary basis does not have to be a family of polynomials. However, to simplify the design and analysis of the reconstruction method, we will consider only polynomial choices.

3. A Robust Gibbs Complementary Basis.

Consider a function $f(x) \in L^2[-1,1]$ that is analytic on an interval $[a,b] \subset [-1,1]$. We seek a high resolution approximation of $f(x)$ for $x \in [a,b]$ from either its truncated Fourier series (2.1) or trigonometric interpolant (2.2). Rather than using Gegenbauer polynomials, we formulate the reconstruction in terms of a general localized reprojection. To simplify the exposition we focus on the recovery from the truncated Fourier series (2.1). Later in §3.1 we detail the minor modifications required when the given data is the trigonometric interpolant (2.2).

We consider a family of robust Gibbs complementary bases $\{\Psi_l^n\}_{l=0}^M$ which are orthonormal for the $w^n(\cdot)$ weighted $L^2[-1,1]$ inner product. The truncated series expansion of a function in terms of a robust Gibbs complementary basis is given by

$$P_M^n(f)(x) := \sum_{l=0}^{M} \hat{f}^n(l)\Psi_l^n(x), \tag{3.1}$$

where

$$\hat{f}^n(l) := \int_{-1}^{1} \Psi_l^n(x) w^n(x) f(x) dx. \tag{3.2}$$

Using the linear transformation (2.10), we reproject the Fourier series (2.1) to the local region of smoothness $[a,b]$ to obtain $P_M^n(S_N f)(\xi(x))$. The error after the reprojection can then be decomposed to separate the effects of the limited information in the original spectral projection, $S_N f(x)$, and the convergence properties of the new basis, $\{\Psi_l^n(\cdot)\}_{l=0}^M$. Specifically, by adding and subtracting the truncated approximation of the exact function in terms of the new basis, we arrive at

$$\begin{aligned} Err_{[a,b]}(M,N,f,n) &:= f - P_M^n(S_N f) = f - P_M^n f + P_M^n(f - S_N f) \\ &=: Trun_{[a,b]}(M,f,n) + Orth_{[a,b]}(M,N,f,n). \end{aligned} \tag{3.3}$$

The first error component, the truncation error, is controlled entirely by the convergence properties of the new basis, not on the degree of the original projection N. (This error was originally called the regularization error in [18].) The second component is a measure of the near orthogonality of the spaces P_M^n and $I - S_N$.

To bound the truncation error, recall that the first requirement of a Gibbs complement implies that for every $f(\cdot)$ analytic on $[a,b]$ there exists some $\rho(f,n) \equiv \rho_n > 1$ and $C(M,n)$ such that

$$\left|Trun_{[a,b]}(M,f,n)\right| \leq \max_{\xi\in(-1,1)} |f(\xi) - P_M^n(f(\xi))| \leq C(M,n)\cdot\rho_n^{-M}, \tag{3.4}$$

where $C(M,n)$ is at most $\mathcal{O}(\max(M,n)^\beta)$ for finite $\beta > 0$. Examples of a basis where this exponential convergence is obtained include the Gegenbauer polynomials for a fixed weight order λ, or more generally, any basis where $\Psi_l^n(\cdot)$ is a polynomial of degree l that is orthogonal under a weight that is strictly positive for all but a set of measure zero in $[-1,1]$, i.e., $w^n(\xi) > 0$ a.e., [9].

We now turn to the orthogonality error which quantifies the effects of possessing only a limited amount of information about the function of concern, $f(\cdot)$. This is realized by measuring the orthogonality of the reprojection space P_M^n and the space containing the information about the function we seek to recover that is not known, $I - S_N$, which being small indicates that the reprojection does not attempt to utilize the unknown information. Hence, the second requirement for a Gibbs complementary indicates that the orthogonality error will decay exponentially. To enforce this decay for a given reprojection weight we seek to bound the error

$$\begin{aligned} Orth_{[a,b]}(M,N,f,n) &:= P_M^n(f - S_N f) \\ &= \sum_{l=0}^{M} \Psi_l^n(\xi)\int_{-1}^{1} w^n(y)\Psi_l^n(y)(f(x(y)) - S_N f(x(y)))dy \\ &= \sum_{l=0}^{M}\sum_{|k|>N} \hat{f}_k \Psi_l^n(\xi)\int_{-1}^{1} e^{i\pi k x(y)} w^n(y)\Psi_l^n(y)dy. \end{aligned} \tag{3.5}$$

Note that the unacceptably slowly decaying Fourier coefficients, $\hat{f}_k$, are weighted by the inner product

$$\int_{-1}^{1} e^{i\pi k x(y)} w^n(y)\Psi_l^n(y)dy = e^{i\pi k(b+a)/2}\int_{-1}^{1} e^{i\frac{k\pi(b-a)}{2}y} w^n(y)\Psi_l^n(y)dy, \tag{3.6}$$

which is simply the Fourier coefficient of $w^n(y)\Psi_l^n(y)$,

$$\widehat{w^n\Psi_l^n}(\kappa) = \int_{-1}^{1} w^n(y)\Psi_l^n(y)e^{-i\pi\kappa y}dy. \tag{3.7}$$

Here we have defined an effective coefficient number $\kappa := -k\frac{b-a}{2}$, which is k scaled by the fraction length of the interval, $\frac{b-a}{2}$. The orthogonality error is then dictated by

$$\left|Orth_{[a,b]}(M,N,f)\right| \leq \sum_{l=0}^{M} \sum_{|k|>N} |\hat{f}_k| \, \|\Psi_l^n\|_{L^\infty[-1,1]} \left| \widehat{w^n\Psi_l^n}\left(-k\frac{b-a}{2}\right) \right|, \tag{3.8}$$

where the decay of $|\widehat{w^n\Psi_l^n}(\kappa)|$ can be controlled by the smoothness of the underlying weight function, [14]. As stated in §2, the portion of the Gegenbauer reconstruction error which corresponds to the orthogonalization error (2.12) only decays exponentially in N if the Gegenbauer weight order $n = \lambda$ is selected proportional to N. This is a direct consequence of the finite regularity of the Gegenbauer weight (2.13) when extended periodically by zero. Hence there is only a finite order decay of (3.7) for fixed λ. As stated previously, the Gegenbauer series series approaches the power series as λ increases with N [3]. As an overall consequence, the limit of the decay constants in the truncation error (3.4) approaches one, i.e., $\rho_\lambda \downarrow 1$ as $\lambda \uparrow \infty$. Moreover, this implies that the Gegenbauer reconstruction method is only guaranteed to converge in the reprojected interval $[a,b]$ if the function is analytic in the complex domain disk $\{z : |z-(b+a)/2| \leq (b-a)/2\}$, rather than just being analytic on the strip $[a,b]$.

To avoid this pitfall we impose an additional constraint in constructing a robust Gibbs complementary; that is that the non-negative weight functions $w^n(\cdot)$ converge to a weight function w^∞ whose associated orthogonal polynomials $\{\Psi_l^\infty\}_{l=0}^M$ form a basis that satisfies the first requirement of being a Gibbs complement. This requirement enforces that the convergence properties of the new reprojection basis converge to those of polynomials which are orthogonal with respect to the weight w^∞, i.e., in (3.4) we have $\rho_n \to \rho_\infty > 1$. This is easily proven inductively using the repeated application of the dominated convergence theorem, [8], and relying on the property of orthogonal bases that every fixed element of the Gibbs complementary basis and limiting basis is bounded in $[-1,1]$. However, the proof does not hold for the Gegenbauer basis as $\lambda \uparrow \infty$ since the weights (2.13) approach zero almost everywhere; and as a result, the limiting weight does not have an associated set of orthogonal polynomials. Rather, it was shown in [3] that as λ increases, the Gegenbauer polynomials $\Phi_l^\lambda(x)$ converge to $c_l x^l$ for some constant c_l, and the Gegenbauer coefficients $\hat{f}_G^\lambda(l)$ converges to $\frac{1}{l!c_l}f^{(l)}(0)$. These combined results show that for increasing λ, the Gegenbauer series expansion of a function converges to its power series, resulting in the generalized Runge phenomenon, [3].

