Twisted $K$-theory 3

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(0.1) Notation: Fix a separable Hilbert space $\mathcal{H}$ and denote by $PU$ the projective unitary group of $\mathcal{H}$, by Fred the space of Fredholm endomorphisms of $\mathcal{H}$ (both in the norm topology, unless otherwise specified) and by $U_K$ the group of unitaries of the form $\text{Id} + K$, where $K$ is a compact operator. Recall that Fred is homotopy equivalent to $\mathbb{Z} \times BU(\infty)$ while $U(\infty) \sim U_K$, whereas $U$ is contractible; therefore, $PU$ has the homotopy type of $BU(1)$ and its conjugation action on Fred and $U_K$ corresponds to the tensor action of line bundles on the classifying space of $K$-theory. The ordinary $K^0$ groups of a space $X$ are the components of the space of continuous maps from $X$ to Fred, whereas $K^1$ is that of maps to $U_K$. (Of course, we can also define them as the homotopy groups of the space of maps from $X$ to Fred.) Twisted groups will be defined from sections of Fred-bundles over $X$.

When considering the equivariant theory under a compact Lie group $G$, we will assume that $\mathcal{H}$ contains every irreducible $G$-representation with infinite multiplicity; there is an induced $G$-action on the spaces Fred, $U(\infty)$ and $U_K$, and the equivariant $K$-groups are the homotopy groups of the equivariant mapping spaces.

1. Australian model for twistings and gerbes

(1.1) Twistings. Recall that we have defined twistings of $K$-theory as projective $\mathcal{H}$-bundles, or equivalently principal $PU$-bundles. For these purposes, we use the norm topology on $PU$, which makes it into a manifold; this is important for the differentiable model for the Chern character which we discuss in the next lecture. (Equivariant twistings require an adjustment to the topology.) For a twisting $\tau$ of $X$ represented by a $PU$-bundle $P \to X$, the groups $\tau K^0,1$ are then the homotopy groups of $PU$-equivariant maps from $P$ to Fred, respectively $U_K$. A quasi-isomorphism of twistings, by which we mean an inclusion of projective bundles, induces an isomorphism between twisted $K$-groups; so therefore does a weak equivalence of twistings, which is a zig-zag of quasi-isomorphisms and their formal inverses. This way, twisted $K$-theory, defined functorially on the category of spaces with twistings, descends to the homotopy category of spaces and twistings up to weak equivalence. The homotopy category of twistings over a given space $X$, with weak equivalences added is a groupoid with set of objects in bijection with $H^3(X;\mathbb{Z})$, given by the characteristic class (Dixmier-Douady invariant) of a twisting, and where each object has the group $H^2(X;\mathbb{Z})$ as a set of automorphisms.

The analogous result in the equivariant setting is not-completely-obvious: the difficult part is that every equivariant class in $H^3_G(X;\mathbb{Z})$ is represented by an equivariant $P\mathcal{H}$-bundle, where the Hilbert space $\mathcal{H}$ is now assumed to contain every representation of $H$ infinitely many times. In fact, such bundles exist in the norm topology on $PU$; as a result we could not use the norm topology on Fredholms either (but $U_K$ would be all right); the needed adjustment are discussed by Atiyah and Segal [AS].

(1.2) The gerbe model. This smaller model replaces the $PU$-bundle with a 1-cocycle valued in line bundles (the Picard group). Strictly speaking, we must use hermitian line bundles to obtain the counterpart of the $P\mathcal{H}$ notion above, without that we get the analogue of $PGL$-bundles; but over a space $X$ a compatible Hermitian structure can always be found, and the space of choices is contractible. (The same applies in the presence of a compact group action.) Given a Čech cover $\coprod U_i \to X$, the data comprises a line bundle $L_{ij}$ on each overlap $U_i \cap U_j$, a family of isomorphisms $\varphi_{ijk} : L_{ij} \otimes L_{jk} \to L_{ik}$ on triple overlaps, with a coherence condition $\varphi_{ijk} \varphi_{ijl}^{-1} \varphi_{ikl} \varphi_{jkl}^{-1} = 1$ on
quadruple overlaps. An equivalence of such cocycles is induced in an obvious way from a family of line bundles $\Lambda_i$ on the $U_i$. It is an exercise to show that the resulting groupoid of cocycles has $H^3(X;\mathbb{Z})$ as isomorphism classes and $H^2(X;\mathbb{Z})$ as automorphisms, and in this case the equivariant analogue is easy, from the exponential isomorphism $H^2_G(X;\mathbb{C}^\times) \cong H^3_G(X;\mathbb{Z})$ and the Čech construction of these groups.

