We now review another computational technique for twisted $K$-theory, the Chern character, which reduces it to cohomology calculations. The price to pay is the loss of torsion, and in fact a concrete model lands in (twisted) de Rham cohomology, which takes real coefficients. The traditional (non-equivariant) Chern character is well-captured by Chern-Weil theory, and the twisted version was successfully developed in [MS]. The equivariant theory is more complicated and the standard approach loses information; a ‘global’ version of the Chern character, which restores missing information for compact Lie groups, was attempted in [BBM], developed in [BG] using Hochschild techniques and then more explicitly in [R]. A twisted version was worked out in [FHT].

1. The Chern character in ordinary $K$-theory

(1.1) Construction via the splitting principle. Recall that for a finite CW complex $X$, we can define a ring isomorphism

$$ch : K^{0/1} \otimes \mathbb{Z} \mathbb{Q} \rightarrow H^{ev/odd}(X; \mathbb{Q})$$

(1.2)

by declaring that $ch(L) = \exp(c_1(L))$ for line bundles $L$, and invoking the splitting principle. The latter says that, for any vector bundle $E \rightarrow X$, the pull-back along the projection $\pi : Fl(E) \rightarrow X$ in cohomology or $K$-theory is injective (with image which can be identified). Since $\pi^*E$ splits into a sum of line bundles over $Fl(E)$, we know from additivity what $ch(E)$ must be and one only needs to check multiplicativity and well-definedness of $ch$.

Unfortunately the splitting principle cannot be applied to twisted $K$-theory classes, as they are usually represented by infinite-dimensional bundles. There is a general nonsense argument that there must exist a twisted version of the Chern isomorphism,

$$\nu ch : \tau K(X) \otimes \mathbb{Z} \mathbb{Q} \rightarrow (\text{twisted version of}) \ H^{ev/odd}(X; \mathbb{Q}),$$

the only trouble is to identify the right-hand side and ideally construct the map explicitly.

It turns out that twisted cohomology has a very concrete de Rham model, and a twisted Chern character living there can be constructed by a variation of the Chern-Weil construction of $ch$, which we review next; but first, let us spell out the general nonsense.

The Chern isomorphism (1.2) asserts a homotopy equivalence between the rationalised ring spectra $(\mathbb{Z} \times BU) \otimes \mathbb{Z} \mathbb{Q}$ and the Eilenberg-MacLane spectrum $\prod_{n \geq 0} K(\mathbb{Q}; 2n)$. For a finite complex $X$, the groups $\tau K^{0/1}(X) \otimes \mathbb{Z} \mathbb{Q}$ are the homotopy groups of a bundle of spectra over $X$ with fibre $BU \otimes \mathbb{Z} \mathbb{Q}$. Using the equivalence, we can similarly construct the twisted bundle of spectra with fibre $\prod_{n \geq 0} K(\mathbb{Q}; 2n)$, which will define the twisted rational cohomology: locally, once the twisting has been trivialised, this is equivalent to ordinary rational cohomology. To understand the construction of this spectrum, note that in the gerbe model, the local copies of $\mathbb{Z} \times BU$ are glued together on overlaps $U_i \cap U_j$ by means of the tensor action of line bundles $L_{ij}$. Under $ch$, this goes over into the multiplication by cohomology classes of the form $\exp(\omega_{ij})$, for 2-cocycles $\omega_{ij}$.

Now the cocycle condition on the $\omega_{ij}$ ensures that we can write them as $\nu_j - \nu_i$ for some 2-cochains $\eta_i$ on the $U_i$. Furthermore, $\delta \nu_i = \delta \nu_j$ on overlaps, so the 3-cochains $\delta \nu_i$ assemble to a 3-cocycle $\eta$ on the space. Of course, this $\eta$ is a representative for the Dixmier-Douady invariant of the gerbe. We can straighten the copies of $\prod_{n \geq 0} C^{2n}(\mathbb{Q})$ in our cochain model for cohomology
multiplying them by \(\exp(\nu_i)\), and then their patchings over the \(U_{ij}\) become trivial. The price to pay was that the differential \(\delta\) in the complex has been conjugated to

\[
\exp(\nu_i) \circ \delta \circ \exp(-\nu_i) = \delta - \delta(\nu_i) \cup = \delta - \eta \cup,
\]

which gives the model for equivariant cohomology. See [FHT] for a detailed execution of the argument, but this is the gist of it.