On the other hand, the convergence properties of a robust Gibbs complementary basis approach those of the limit basis which does not suffer from the Runge phenomenon. Hence for sufficiently large original projection order N, the reprojection will yield an accurate approximation of a function once the reprojection polynomial order M is sufficiently large to resolve it. We compile the properties of a robust Gibbs complement in the following definition:

Definition 3.1. **A robust Gibbs complementary basis** *satisfies the following properties:*

1. *For a function analytic on the interval* $[-1,1]$*, the expansion of the function in the orthogonal reprojection basis is exponentially convergent.*
2. *The projection of the high modes in the original basis on the low modes in the new basis is exponentially small.*
3. *As the order of the original projection* N *increases, the weight function of the*

orthogonal reprojection basis converges to a weight whose associated orthogonal polynomial basis satisfy the first requirement.

We note in the third requirement that the weight function has parameters that depend on the order of original projection terms N. Hence we are really referring to a family of weight functions rather than one particular weight function.

Before constructing examples of robust Gibbs complements we detail the minor modification in the analysis when given the trigonometric interpolant (2.2) rather than the spectral projection (2.1).

3.1. Approximation of a piecewise smooth function from its equidistant samples. The reprojection method proposed above for the recovery of a function from its truncated Fourier series works equally well when the given information consists of equidistant samples, or equivalently the trigonometric interpolant of the function, (2.2). The reprojection $P^n_m(S_N f)(x(\xi))$ is simply replaced by

$$P^n_M(I_N f(x(\xi))) = \sum_{l=0}^{M} \widehat{I_N f^n_l}\,\Psi^n_l(x(\xi)), \tag{3.9}$$

where

$$\widehat{I_N f^n_l} := \int_{-1}^{1} \Psi^n_l(\xi) w^n(\xi) I_N f(x(\xi)) d\xi.$$

The error is again decomposed into the truncation and orthogonalization error in the same fashion as (3.3). The truncation error is unchanged, but $\hat{f}_k$ is replaced by $\tilde{f}_k$ in the orthogonalization error bound (3.8). As before, the fundamental issues determining the convergence are the convergence properties of the new basis and the near orthogonality of the spaces P^n_M and $I - I_N \equiv I - S_N$. In fact, a slightly larger orthogonality error bound achieved by using $|\hat{f}_k|, |\tilde{f}_k| \leq \|f\|_{L^1[-1,1]}$,

$$\left|Orth_{[a,b]}(M,N,f,n)\right| \leq \|f\|_{L^1[-1,1]} \sum_{l=0}^{M} \sum_{|k|>N} |\, \|\Psi^n_l\|_{L^\infty[-1,1]} \left|\widehat{w^n \Psi^n_l}\left(-k\frac{b-a}{2}\right)\right|, \tag{3.10}$$

is valid for either the recovery from the truncated Fourier series or the trigonometric interpolant. We now turn to constructing an example of a robust Gibbs complement which suffers little from round-off errors.

3.2. An example of a robust Gibbs complement. Before constructing an example of a robust Gibbs complement we list two additional desirable properties for the weight function of a reprojection basis: first, that it utilizes as much of the region of smoothness as possible, and second, that the maximum amplitudes of the associated low order reprojection polynomials ($l = 0, 1, \ldots, M$) increase at most only slowly with the order of the polynomial. The reason for the first property is to incorporate in the reprojection as much information about the smooth portion of the function as possible. The second property is selected both to decrease the orthogonality error in the bound (3.8), as well as to make the reprojection less susceptible to numerical round-off error. In particular, for the Gegenbauer polynomials it has been shown that the rapid decay of the coefficients (3.7) is sufficient to overcome the growth of the Gegenbauer polynomials magnitude, resulting in an exponentially decaying orthogonality error, [16]. Yet in numerical implementations, round-off error causes the decay of (3.7) to be truncated at machine epsilon (see e.g. [12]). This limits the number of terms available in the reprojection, M, and as a result reduces the achievable accuracy in the reconstruction. Moreover, for moderately oscillatory functions the number of terms required to resolve the function can become large, resulting in the poorly conditioned polynomials due to $\lambda \sim N$, as is apparent from Figure 2.1(b).

These two desirable properties are intimately related, and in addition to satisfying the third requirement of a robust Gibbs complement, can be achieved by selecting a weight function for a reprojection basis that will approach $\chi_{(-1,1)}$ as the original projection order N increases. As the weight goes toward $\chi_{(-1,1)}$, the convergence properties of the robust Gibbs complement approaches that of the Legendre polynomials, $w_G^{1/2} = \chi_{(-1,1)}$, which is well known to yield exponential convergent truncated series approximations for analytic functions on $[-1, 1]$. As a consequence of this nearly uniform weight, the corresponding polynomials also maintain a significantly smaller maximum amplitude than do the Gegenbauer polynomials. Moreover, later we construct a weight where the maximum amplitude for a fixed degree polynomial *decreases* as the projection order N increases. This property then suggests that the growth rate of the reprojection basis, where the degree of the polynomial M grows with N, is "moderate enough" so that round-off error does not become an inhibiting factor. We note that although clearly some classical orthogonal polynomials have weights which nearly uniformly weight the region of smoothness, such as the Legendre polynomials with weights (2.13) for $\lambda = \frac{1}{2}$, they do not satisfy the second requirement of being Gibbs complementary [16]. Rather than use the Legendre weight directly, we develop a family of weights that converge to the Legendre weight, yet which satisfy the second property of a Gibbs complement.

3.2.1. Reconstruction bases from finitely regular weight functions. Before constructing a robust Gibbs complementary basis, we briefly describe how a family of alternative finite order bases can be developed, i.e., we relax the first requirement of a Gibbs complement and seek to recover only finite order accuracy. As mentioned before, although it is attractive to use the Gegenbauer weight (2.13) and corresponding Gegenbauer polynomials as a family of reprojection bases, they are not robust in reconstruction. This lack of robustness becomes most striking when considering the diagonal limit, where the weight and polynomial orders, λ and M respectively, grow proportionally with the degree of the original projection N [3]. As proposed in [12], this problem can be alleviated by relaxing the second condition of the Gibbs complement from being exponentially convergent to being only finitely convergent. The Gegenbauer polynomials are still used in reconstruction, but m and λ are limited to reduce the effects of round-off error, resulting in fixed finite order approximations. Although this strategy can be implemented successfully, using the Gegenbauer reconstruction method for finite order accuracy is computationally inefficient. From the discussion above we recognize that other weights can be constructed which will in addition have the advantage of satisfying the third requirement of a robust Gibbs complement. The reconstruction methods produced from these finite order weights are better conditioned and less susceptible to round-off error than the Gegenbauer weights (2.13). Examples of weights that generate such finite order robust Gibbs complements are

$$w^c(\xi) := \begin{cases} 1, & |\xi| \leq \xi_0, \\ c(\xi), & \xi_0 < |\xi| \leq 1, \end{cases} \tag{3.11}$$

with $c(\cdot)$ selected to smoothly connect one for $|\xi| \leq \xi_0$ to zero for $|\xi| > 1$. Such functions have been developed in the construction of classical filters, with the most common being the raised cosine, $c_{rc}(\xi) := \frac{1}{2}\left(1 + \cos\left(\pi\frac{\xi-\xi_0}{1-\xi_0}\right)\right)$, and sharpened raised cosine, $c_{src}(\xi) := c_{rc}^4(\xi)(35 - 84c_{rc}(\xi) + 70c_{rc}^2(\xi) - 20c_{rc}^3(\xi))$. Furthermore, the weight can be designed to approach $\chi_{(-1,1)}$ by selecting the translate ξ_0 to approach one as the original projection order increases, i.e. $\xi_0 \uparrow 1$ as $N \uparrow \infty$.