(1.3) The Australian models. A model for twistings over a space $X$ which simultaneously generalises $PU$-bundles and Picard-valued cocycles are the bundle gerbes of [BCMPS]. In the language of stacks, they are central extensions of (a groupoid locally equivalent to) $X$; this description makes immediate the extension to the equivariant theory (or even to stacks in the topological category), as we only need to use the action groupoid $G \times X \rightrightarrows X$ in lieu of the space $X$, holding for a compact group $G$. One benefit is the immediate generalisation to topological stacks which are locally, but not globally, quotient stacks; see the appendix of [FHT1]. This is not merely for generality’s sake; the example of $G$ acting on itself by conjugation can be enhanced to include a certain circle action, at which point the stack which is secretly being considered is no longer a global quotient by a compact group, and this is relevant to elliptic cohomology.

Specifically, we are looking for a groupoid $X_\bullet := [X_1 \rightrightarrows X_0]$ equipped with a map $X_\bullet \rightarrow X$ which is a local equivalence. Concretely, that means that the map $\varepsilon : X_0 \rightarrow X$ has local sections at every point and that $X_1 = X_0 \times X_0$. (The best setting here is the category of sheaves over the site of topological spaces, and the map to $X$ is then an equivalence on stalks.) The two projections $p_{0,1}$ to $X$ are also denoted $s,t$ (source and target). In most differential-geometric applications, we expect that $\varepsilon$ should be a submersion. A central $U(1)$-extension of $X_\bullet$ is a groupoid $\tilde{X}_1 \rightrightarrows X_0$, where $\tilde{X}_1 \rightarrow X_1$ is a principal $U(1)$-bundle compatible with the groupoid structure. The same information is encoded in a multiplicative hermitian line bundle $L \rightarrow X_1$, multiplicativity meaning an isomorphism of line bundles $p_1^*L \rightarrow p_0^*L \otimes p_2^*L,$ for the three maps $p_{0,1,2} : X_2 \rightarrow X_1$, where $X_2 := X_1 \times_{X_0} X_1$. (The map $p_1$ is the composition law in the groupoid.) This multipication isomorphism must satisfy an associativity constraint over the next space $X_3$ in the simplicial continuation of the groupoid, which I leave to the reader to write out.

Quasi-isomorphisms in the category of twistings are equivalences of groupoids $Y_\bullet \rightarrow X_\bullet$ over $X$ with compatible isomorphisms of the central extensions $\tilde{Y}_1, \tilde{X}_1$ and they generate the weak equivalences of twistings. By using Čech charts, one can show that we obtain a groupoid of twistings over $X$ equivalent to the one above.

From a principal $PU$-bundle $\mathcal{P} \rightarrow X$, we obtain an Australian model in the guise of the action groupoid $U \times \mathcal{P} \rightrightarrows \mathcal{P}$. A hermitian Čech-Picard cocycle gives rise to the groupoid $\coprod U_i \cap U_j \rightrightarrows \coprod U_i$ over $X$, whose central extension is given by the unit circle bundles in the $L_{ij}$.

Given an Australian model, a twisted $K^0$-class would like to be a vector bundle $E_\bullet := E_1 \rightrightarrows E_0$ over $\tilde{X}_1 \rightrightarrows X_0$, $U(1)$-equivariant with respect to the center and the natural $U(1)$-action on the fibers. Equivalently, that is determined by the bundle $E_0 \rightarrow X_0$ and a multiplicative isomorphism between pull-backs, $L \otimes s^*E_0 \sim t^*E_0$ over $X_1$; multiplicativity is checked on $X_2$. However, there will be no such vector bundles in general, when the Dixmier-Douady class of the extension is not torsion: all twisted $K^0$ classes have virtual dimension zero in that case. Instead, $E_0$ must be replaced by a Fredholm endomorphism $F : H_0 \rightarrow H_0$, where $H \rightarrow X_0$ is a Hilbert bundle with the same structure as described for $E_0$. There is a more general construction with Fredholm complexes on more general topological vector bundles, but the analysis is easier to control in the Hilbert space setting. So the softer model of twistings does not on its own avoid the infinite-dimensional model for twisted $K$-theory.

A finite-dimensional model, with a generalisation of the notion of vector bundle is offered in [G].

(1.4) Connections. A nice feature of the differentiable setting is the de Rham model of differential forms for cohomology (generalised to the Cartan model, in the equivariant setting), and the Chern-Weil construction of characteristic classes by means of connections on vector bundles. We will review these in the next lecture; here we just show how connections can be added in the Australian model. We assume that the map $X_0 \rightarrow X_1$ is a submersion, and that the groupoid is locally
equivale to the quotient groupoid of a manifold by a compact Lie group.