**1.3 The Chern-Weil model.** For a rank \(r\) vector bundle \(E \to X\), we can choose a connection \(\nabla : C^\infty(X; E) \to \Omega^1(X; E)\). This extends by linearity to operators \(\Omega^p(E) \to \Omega^{p+1}(E)\) and satisfies \(\nabla^2 = F_E\). For some zeroth order operator \(F_E \in \Omega^2(X; \text{End}(E))\), the curvature of \(\nabla\). Chern-Weil theory assures us that, for any invariant polynomial \(\Phi\) on \(X\), which gives the model for equivariant cohomology. See [FHT] for a detailed execution of the

1.4 Theorem. The differential form \(\text{Tr} \exp(iF_E/2\pi)\) represents \(ch(E)\) in de Rham cohomology.

One proves it by noting that the formula is additive and multiplicative in \(E\) and represents \(\exp(c_1)\) on line bundles.

2. Equivariant Chern character and the Completion Theorem.

The equivariant case illuminates the close relation of the Chern character with the character of a group representation. More precisely, it is analogous to the formal germ of the group character near the identity of the group. This allows one to see immediately that

(i) The equivariant Chern character could (and in fact does) lose information

(ii) Recovering the information requires a global version, in particular, we need to understand its analogue near every point of the group.

The target space of the equivariant Chern character is the equivariant cohomology \(H^*_G(X; \mathbb{Q})\), which by definition is the cohomology of the homotopy quotient \(X_{hG} := (EG \times X)/G\). Here, \(EG\) is the universal principal \(G\)-bundle, a contractible space with free \(G\)-action. Because \(EG\) can be realised as a \((G\)-equivariant\) CW complex, a colimit of finite ones, we have an isomorphism

\[
ch : K^*(X_{hG}; \mathbb{Q}) \sim \to H^*(X_{hG}; \mathbb{Q})
\]

where we have agreed to collapse the grading mod 2 and define cohomology and \(K\)-theory with \(\mathbb{Q}\) coefficients by taking the limit of the rationalised theory over finite subcomplexes. (Caution, this will no longer agree with \(K(X_{hG}) \otimes_\mathbb{Z} \mathbb{Q}\).)

Now an equivariant vector bundle over \(X\) certainly descends to a bundle over \(X_{hG}\), and this assignment preserves sums and products, so we do get by composition a Chern character

\[
ch : K^*_G(X) \to K^*(X_{hG}; \mathbb{Q}) \sim \to H^*(X_{hG}; \mathbb{Q})
\]

but this map is not an isomorphism, even if we rationalise the coefficients in the first space. In fact, the relation between \(K^*_G(X)\) and \(K^*(X_{hG})\) is precisely captured by the
2.1 Theorem (Atiyah-Segal completion theorem). $K^*(X_{hG})$ is the completion of $K^*_G(X)$ at the augmentation ideal $I_G \subset R_G$.

This augmentation ideal is the kernel of the dimension homomorphism $R_G \to \mathbb{Z}$, and we use the $R_G$-module structure of $K^*_G(X)$ (by tensoring with representations of $G$) in the statement.

2.2 Remark. After complexifying passing to characters, $R_G$ becomes the ring of polynomial functions on the space $Q := G_C//G_C$ of conjugacy classes of $G_C$, and $I_G$ becomes the defining ideal of the identity class. So we are completing the $\mathbb{C}[Q]$-module $K^*_G(X; \mathbb{C})$ at the identity class.

2.3 Remark (Example: $G = \mathbb{S}^1$, $X = \text{pt.}$). Now, $BS^1 = \text{colim} \mathbb{C}P^N$ and so

$$K^0(BS^1) = \lim K^0(\mathbb{C}P^N) = \lim \mathbb{Z}[L]/(L-1)^{N+1} = \mathbb{Z}[[L-1]]$$

which agrees with the formal completion of $R_{S^1} = \mathbb{Z}[L^{\pm 1}]$ at $L = 1$.

(2.4) Equivariant Chern-Weil formula. The equivariant case requires us to agree on a model for equivariant de Rham cohomology. For this we can use the Cartan model, the complex $(\Omega^*(X)[g])^G$ of $G$-invariant power series on the Lie algebra $g$ with values in the differential forms on $X$. This complex carries the Cartan differential

$$d_C := d + \xi^a \otimes \iota(\xi_a)$$

(2.5)

in a basis $\xi_a$ of $g$ and the dual basis $\xi^a$ for linear functions on $g$, and with $\iota(\xi_a)$ designating contraction with the associated vector field. With $L_a$ denoting the Lie derivative and $\text{ad}^*_a$ the adjoint action on function on $g$, we have, from the Lie formula,

$$d_C^2 = \xi^a \otimes L_a = \xi^a \otimes L_a + \xi^a \circ \text{ad}^*_a$$

(the last term vanishes for symmetry reasons), and so $d_C^2 = 0$ on invariant forms. The cohomology of the Cartan complex is isomorphic to the real cohomology of the homotopy quotient $X_{hG}$.