We add that such a reprojection basis designed to recover a finite order approximation has certain advantages over the classical non-adaptive filtered Fourier reconstruction. Specifically, the L_∞ error will maintain the finite order of the reconstruction throughout the interval, i.e., the approximation near the end points will not deteriorate as it does for filtered reconstructions. While there are many applications where a finite order reconstruction is "good enough", our goal here is to design weights to satisfy all of the robust Gibbs complement requirements.

3.2.2. Reconstruction bases from Gevrey regular weight functions. Another possible weight function to consider is an infinitely differentiable compactly supported cutoff function. For example, it was shown in [23] that

$$w^G(\xi) := \exp\left(\frac{\xi^2}{\xi^2 - 1}\right) \tag{3.12}$$

has Fourier coefficients that decay at the root exponential rate,

$$\hat{w}_k^G \leq Const\ \exp(-\eta\sqrt{|k|}), \qquad \eta > 0,$$

and consequently (3.7) also decays root exponentially (details are presented in Appendix §A). The space of compactly supported infinitely differentiable functions is usually catalogued in terms of Gevrey regularity, with (3.12) serving as an example of a Gevrey regular function. Using such compactly supported weights will only allow an overall root exponential accuracy in the reprojection, instead of the desired true exponential accuracy which is achievable in theory through Gegenbauer reconstruction. Hence we will not pursue their construction further. We note, however, that there are certain advantages to using compactly supported weight functions. In particular, they would allow for more straight forward mathematical manipulation, i.e., no boundary terms in (3.16).

3.2.3. Reconstruction bases from Freud weight functions. We now proceed with the development of a family of weights which will yield an exponentially convergent robust Gibbs complement. Following the line of thought in [24], we abandon strict compact support, and illustrate how a properly localized analytic weight allows for true exponential accuracy. Although we are not aware of the optimal weight for the robust Gibbs complement, we propose a weight that converges to $\chi_{(-1,1)}$ and yields an overall exponential error decay. In this way, not only will the occurrence of the Runge phenomenon be completely removed, but we also hope to limit the growth of the corresponding polynomials, which will reduce the potential of round-off error. Throughout the remainder of this paper we focus on the family of Freud weights,

$$w_F^n(\xi) := e^{-c\xi^{2n}} \qquad \text{for} \quad n \in \mathbb{Z}^+. \tag{3.13}$$

The orthogonal polynomials resulting from the Freud weights have been extensively studied since the early 1970s when Freud proposed them in [9] as the natural extension of Hermite polynomials ($n = 1$). However they remain much less understood than the Gegenbauer polynomials. In particular, the precise behavior of the convergence rate constants, $\rho(f, w_F^n)$ in (3.4), and the three term recursion relationships for their iterative construction are not known for general n. Moreover, although we are considering reprojection bases that are orthonormal over the finite interval, the Freud polynomials are taken to be orthogonal over the real line. Despite these complications, by properly selecting the parameters c and n of the Freud weights, we can obtain an orthogonal polynomial basis that satisfies the properties of a robust Gibbs complementary basis. It is beyond the scope of this paper to fully develop the properties of the Freud polynomials, instead we seek to illustrate the relevant properties for moderate order polynomials analytically, and where necessary numerically.

Let us consider the Freud weight (3.13) with parameters

$$n \equiv n(N) := \text{round}\left(\sqrt{N\frac{b-a}{2}} - 2\sqrt{2}\right), \text{ and } c := -\ln(\epsilon). \tag{3.14}$$

Here $\epsilon \ll 1$ is the amplitude of the weight at $\xi = \pm 1$, and the term $\frac{b-a}{2} \leq 1$ accounts for the dilation to the region of smoothness $[a, b]$ to give the effective number of wavelengths found in the smooth interval. The first requirement that the reprojection basis yield an exponentially convergent approximation for analytic functions is satisfied due to the weight being non-negative, [9]. The third requirement is clearly satisfied as the weights approach the Legendre weight $w_G^{1/2} := \chi_{(-1,1)}$ as $N \uparrow \infty$. Before turning to the remaining (second) requirement, we illustrate the evolution of the Freud weight for increasing N in Figure 3.1(a), and show the corresponding growth rate of the polynomials in Figure 3.1(b). In contrast to the Gegenbauer weight, which becomes increasingly narrow as N increases, notice that the Freud weight with parameters (3.14) is increasingly uniform over $(-1, 1)$ and converges to the Legendre weight. Moreover, for a fixed order polynomial l, the maximum amplitude of its corresponding polynomial *decreases* as n increases. This self-regularizing property dramatically reduces the round-off error in numerical implementations.

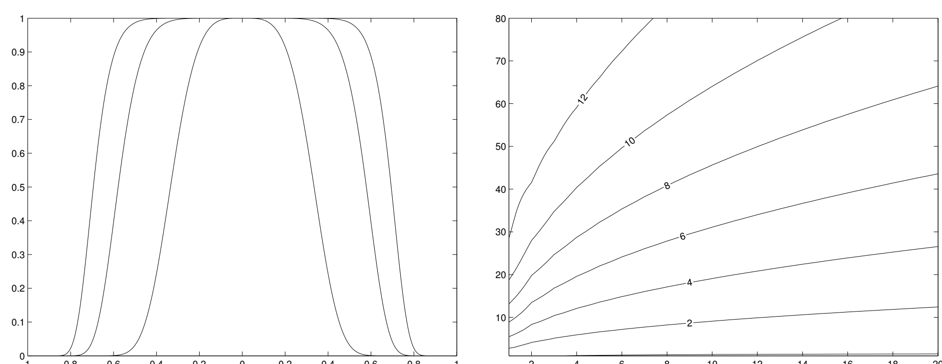

Fig. 3.1. *(a) The Freud weight* $\exp(-c\xi^{2n})$ *with parameters (3.14) and* $\epsilon = 10^{-24}$ *becomes increasingly wider for increasing* $n = 2, 4,$ *and* 6. *(b) Contour lines of* $\log_{10} \max |\Psi_l^n(\xi)|$ *for* $\xi \in [-1, 1]$ *with horizontal axis the weight order* n *and vertical axis the polynomial order* l. *Note that for* l *fixed, the maximum amplitude decreases as* n *increases.*

To establish overall exponential convergence it remains to establish that for sufficiently low order the Freud polynomials, which will serve as the reprojection basis, are nearly orthogonal to the original basis, the complex exponentials $\{\exp(i\pi kx)\}_{|k|\le N}$. More precisely, we must establish that (3.7) is exponentially small for $l = 0, 1, \dots, M$ and $|k| \ge N$. Unfortunately, the incomplete knowledge about the Freud polynomials prevents us from proving this result directly. Yet rather than use a Gevrey regular weight (3.12), which allows a fully rigorous proof of root exponential convergence, or a finitely regular weight (3.11), which yields a proof of finite order convergence, we submit both analytical and numerical evidence that strongly suggest the true exponential decay of (3.7) using the Freud weight (3.13) with properly selected parameters (3.14).