1.5 Proposition.

(i) A multiplicative hermitian line bundle \( L \to X_1 \) admits a multiplicative connection.

(ii) A \( U(1) \)-equivariant vector bundle \( E \) over \( \tilde{X}_1 \to X_0 \) admits a connection compatible with the connection on \( L \) in isomorphism \( L \otimes s^\ast E_0 \to t^\ast E_0 \) over \( X_1 \).

Proof. The case of the groupoid obtained by a Lie group acting on a point is interesting. In that case, the multiplicative line bundle \( L \to G \) is produced from the \( U(1) \) central extension \( \tilde{G} \). A multiplicative connection on it is determined by its value over the identity, that is, a linear splitting of the Lie algebra sequence

\[ \mathbb{R} \to \tilde{g} \to g \]

by using the left action of the group to transport that splitting everywhere. However, we could have used the right splitting instead, which amounts to saying that the splitting is Ad-invariant. In other words, we need a Lie algebra splitting. Of course these always exist for \( U(1) \)-extensions coming from a compact Lie group; we have \( H^2_G(C^\infty(U(1))) \cong H^3(BG; B\mathbb{Z}) \) by the exponential sequence, and the last group is torsion, so the class comes from some roots-of-unity subgroup.

I leave it to the reader to generalise this to a smooth groupoid in the following form: a multiplicative connection on \( L \) is equivalent to a splitting of the central extension \( T\tilde{X}_1 \to TX_1 \) of Lie algebroids defined by the tangent spaces along the identity \( X_0 \subset X_1 \). By \( C^\infty \) partitions of unity, this is a local argument, and locally we are reduced to the case of a Lie group and its central extension acting on a space; and the extension splits for the reason above.

Finally, placing a compatible connection on \( E \) is a cohomology vanishing argument exploiting the softness of the sheaf of connections. We can choose a generic connection on \( X_0 \). While it will fail to satisfy multiplicativity over \( X_1 \), the difference will give a 1-cocycle on the stack \( X \) with coefficients in the sheaf of \( \text{End}(E) \)-valued 1-forms. Because of the local presentation as manifold over compact group, cohomology with coefficients in this vanishes, and the correcting 1-coboundary lets us build the matching multiplicative connection on \( E \).

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(1.6) General properties of twisted \( K \)-theory. We assemble some general facts which underlie the first computational technique for twisted \( K \)-groups, the Mayer-Vietoris sequence.

1.7 Theorem (Properties of twisted \( K \)).

(i) Twisted \( K \)-theory, \( (X, \tau) \to \tau K^{0,1}(X) \) is a homotopy functor from the category of (topological spaces with twistings) to graded abelian groups.

(ii) An isomorphism \( \tau K^{0,1}(X) \cong K^{0,1}(X) \) is induced by a trivialisation of the twisting \( \tau \), that is, an isomorphism with the zero twisting. Two trivialisations differ by a class in \( H^2 \), and the said isomorphisms will differ by tensoring with the line bundle with that Chern class.

(iii) \( \tau K^{0,1}(X) \) is, functorially, a graded module over \( K^*(X) \).

(iv) For a closed \( A \subset X \) and a trivialisation of \( \tau \) over \( A \), define

\[ \tau K^*(X;A) := \ker(\tau K^*(X/A) \to K^0(A/A) \cong \mathbb{Z}) \]

Then, the twisted \( K \)-groups satisfy the Eilenberg-Steenrod axioms for generalised cohomology on the category of CW pairs \( (X,A) \) equipped with twistings trivialised over \( A \).

(v) There is a natural cup-product \( \tau K^*(X) \otimes \tau' K^*(X) \to \tau + \tau' K^*(X) \).

1.8 Remark.

(i) Homotopy functor means that homotopic maps induce identical group homomorphisms. We need, of course, a homotopy of maps between spaces with twistings.
having identified $K$ leads to the exact sequence

$$Z \oplus \mathbb{Z}(L - 1) \to nK^1(S^3) \to 0;$$

the matrix $M$ is determined by the restriction maps for which we have to track the trivialisations of the twisting: if we choose the first summand $K^0(U)$ to restrict to the trivial line bundle, then the generator of the second summand $K^0(V)$ must restrict to $L^\otimes n = (1 + (L - 1))^\otimes n = 1 + n(L - 1);$ so we have

$$M = \begin{bmatrix} 1 & 1 \\ 0 & n \end{bmatrix},$$

and conclude that $nK^0(S^3) = 0$ and $nK^1(S^3) = \mathbb{Z}/(n)$.