A $G$-equivariant vector bundle $E \to X$ carries a $G$-invariant connection $\nabla$; one such can be obtained by averaging a general connection. As before, let $F_E$ be its curvature. A vector field $\xi$ of the $g$-action on $X$ carries two liftings to $E$:

(i) The lifting $L_\xi$ by the Lie group action on $E$

(ii) The lifting $\nabla_\xi$ by the connection

These lifts differ by a zeroth-order endomorphism $\mu(\xi) := L_\xi - \nabla_\xi \in C^\infty(X; \text{End}(E))$.

2.6 Definition. The equivariant curvature of $\nabla$ is $F_E - \mu$, an equivariant 2-form valued in the endomorphism bundle $\{\Omega^2(X; \text{End}(E)) \otimes C^\infty(X; \text{End}(E)) \otimes g^*\}^G$.

This is not a random definition. By analogy with (2.5), we define

$$\nabla_C := \nabla + \xi^a \otimes \iota(\xi_a); \quad \text{then,} \quad \nabla_C^2 = F_E + \xi^a \otimes \nabla_{\xi_a} = F_E - \mu + \xi^a \otimes L_a$$

and the vanishing as before of $\xi^a \otimes L_a$ on invariant sections of the endomorphism bundle shows that $\nabla_C^2 = F_E - \mu$ there. As before, one proves the Cartan-closure of the equivariant forms $\text{Tr} \{(F_E - \mu)^k\}$, and the independence of their cohomology classes on the connection.

The Cartan model of the equivariant Chern character is defined by $ch(E) := \text{Tr} \{\exp \frac{i}{2\pi}(F_E - \mu)\}$. Note that it gives a function on all of $g$, but not a polynomial function. It captures the class of $E$ in $K^*_G(X) \otimes \mathbb{C}$ localised near the identity of $G$, transported to $g$ via the exponential map $\xi \to \exp(\xi/2\pi i)$. 

3
3. The globalised Chern character

Equivariant $K$-theory $K_G(X)$ is a module over $R_G$. When complexifying, this is a sheaf over the space $Q := G_C//G_C$ of conjugacy classes, and the Chern character sees the neighborhood of the identity. This loss of information must be remedied by constructing an analogue near each (semi-simple) conjugacy class $q \in Q$. This can be done explicitly by means of the fixed-point theorems. I refer to [FHT], Part I for the proofs, which are quite easy in the untwisted case but get more involved as we twist.

(3.1) Finite groups. The case of finite groups is illuminating.

3.2 Theorem. [AS2] For a finite group $G$ acting on $X$,

$$K^*_G(X) \otimes \mathbb{C} \cong \bigoplus_{g \in G} H^*(X^g; \mathbb{C}) \cong \bigoplus_{q \in Q} H^*_{Z(g)}(X^g; \mathbb{C});$$

in the second sum $g$ is a representative of the conjugacy class $q$. The map sends a vector bundle $V \to X$ to

$$\bigoplus_{g \in G} \text{Tr}\left\{ g \cdot \exp \left( \frac{iF_V}{2\pi} \bigg|_{X^g} \right) \right\}$$

(3.3)

In the absence of a space, this is just the character of a representation. An equivalent formulation of the map is that, over each fixed-point set $X^g$, we decompose the bundle into eigenvalues by the fiberwise action and sum the Chern characters of its components weighted by the eigenvalues. The isomorphism requires, first, the statement that the restriction $K_G(X) \to K_{Z(g)}(X^g)$ is an isomorphism after complexification and localisation at the conjugacy class of $g$; this is tested on spaces $X$ of the form $G/H$, for all subgroups $H$. Then, the usual equivariant Chern character gives the result after eigenvalue decomposition of $V \to X^g$.

(3.4) Compact Lie groups. The second isomorphism is the one that generalises well. For each semi-simple conjugacy class $q \in Q$, there exists a minimal topologically cyclic subgroup $\langle g \rangle \subset G$ whose complexification contains $q$. Let, slightly abusively, $X^g$ denote the fixed-point set under that subgroup, and $Z(g)$ the centraliser in $G$.

3.5 Theorem (See for instance [FHT]). For a compact Lie group $G$ acting on $X$, we have an identification of the formal completion

$$K^*_G(X; \mathbb{C})^\wedge_q \cong H^*_Z(X^g; \mathbb{C})$$

with the equivariant cohomology of the fixed-point set; the map is given by

$$V \mapsto \text{Tr}\left\{ g \cdot \exp \left( \frac{i(F_V - \mu Z)}{2\pi} \bigg|_{X^g} \right) \right\}$$

having used the equivariant Chern-Weil character with respect to $Z(g)$.