To establish the decay of (3.7) for the Freud weight, we separate the effects of smoothness and the localization to $|\xi| < 1$ for the quantity $w_F^n(\xi)\Psi_l^n(\xi)$. First we consider the integral taken over the real line and apply s consecutive integration by parts to obtain

$$\int_{-\infty}^{\infty} w_F^n(y)\Psi_l^n(y)e^{-i\pi\kappa y}dy = (-i\pi\kappa)^{-s}\int_{-\infty}^{\infty} e^{-i\pi\kappa y}\frac{d^s}{dy^s}\left(w_F^n(y)\Psi_l^n(y)\right)dy, \tag{3.15}$$

where $\kappa = -k\frac{b-a}{2}$ has been previously defined. This integral can be bounded by separating the right hand side into integrals for $|y| \le 1$ and $|y| > 1$ yielding

$$(-i\pi\kappa)^{-s}\int_{-\infty}^{\infty} e^{-i\pi\kappa y}\frac{d^s}{dy^s}\left(w_F^n(y)\Psi_l^n(y)\right)dy = I_1 + I_2,$$

where

$$I_1 := (-i\pi\kappa)^{-s}\int_{|y|\le 1} e^{-i\pi\kappa y}\frac{d^s}{dy^s}\left(w_F^n(y)\Psi_l^n(y)\right)dy$$

and

$$I_2 := (-i\pi\kappa)^{-s}\int_{|y|> 1} e^{-i\pi\kappa y}\frac{d^s}{dy^s}\left(w_F^n(y)\Psi_l^n(y)\right)dy.$$

As explained in Appendix §A, for $w_F^n(y)\Psi_l^n(y)$ analytic, the portion where $|y| \le 1$ can be controlled by its regularity. Specifically, for $\eta > 0$ we have

$$(-i\pi\kappa)^{-s}\int_{|y|\le 1} e^{-i\pi\kappa y}\frac{d^s}{dy^s}\left(w_F^n(y)\Psi_l^n(y)\right)dy \le Const\cdot|\kappa|^{\frac{1}{2}}e^{-\pi\eta|\kappa|}.$$

We can then bound (3.15) by

$$\left|\int_{-\infty}^{\infty} w_F^n(y)\Psi_l^n(y)e^{-i\pi\kappa y}dy\right| \le Const\cdot|\kappa|^{\frac{1}{2}}e^{-\pi\eta|\kappa|}+(\pi|\kappa|)^{-s}\int_{|y|>1}\left|\frac{d^s}{dy^s}\left(w_F^n(y)\Psi_l^n(y)\right)\right|dy.$$

Since the left hand side can be bounded from below by

$$\left|\int_{-1}^{1} w_F^n(y)\Psi_l^n(y)e^{-i\pi\kappa y}dy\right|-\left|\int_{|y|>1} w_F^n(y)\Psi_l^n(y)e^{-i\pi\kappa y}dy\right| \le \left|\int_{-\infty}^{\infty} w_F^n(y)\Psi_l^n(y)e^{-i\pi\kappa y}dy\right|,$$

the final bound for (3.7) is obtained by

$$\begin{aligned}&\left|\widehat{w_F^n\Psi_l^n}(\kappa)\right| \le Const\cdot|\kappa|^{\frac{1}{2}}e^{-\pi\eta|\kappa|} \\ &+Const\cdot\int_{y=1}^{\infty}|w_F^n(y)\Psi_l^n(y)|\,dy + (\pi|\kappa|)^{-s_{min}}\int_{|y|>1}\left|\frac{d^{s_{min}}}{dy^{s_{min}}}\left(w_F^n(y)\Psi_l^n(y)\right)\right|dy.\end{aligned} \tag{3.16}$$

Here $s_{min} := \pi\eta|\kappa|$ where $\eta > 0$ is determined using (A.1).

Lacking more precise knowledge about the Freud polynomials, such as the three term recursion relationship coefficients, the authors are not aware of a technique to bound the decay of $w_F^n(y)\Psi_l^n(y)$ for $y > 1$, and as a result prove that $\left|\widehat{w_F^n\Psi_l^n}(\kappa)\right|$ is exponentially small for $l = 0, 1, \ldots, M$ and $|k| \geq N$. Nevertheless, it is intuitively clear that for the parameter c in (3.14) sufficiently large compared to the size of M, the integrals on the right hand side of (3.16) can be made exponentially small by connecting parameters c to N. Alternatively, the integrals can be forced to be smaller than machine epsilon so that they will not interfere with the exponential convergence in numerical implementations.

To achieve exponential decay for the truncation error (3.4) we select the number of terms in the reprojection basis so that M grows with N, where N is the number of terms in the original basis. Additionally, in order that the spaces P_M^n and $I - S_N$ are nearly orthogonal, we incorporate a gap between the wavelengths in $I - S_N \equiv I - I_N$ and P_M^n by selecting $M \leq \frac{N}{4}$. Since the reprojection is taken only over the largest region of smoothness, $[a, b]$, which in general is not of the full interval of the original projection (defined here as $[-1, 1]$), we weight the number of terms in the reprojection as

$$M := \frac{N}{4} \cdot \frac{b-a}{2}. \tag{3.17}$$

This weighting by the fractional length $\frac{b-a}{2}$ allows for the proper decay of (3.7) for the polynomial orders $l = 0, 1, \ldots, M$, and the exponential powers $\kappa := -k\frac{b-a}{2}$ for $|k| > N$. It should be noted that unlike the Gegenbauer case, the selection of M here is not function dependent. Hence using the Freud weight based orthogonal polynomials as the reprojection bases is a "black box" reconstruction algorithm.

We now turn to numerically illustrate that the reprojection basis based on the Freud weights with parameters (3.14) satisfies the remaining (second) requirement, that is that the near orthogonality of the reprojection space, P_M^n, and the space in which information about the underlying function is not known, $I - S_N$. Specifically, it is necessary to show that (3.7) is exponentially small for $l = 0, 1, \ldots, M$ and $|k| \geq N$. Figure 3.2 illustrates the magnitude of (3.7) for the interval $[a, b] = [-1, 1]$ with $N = 64$ and 128. The horizontal axis consists of the first $\frac{N}{2}$ complex exponential powers beyond those given in the original projection, $k = N + 1, N + 2, \ldots, \frac{3N}{2}$, and the vertical axis consists of the order of the reprojection polynomials, $l = 0, 1, \ldots, M$ where $M = \frac{N}{4}$. We select $\epsilon := 10^{-24}$ so that the weight smoothly connects to zero, although we remark that other values moderately below machine epsilon also work well. Note the rapid (conjectured to be exponential) decay of the magnitude of (3.7), which decreases by approximately 10^{-2} as N is doubled.

It is important to realize the essential component of the orthogonality error, the decay of (3.7), is not dependent on the function being recovered, so that the Freud polynomial bases have no function dependent parameters to be estimated by the user. As noted previously, this is a significant advantage over Gegenbauer reconstruction, where its success is heavily reliant on the proper selection of both its weight parameter λ and reprojection order M [13].