(ii) In the projective model, a trivialisation of the twistings gives an ‘unprojectivisation’ of the projective Hilbert bundle $\mathcal{P} \to X$ and we are led to the standard model of untwisted $K$-theory using the space of maps to Fredholm operators. Two unprojectivisations differ by tensoring with a line bundle $L \to X$, which acts on $K$-theory classes in the obvious way (the kernel and cokernel of the Fredholm operator get tensored with $L$).

(v) The cup-product is induced by the tensor product of (projective) Fredholm complexes; it should be clear how to define that, using the Hilbert space tensor product.

2. Mayer-Vietoris examples

Because of 2-periodicity of the groups, the Mayer-Vietoris sequence in (twisted) $K$-theory is always a cyclic 6-term sequence. We compute some examples.

(2.1) The 3-sphere $S^3$. Choose a twisting $n \in \mathbb{Z} \cong H^3(S^3; \mathbb{Z})$. Let $U, V$ be the two (small open neighbourhoods of the) hemispheres, intersecting along (an open annulus around) the equatorial 2-sphere. We can trivialise the twisting $n$ on $U$ and $V$ separately, but of course the difference of trivialisations gives a 2-cocycle on $U \cap V$ representing $n$ times the area. This is even more obvious in the Picard gerbe model, where the line bundle with Chern class $n$ on the equatorial $S^2$ represents the twisting $n$ on $S^3$. The MV sequence is

$$nK^0(S^3) \to K^0(U) \oplus K^0(V) \to K^0(S^2) \to K^1(S^2) \to K^1(U) \oplus K^1(V) \to nK^1(S^3)$$

The bottom left and middle $K^1$ groups vanish, whereas the middle top group is $\mathbb{Z} \oplus \mathbb{Z}$ and the top right is $\mathbb{Z} \oplus \mathbb{Z}(L - 1)$, where $L \to S^2$ is the degree one line bundle, satisfying $(L - 1)^2 = 0$. This leads to the exact sequence

$$0 \to nK^0(S^3) \to \mathbb{Z}^2 \xrightarrow{M} \mathbb{Z} \oplus \mathbb{Z}(L - 1) \to nK^1(S^3) \to 0;$$

the matrix $M$ is determined by the restriction maps for which we have to track the trivialisations of the twisting: if we choose the first summand $K^0(U)$ to restrict to the trivial line bundle, then the generator of the second summand $K^0(V)$ must restrict to $L^\otimes n = (1 + (L - 1))^\otimes n = 1 + n(L - 1);$ so we have

$$M = \begin{bmatrix} 1 & 1 \\ 0 & n \end{bmatrix},$$

and conclude that $nK^0(S^3) = 0$ and $nK^1(S^3) = \mathbb{Z}/(n)$.

(2.2) The product $S^1 \times S^2$. Again, $H^3 \cong \mathbb{Z}$ and we choose a twisting $n$. We could cut the 2-sphere into two disks, but instead we make a single cut on the circle to obtain an interval and obtain a self-gluing version of the MV sequence

$$nK^0(S^1 \times S^2) \to K^0(S^2) \to K^0(S^2) \to K^1(S^2) \to K^1(S^2) \to nK^1(S^1 \times S^2)$$

having identified $K^\ast([0, 1] \times S^2)$ with $K^\ast(S^2)$. Again this simplifies to

$$0 \to nK^0(S^1 \times S^2) \to \mathbb{Z} \oplus \mathbb{Z}(L - 1) \xrightarrow{M'} \mathbb{Z} \oplus \mathbb{Z}(L - 1) \to nK^1(S^1 \times S^2) \to 0$$
and the key is to identify $M'$, which is the difference $r_1 - r_0$ of the two restrictions at the endpoints 0, 1. We can choose $r_0$ to be the identity, but then $r_1$ is tensoring with $L^\otimes n = 1 + n(L - 1)$, so $r_1 - r_0 = n(L - 1)\otimes$ and

$$M' = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix};$$

this gives $^nK^0 \cong \mathbb{Z}$ and $^nK^1 \cong \mathbb{Z} \oplus \mathbb{Z}/(n)$. If you wish to be more specific, the groups are really $\mathbb{Z}(L - 1)$ and $\mathbb{Z} \cdot 1 \oplus \mathbb{Z}/(n) \cdot (L - 1)$, where we cup the respective generators of $K^0(S^2)$ with the basis of $K^*(S^1)$ in the Künneth isomorphism.