4. Numerical examples. Lacking the three term recursion relationship of the Freud polynomials, we rely on a numerical technique to construct the reprojection bases. Specifically, we utilize the Stieltjes procedure outlined in [11] to generate

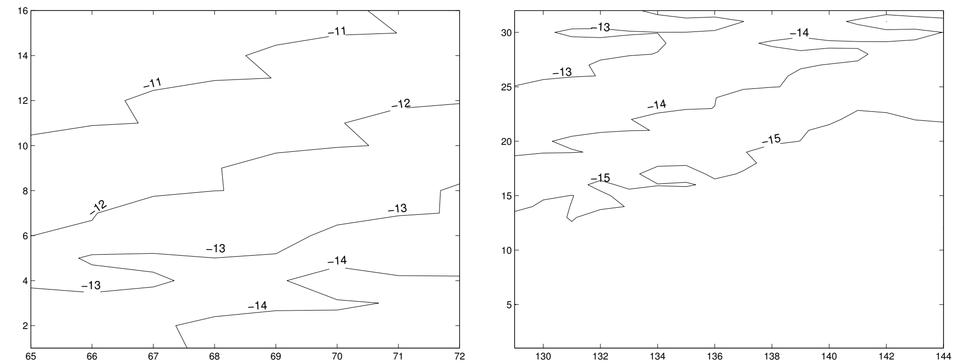

FIG. 3.2. *The magnitude of the inner product (3.7) for the interval* $[a, b] = [-1, 1]$ *with (a)* $N = 64$ *and (b)* $N = 128$*. The horizontal axis consists of the first* $\frac{N}{2}$ *complex exponential powers beyond those given in the original projection,* $k = N+1, N+2, \ldots, \frac{3N}{2}$*, and the vertical axis consists of the order of the reprojection polynomials,* $l = 0, 1, \ldots, M$*, where* $M = \frac{N}{4}$*. We select the parameter* $\epsilon := 10^{-24}$ *so that the weight smoothly connects to zero.*

approximations to the recursion relationships where the weight is taken over $[-1, 1]$. A brief description of the Stieltjes procedure is outlined in Appendix §B, but we point the interested reader to the comprehensive text [11] where the algorithm is discussed in detail and a computer code is provided.

Before presenting the numerics we also address an important practical consideration, that of a function being fully resolved within machine accuracy with fewer than the designated order of polynomials, $l = 0, 1, \ldots, M_{lim}$ for some $M_{lim} < M$. With the number of terms in the original projection increasing, the number of terms available in the reprojection basis will inevitably be more than is necessary to resolve the function numerically. Once this happens, due to round-off error, the reprojection coefficients $\widehat{S_N f^n}(l)$ will become limited to near machine epsilon. Combining these artificially large (machine epsilon) coefficients with polynomials of increasing magnitude results in the degradation of the approximation quality. To overcome this practical numerical concern, we additionally limit the number of terms in the reprojection basis at the first occurrence where the average of three consecutive coefficients is below some tolerance, Tol. Specifically, if we let

$$\widehat{S_{ave}(l)} := \frac{1}{3}(\widehat{S_N f^n}(l-1) + \widehat{S_N f^n}(l) + \widehat{S_N f^n}(l+1)) \tag{4.1}$$

we can define M_{lim}^n as

$$M_{lim}^n := \min\Big(M, \min\{l \text{ such that } \widehat{S_{ave}(l)} < Tol\}\Big). \tag{4.2}$$

In the following numerical examples we contrast Gegenbauer reconstruction with parameters $\lambda = \frac{N}{8} \cdot \frac{b-a}{2}$ and the Freud robust Gibbs complement with parameters (3.14) where $\epsilon = 10^{-24}$. For both reprojection bases, the number of terms M is selected as in (3.17), with the above limiting where $Tol := 10^{-14}$. As a result, accuracy beyond this threshold can not be expected. Standard trapezoidal quadrature with spacing $\frac{1}{2N}$ was used in computing $\widehat{S_N f^n}(l)$ whereas $\widehat{I_N f^n}(l)$ was computed using the trapezoidal quadrature with only the given equidistant sample, $\{f(\frac{\nu}{N})\}_{\nu=-N}^{N-1}$.

This rather course quadrature is permissible due to the exponential accuracy of the trapezoidal sum (see e.g. [4, 14]).

One of the primary motivations for the development of the robust Gibbs complements was to find a reprojection basis that, rather than be extrapalatory, utilizes as much of the region of smoothness as possible while still satisfying the second requirement of a Gibbs complement. In doing so, the robust Gibbs complement also avoids the Runge phenomenon, allowing the recovery of any piecewise analytic function once N is sufficiently large to resolve it. As the following examples illustrate, not only is the Runge phenomenon eliminated, but the effects of round-off error are clearly reduced.

We begin by considering the reconstruction of a function suggested in [3] to measure the Runge phenomenon,

$$f^{symm,pole}(x, z_s) := [\Im(z_s)]^2 \left\{ \frac{1}{[\Im(z_s)]^2 + (x - \Re(z_s))^2} + \frac{1}{[\Im(z_s)]^2 + (x + \Re(z_s))^2} \right\}, \tag{4.3}$$

where z_s is taken to be a fixed constant and $f^{symm,pole}$ is a function of x with a pole at z_s. We then measure the ability of the Gegenbauer and Freud reprojection bases to recover $f^{symm,pole}$ depending on the location of the pole z_s. Figure 4.1(a-c) illustrates that the region of failed convergence due to Runge phenomenon is not decreasing for the Gegenbauer weight $\lambda = \frac{N}{8}$, which although not optimal, satisfies both theoretical and numerical concerns from [12, 13]. However, Figure 4.1(d-f) shows that the region of convergence consistently increases for the Freud bases. In fact, the Runge phenomenon is not apparent at all. Rather, the region of failed convergence is a result of the function not being fully resolved with the limited amount of information in the known Fourier coefficients, $\{\hat{f}_k^{symm,pole}\}_{|k|\leq N}$. This example emphasizes how critical the choice of parameters is to the Gegenbauer reconstruction. Specifically, we note the degradation of results from Figure 4.1(b) to Figure 4.1(c). As was shown in [3, 13], the reprojection polynomial degree M and weight order λ must become increasingly smaller proportional to N as the off-axis singularity shrinks to the origin.

Having established the advantages of the robust Gibbs complement in overcoming the Runge phenomenon, we now compute an approximation of another test function put forth in [23] as a challenging function due to its sharp peak and the different regularity constants for the left and right regions:

$$f_2(x) = \begin{cases} (2e^{2\pi(x+1)} - 1 - e^{\pi})/(e^{\pi} - 1) & x \in [-1, -1/2) \\ -\sin(2\pi x/3 + \pi/3) & x \in [-1/2, 1) \end{cases}$$

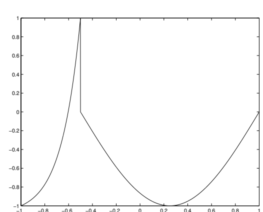

Figure 4.2 shows the behavior of both the Gegenbauer and Freud polynomial reconstructions in each region of smoothness. For the region $(-1, -\frac{1}{2})$, both reconstructions continue to converge in similar fashions due to the region not being fully resolved below machine epsilon for $N \leq 256$. However for the interval $(-\frac{1}{2}, 1)$, both methods have enough terms to fully resolve the function. Unfortunately, after the Gegenbauer method has nearly resolved the function with $N = 128$, it continues to increase the weight parameter, $\lambda \sim N$, causing the reconstruction to become more extrapalatory. As a result, the error in the Gegenbauer reconstruction increases due

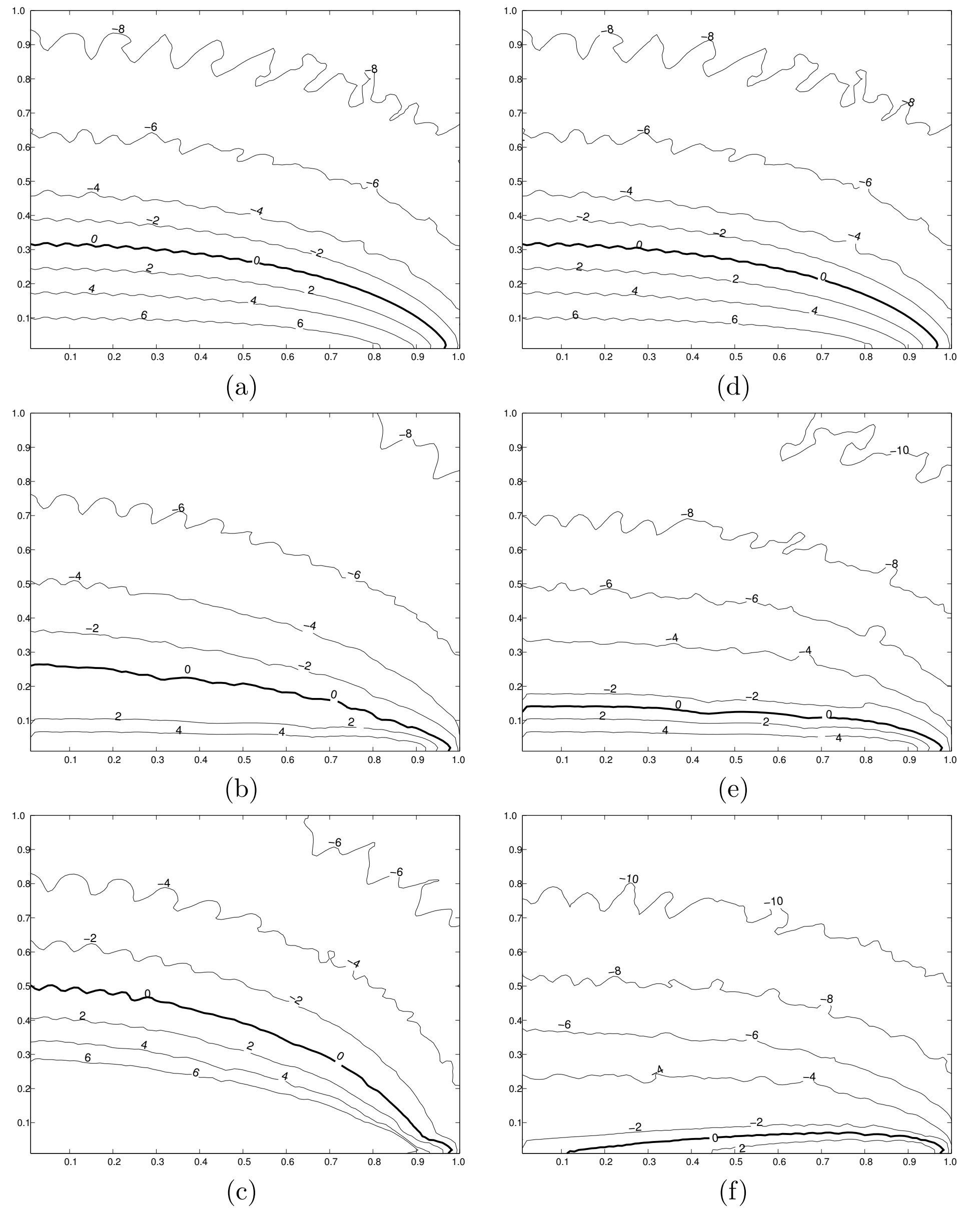

Fig. 4.1. *The* $\log_{10}$ *of the* $L^\infty[-1,1]$ *error in recovering* $f^{symm,pole}$ *for* z_s *in the upper right quadrant of the complex plane, i.e.,* $[\Im(z_s)], [\Re(z_s)] \in (0,1]$*, from its truncated Fourier coefficients, with* $N = 64$ *(a,d),* $N = 128$ *(b,e), and* $N = 256$ *(c,f).* $\Re(z_s)$ *and* $\Im(z_s)$ *make up the respective horizontal and vertical axes. Results are from the Gegenbauer reprojection basis with* $\lambda = N/8$ *(left) and the Freud robust Gibbs complement (right) with parameters (3.14) where* $\epsilon = 10^{-24}$*. In each plot the thick contour line designating error of unit amplitude can be viewed as separating the region where an approximation is recovered from the region where the reconstruction fails. Note that the region for which the Gegenbauer reprojection fails to converge does not decrease with increasing* N*, whereas the Freud reprojection yields not only an increasingly accurate reconstruction, but also the region where the function is fully resolved from the given information is also increasing.*

to the round-off errors and inherently poor conditioning of the Gegenbauer polynomials (Figure 2.1(b)). Methods have been developed in [12] which attempt to overcome this effect by properly selecting the Gegenbauer weight parameter λ. The increasingly poor conditioning can be ameliorated by limiting λ, but only by accepting a reduced rate of convergence. This is evident in Figure 4.2(a) for $x \in (-1/2, 1)$, where the accuracy of the Gegenbauer reconstruction visibly decreases as the original projection order N increases. On the other hand, as is evident in Figure 3.1(b), the Freud robust Gibbs complement actually provides increasingly better conditioned bases as N increases, rather than just limiting the poor conditioning. This is further exhibited in Figure 4.2(b), where it is clear that in contrast to to the increasing error in the Gegenbauer reconstruction, the Freud basis yields increasing accuracy in both regions of smoothness. This effect is further illustrated in Table 4.1 where the maximum L_∞

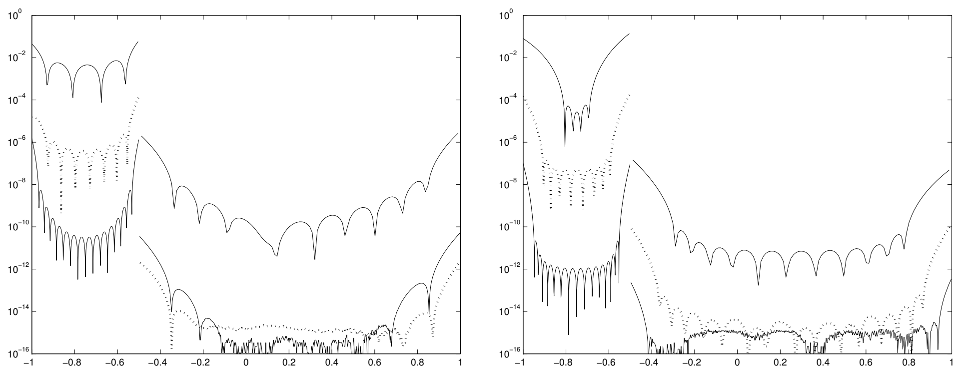

Fig. 4.2. *The error in recovering f_2 using the (a) Gegenbauer and (b) Freud reprojection bases with the same parameters as in Figure 4.1, except for the scaling by $\frac{b-a}{2}$ due to the length of the two intervals of smoothness, $(-1, -\frac{1}{2})$ and $(-\frac{1}{2}, 1)$. The given data were the global, taken over $[-1, 1]$, Fourier series, with $N = 64$ (upper solid), 128 (dotted), and 256 (lower solid).*

error, excluding the discontinuities $x = -1, -\frac{1}{2}$, and 1, are measured. The Freud basis resolves the function and then maintains the accuracy at the size of the user selected Tol, whereas the Gegenbauer weight becomes more extrapalatory. The resulting poorly conditioned Gegenbauer polynomials cause the overall error to increase after the function is fully resolved.