(2.3) The product $S^1 \times \mathbb{C}P^\infty$. We repeat the calculation above for a twisting $n \in H^3(S^1 \times \mathbb{C}P^\infty) \cong \mathbb{Z}$, but note that $K^0(\mathbb{C}P^\infty) \cong \mathbb{Z}[(L - 1)]$, from the computation $\lim K^0(\mathbb{C}P^N) = \lim_{N} \mathbb{Z}[L]/(L - 1)^{N + 1}$. ($K^1 = 0$.) Let $\alpha := (L - 1)$; we get

$$0 \to ^nK^0(S^1 \times \mathbb{C}P^\infty) \to \mathbb{Z}[\alpha] \xrightarrow{(L^\otimes n - 1)\otimes} \mathbb{Z}[\alpha] \to ^nK^1(S^1 \times \mathbb{C}P^\infty) \to 0$$

giving $^nK^0 = 0$, but $K^1$ is not immediately obvious to compute integrally (see Remark 2.6). But if we invert $n$ in the coefficients, then

$$L^\otimes n - 1 = (1 + \alpha)^n - 1 = \alpha (n + O(\alpha))$$

generates the ideal $(\alpha)$ in $\mathbb{Z}[\alpha]_n$ and the quotient $^nK^1 \cong \mathbb{Z}[\alpha]_n$.

(2.4) The equivariant twisted $^nK^*_S(S^1)$. Recall that $\mathbb{C}P^\infty$ is the classifying space of $S^1$: we replace that by the ‘stack’ $BS^1$, which means that we compute the equivariant $K$-groups of $S^1$ acting trivially on itself. The answer is cleaner than above. Indeed, $K^0_S(pt) = \mathbb{Z}[L^\pm]$, the Laurent polynomial ring in the standard representation $L$, and $K^1 = 0$. The sequence now becomes

$$0 \to ^nK^0_S(S^1) \to \mathbb{Z}[L^\pm] \xrightarrow{(L^\otimes n - 1)\otimes} \mathbb{Z}[L^\pm] \to ^nK^1_S(S^1) \to 0$$

and again $^nK^0 = 0$, but $K^1$ is now the free abelian group $\mathbb{Z}[L]/(L^n - 1)$ based on the $n$th roots of unity.

2.6 Remark. The completion theorem of Atiyah and Segal relates the answers to the computations of §2.3 and §2.4: specifically, the former is the formal completion of the latter at the augmentation ideal $(L - 1)$. When $n$ is a prime, the integral answer to §2.3 is $K^1 \cong \mathbb{Z} \oplus \mathbb{Z}/(n) \mathbb{Z}$, the fibered sum of $\mathbb{Z}$ and its $n$-completion $\mathbb{Z}_n$.

(2.7) Generalisation of §2.4 to a torus $T$. We replace $S^1$ with a torus $T$ acting trivially upon itself. Let $\lambda := \text{Hom}(T; S^1)$ be the character lattice, and choose a twisting class $[\tau] \in \Lambda \otimes \Lambda \cong H^2(T) \otimes H^1(T) \subset H^2_T(T)$ which is rationally non-degenerate. We have $K^0_T(pt) = \mathbb{Z}(\Lambda)$, the group ring. When $\tau$ is a product class coming from a decomposition of $T$ into circle factors, we can compute $\tau K_T(T)$ by using the Künneth theorem and invoking §2.4. In general, we need to cut up the torus $T$ into a cube using a basis $\xi_k$ of $H^1(T)$, to trivialise the twisting, and the exact sequence (2.5) is replaced by an Atiyah-Hirzebruch spectral sequence with $E_2$ page the sequence

$$\mathbb{Z}(\Lambda) \to \mathbb{Z}(\Lambda) \otimes \Lambda \to \mathbb{Z}(\Lambda) \otimes \Lambda^2 \to \cdots \to \mathbb{Z}(\Lambda) \otimes \Lambda^{\text{top}} \Lambda$$

and second differential $\sum_k \xi_k \otimes (\xi_k \triangleright \tau)$. This complex computes the group homology of $\pi_1(T)$ acting on $\mathbb{Z}(\Lambda)$ by translation mediated by $\tau : \pi_1(T) \to H^2(BT)$, and is confined to the top degree where it computes the free abelian group on $\Lambda/\pi_1(T) \triangleright \tau$. This makes the spectral sequence collapse after the second page, and computes $\tau K_T(T)$ as a $K_T(pt)$-module isomorphic to $K_F(pt)$, where $F \subset T$ is the subgroup which is the kernel of the isogeny $T \to T^\tau$ defined from $\tau$. 

5
References