N	Gegenbauer	Freud
32	6.05(-1)	8.90(-1)
64	5.73(-1)	1.37(-1)
128	1.34(-4)	1.84(-4)
256	1.52(-6)	1.01(-7)
512	2.16(-9)	9.33(-13)
1024	1.43(-7)	5.27(-13)
2048	8.99(-7)	5.23(-14)
4096	1.23(-6)	6.59(-14)

Table 4.1

The L_∞ error for the approximation of $f_2(x)$. Here we use the notation $z(-r) := z \times 10^{-r}$. The Gegenbauer reprojection bases becomes increasingly extrapalatory, resulting in increasing round-off error for $N > 512$. In contrast, the Freud robust Gibbs complement resolves the function by $N = 512$, and automatically maintains the accuracy at about the user defined limiting tolerance level, Tol.

5. Summary and Future Work. Gegenbauer reconstruction with suitably selected weight parameter λ and reprojection order M has been shown to recover a function from its (pseudo-)spectral data with exponential accuracy up to the discontinuities. Unfortunately, as a result of the function dependent parameters M and $\lambda \sim N$, Gegenbauer reconstruction suffers both from numerical round-off errors as well as the Runge phenomenon [3, 13]. Fortunately, this limitation is not due to the underlying approach of reprojecting the available (pseudo-)spectral data with a Gibbs complementary basis. Rather, these problems come directly from using the Gegenbauer polynomials as the Gibbs complement.

Here we introduced a more general alternative error decomposition which make the desirable traits in a Gibbs complement more transparent. This insight allows the proposition of an additional requirement for the Gibbs complement. Specifically, we impose that the weight of the new orthogonal reprojection basis approaches a weight whose associated orthogonal polynomials yield exponentially convergent series expansions of functions analytic on $[-1, 1]$. We refer to such reprojection bases as *robust Gibbs complements*. The Freud weights as defined in (3.13) satisfy this requirement, and have the additional desirable property of converging to $\chi_{(-1,1)}$. As a result, the convergence properties of their corresponding polynomials approach those of the more familiar Legendre polynomials for the reconstruction of smooth functions in $[-1, 1]$, i.e., they yield spectral convergence.

By satisfying this additional property, the reprojection bases are better conditioned in the sense that the amplitude of the polynomials does not grow too rapidly. Moreover, the weight more uniformly utilizes the region of smoothness, and the resulting reprojection basis approaching the Legendre polynomials which are orthogonal under the limiting weight $w_G^{1/2} = \chi_{(-1,1)}$. Although the optimal robust Gibbs complement is not known, we propose the properly selected Freud polynomials to illustrate the advantages of robust Gibbs complements over the Gegenbauer polynomials. Unfortunately, although the Freud polynomials have been studied extensively since the early 1970's, [10], many of their properties are not known for general parameter n. As a consequence we are so far unable to determine the optimal parameters for the Freud weight, (3.13). Nevertheless, the values selected in (3.14) are numerically shown to satisfy the properties of a robust Gibbs complement, as displayed in Figure 3.2. The numerical examples in §4 illustrate that the Freud polynomials achieve exponential accuracy up to the discontinuities without suffering from the Runge phenomenon or significant round-off errors. It should also be noted that unlike Gegenbauer reconstruction, which requires function dependent parameter tuning, the Freud parameters (3.14) are function independent.

Although the Freud reprojection basis establishes the importance of using a robust Gibbs complement, a great deal of work remains in fully developing this idea. The following topics will be considered in future investigations:

- Ideally, we wish to determine the optimal robust Gibbs complementary basis in that the space P_M^n that is "most orthogonal" to $I - S_N$. If this can not be done explicitly, it would be useful either to determine the properties of the Freud polynomials necessary to rigorously prove the exponential convergence of the reprojection, or possibly to select another basis which allows such a rigorous proof. Such a result will not only further establish the Freud basis as an alternative for Gegenbauer post-processing, but should also allow for the optimal selection of the weight parameters as a function of the number of terms in the given (pseudo-)spectral projection, N.

- It is important to ensure the near orthogonality of $I - I_N$ and P_M^n (second requirement) even when w^n approaches a weight whose space spanned by P_M^∞ is not exponentially orthogonal to $I - I_N$. In particular, we know from [18] and subsequent papers that the Legendre polynomials, to which our polynomials approach in limit, do not constitute a basis that satisfies this requirement. However, by appropriate selection of M as a function of N the second Gibbs complement property can be maintained. With a further understanding of the particular reprojection basis, the precise behavior of M can be established.
- As proposed in [17], the most optimal Gibbs complementary basis may not consist of polynomials. Hence it would be useful to explore the construction a robust Gibbs complement that may not be composed of polynomials.
- The Gegenbauer reconstruction method has also been developed when the original projection basis consists of orthogonal polynomials, specifically Legendre, Chebyshev, and general Gegenbauer polynomial bases. In addition the method has been utilized for spherical harmonics in two dimensions. Robust Gibbs complements should similarly be developed for these commonly used global projections in addition to the Fourier (pseudo-)spectral basis discussed here.
- Finally, we wish to study the application of the Freud reprojection basis to various scientific disciplines. In particular, the Gegenbauer reconstruction method has been successfully applied in a number of areas, including medical imaging and the post-processing of numerical hyperbolic partial differential equations that admit solutions with shocks. Having established several significant advantages of the Freud robust Gibbs complement, we will pursue its effective implementation for various applications.

Appendix A. Gevrey weight functions and root exponential decay. Below we briefly sketch an argument to show that infinitely differentiable compactly supported weight functions, discussed in §3.2.2 will yield a reprojection basis that provides root exponential accuracy. A more detailed analysis can be found in [22, 23].

Gevrey regular functions are a class of compactly supported infinitely differentiable functions, classified in terms of the growth rate of their derivatives. Specifically, a function $\psi(\cdot)$ is Gevrey order alpha is equivalent to the statement

$$\|\psi^{(s)}\|_{L^\infty} \leq Const \cdot \eta^{-s}(s!)^\alpha$$

for some $\eta > 0$ and $\alpha \geq 1$. With this bound, it is straightforward to show that the the Fourier coefficients of a function with Gevrey regular periodic extension decay at the root exponential rate. We sketch the technique for this here.

Consider a Gevrey alpha regular function, $\psi(x)$, compactly supported in $[-1, 1]$. We apply integration by parts s times to its Fourier coefficient (2.1) to yield

$$\hat{f}_k = 2^{-1}(-i\pi k)^{-s} \int_{-1}^{1} \psi^{(s)}(x) \exp(-i\pi kx) dx.$$

Taking the absolute value of each side and passing it inside the integral we obtain

$$|\hat{f}_k| \leq (\pi|k|)^{-s} \|\psi^{(s)}\|_{L^\infty},$$

which is valid for any s. By substituting in the Gevrey regularity bound and using Sterling's inequality,

$$s! \leq Const \cdot \sqrt{s} \left(\frac{s}{e}\right)^s,$$

we have

$$|\hat{f}_k| \le Const \cdot s^{\frac{\alpha}{2}} (\frac{s^\alpha}{e^\alpha \pi \eta |k|})^s.$$

Since this bound is valid for all s, we can arrive at the nearly smallest bound by minimizing the dominant term, $(s^\alpha / e^\alpha \pi \eta |k|)^s$ over s. The resulting minimum bound is

$$|\hat{f}_k| \le Const\sqrt{|k|}\exp(-\alpha(\pi\eta|k|)^{1/\alpha}), \qquad s^\alpha_{min} = \pi\eta|k|, \tag{A.1}$$

illustrating the root exponential decay. The case of true exponential decay, $\alpha = 1$, corresponds to analytic functions which necessarily can not be compactly supported.

Appendix B. The Stieltjes algorithm for computing the Freud reprojection basis. Below we present the Stieltjes algorithm for computing polynomial orthogonal under the discrete quadrature

$$< f, g >_{w^n} := \sum_\nu f(t_\nu) g(t_\nu) w^n(t_\nu) \tag{B.1}$$

where t_ν is a finite stencil on $[-1, 1]$. Here we consider only the case of an even weight $w^n(\cdot)$, which simplifies the three term recursion relationship for the orthogonal polynomials to

$$\Psi^n_{k+1}(t) = t\Psi^n_k(t) - \beta^n_k \Psi^n_{k-1}(t).$$

Applying the inner product for the orthogonal polynomials results in the formula for the recursion coefficient,

$$\beta^n_k := \frac{< \Psi^n_k, \Psi^n_k >_{w^n}}{< \Psi^n_{k-1}, \Psi^n_{k-1} >_{w^n}}. \tag{B.2}$$

The Stieltjes algorithm for computing a family of orthogonal polynomials on a fixed stencil $\{t_\nu\}_\nu$ begins with the base polynomials $\Psi_{-1}(t) := 0$ and $\Psi_0(t) := 1$ and computes the first recursion coefficient β_0. This coefficient is used to compute the values of the next orthogonal polynomial on the stencil t_ν, i.e., $\Psi_1(t_\nu)$. The procedure is repeated inductively to compute the desired number of recursion coefficients $\{\beta_k\}^{M-1}_{k=0}$. A more comprehensive discussion of the Stieltjes and other algorithms for computing orthogonal polynomials is given in [11].

When given $S_N f(\cdot)$ (2.1), the Freud orthogonal polynomials are computed on the mesh $t_\nu := \frac{\nu}{2N}$, where $\nu = -2N, -2N+1, \ldots, 2N-1$. Alternatively, when given the equidistant sampled function values $f(x_j)$ for $x_j = -1 + \frac{j}{2N}$, $j = 0, \cdots 2N-1$, the Freud orthogonal polynomials are computed on the same stencil. We note again that although the mesh is course given the N term (pseudo-)spectral information, a very accurate approximation can be recovered due to the exponential accuracy of the trapezoidal quadrature formula for smooth periodic functions, e.g. [4, 14]. We further note that while the polynomials generated from the Sieltjes algorithm are orthogonal under the discrete inner product (B.1), as N increases the recursion coefficients (B.2) approach those of the family of polynomials which are orthogonal under the continuous inner product. Consequently the generated orthogonal polynomials approach those which are orthogonal under the continuous inner product, [11].

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REFERENCES

[1] M. Abramovitz and I. Stegun, Handbook of Mathematical Functions, Dover Publications, New York, 1965.

[2] H. Bateman, Higher Transcendental Functions, Vol.2, McGraw-Hill, New York, 1953.

[3] J.P. Boyd, Trouble with Gegenbauer Reconstruction for Defeating Gibbs' Phenomenon: Runge Phenomenon in the Diagonal Limit of Gegenbauer Polynomial Approximations, J. of Comp. Phys., in press, (2005).

[4] J.P. Boyd, Chebyshev and Fourier Spectral Methods, Second Edition, Dover Publications, Mineola, New York, 2001.

[5] O.P. Bruno, Fast, High-Order, High-Frequency Integral Methods for Computational Acoustics and Electromagnetics, Topics in Computational Wave Propagation Direct and Inverse Problems Series: Lecture Notes in Computational Science and Engineering, **31** (M. Ainsworth, P. Davies, D. Duncan, P. Martin, B. Rynne, eds.), Springer, 2003, 43-83.

[6] T.A. Driscoll and B. Fornberg, A Padé-Based Algorithm for Overcoming the Gibbs' Phenomenon, Num. Alg. **26** (2001) 77-92.

[7] K.S. Eckhoff, On a High Order Numerical Method for Functions with Singularities, Math. Comp. **67(223)** (1998) 1063-1087.

[8] G. Folland, Real Analysis: Modern Techniques and Their Applications, John Wiley & Sons, New York, 1999.

[9] G. Freud Orthogonal Polynomials, Pergamon press, Oxford, 1971.

[10] G. Freud, On polynomial approximation with the weight $\exp(-x^{2k}/2)$, Acta Math. Acad. Sci. Hungar. **24** (1973) 363-371.

[11] W. Gautschi, Orthogonal Polynomials: Computation and Approximation, Numerical Mathematics and Scientific Computation Series, Oxford University Press, 2004.

[12] A. Gelb, Parameter Optimization and Reduction of Round Off Error for the Gegenbauer Reconstruction Method, J. Sci. Comp. **20(3):** (2004) 433-459,

[13] A. Gelb, and Z. Jackiewicz, *Determining Analyticity for Parameter Optimization of the Gegenbauer Reconstruction Method,* SIAM J. Sci. Comput. submitted (2004).

[14] D. Gottlieb and S. Orszag, Numerical analysis of spectral methods: theory and applications, CBMS-NSF Regional Conference Series in Applied Mathematics, 26, SIAM, 1977.

[15] D. Gottlieb and C.-W. Shu, Resolution properties of the Fourier method for discontinuous waves, Comput. Meth. Appl. Mech. Engin. **116** (1994) 27-37.

[16] D. Gottlieb and C.-W. Shu, On the Gibbs phenomenon and its Resolution, SIAM Review, **30** (1997) 644-668.

[17] D. Gottlieb and C.W. Shu, A general theory for the resolution of the Gibbs phenomenon, Atti dei Convegni Lincei **147** (1998) 39-48.

[18] D. Gottlieb, C.-W. Shu, A. Solomonoff, and H. Vandeven, On the Gibbs Phenomenon I: Recovering Exponential Accuracy from the Fourier Partial Sum of a Nonperiodic Analytic Function, J. of Comp. and Appl. Math. **43** (1992) 81-98.

[19] I. Gradshteyn and I. Ryzhik, Table of Integrals, Series, and Products, Academic Press, 2000.

[20] Z. Jackiewicz, Determination of optimal parameters for the Chebyshev–Gegenbauer reconstruction method, SIAM J. Sci. Comput. **25** (2004) 1187-1198.

[21] J.-H. Jung and B.D. Shizgal, Generalization of the inverse polynomial reconstruction method in the resolution of the Gibbs' phenomenon, J. Comput. Appl. Math. **172** (2004) 131-151.

[22] F. John, Partial Differential Equations, 4th ed., Springer-Verlag, New York, 1982.

[23] E. Tadmor and J. Tanner, Adaptive Mollifiers - High Resolution Recovery of Piecewise Smooth Data from its Spectral Information, J. Foundations of Comp. Math. **2** (2002) 155-189.

[24] J. Tanner, Optimal Filter and Mollifier for Piecewise Smooth Spectral Data, Math. Comp. submitted (2004).