Alternatively, we could have used $g$ to decompose $V$ into eigenbundles over $X^g$ before applying the $Z(g)$-equivariant Chern-Weil character.

Note that when $g = \exp(\xi/2\pi i)$ is near the identity, we can include $\xi$ in the exponential by adding it to $\mu$, and we recover the restriction to $X^g$ of the same formula at $g = 1$. In other words, the localised formulas patch together over the group.

4. The twisted case

We follow [MS], to which we refer for details, for the twisted Chern-Weil theory.

---

1 A topologically cyclic group is a product of a torus and a finite cyclic group.
(4.1) Twisted cohomology and the twisted Chern character. We define a model for the twisted cohomology of a manifold $X$, given a closed 3-form $\eta$, from the de Rham complex $\Omega^*(X)$ but with modified differential $d + \eta \wedge$. Strictly speaking, a better model arises by adjoining a formal element $\beta$ of degree $-2$, the Bott element, and working in $\Omega^*(X)(\langle \beta \rangle)$ with differential $d + \beta \eta \wedge$; this makes it $\mathbb{Z}$-graded and 2-periodic.

Let now $\tilde{X}_1 \Rightarrow X_0$ be a bundle gerbe model for our twisting, as in Lecture 3. As in Lecture 3, §1.4 we place a multiplicative connection on $L$. The curvature $\omega$ of this connection must satisfy $p^*\omega + p_\pi \omega = p_\pi^* \omega$. Descent of differential forms implies the existence of a form $\phi \in \Omega^2(X_0)$ such that $\omega = t^*\phi - s^*\phi$. Choose such a $\phi$; it is determined up to the pull-back of some 2-form from $X$. We also know that $t^*d\phi = s^*d\phi$, which shows that $d\phi$ is the pull-back of a form $\eta$ on $X$. Tracing differentials in the spectral sequence which computes $H^3(X)$ from the groupoid shows that this $\eta$ represents the Dixmier-Douady invariant of the gerbe in de Rham cohomology.

Assume now that we have a multiplicative vector bundle representative $E_\tau$ for our favourite twisted $K$-theory class. (As mentioned earlier, we shall find no such finite-dimensional bundle in general). Placing a multiplicative connection $\nabla$ on $E$ with curvature $F_E$, we have the relation $t^*F_E = s^*F_E + \omega \cdot \text{Id}$. This means that

$$t^*(F_E - \phi \cdot \text{Id}) = s^*(F_E - \phi \cdot \text{Id})$$

in particular, this applies to all invariant polynomials in these forms, so all invariant polynomials applied to $F_E - \phi \cdot \text{Id}$ are pulled back from $X$. We define the twisted Chern character of $E$ as

$$\tau \text{ch}(E) := \exp(-\phi) \wedge \text{Tr} \exp \left( \frac{iF_E}{2\pi} \right)$$

(4.2)

and this satisfies $(d + \eta \wedge)^\tau \text{ch}(E) = 0$, because the Trace factor is closed (by the same arguments as in §3) and $d\phi = \eta$.

The construction must be modified, because we will not succeed in representing classes by finite-dimensional projective bundles, if the twisting is not defined by a torsion class. It is explained [MS] how any twisted $K^0$ class can instead be represented by a difference of projective bundles $E - E'$ which are ‘close enough’ in the sense that their difference is trace class. In that case, it is further shown that all operators $F^k_E - F^k_{E'}$ are also trace class, so the trace of the difference of exponentials in (4.2) is well-defined, and gives a differential form representative of $\tau \text{ch}$. [More details to add if I get a chance...]

(4.3) The twisted version for finite groups In adapting the Atiyah-Segal description [AS2] to a twisting $\tau \in H^3_G(X; \mathbb{Z})$, the obvious change is to replace cohomology by its twisted version, as in deed $\tau$ restricts to a class in $H^3_Z(g)(X^g; \mathbb{Z})$, where the twisted Chern character seems to land. But there is another change: the group element $g \in G$ acts only projectively on the fibres of $V$. Indeed, the $K$-theory twisting restricts over $X^g$ to a $U(1)$-central extension of the cyclic group $\langle g \rangle$. Such an extension of a cyclic group is trivialisable, but not canonically; trivialisations differ by a character $\langle g \rangle \rightarrow U(1)$. These form a discrete set; as a result, the lines over $g$ of these central extensions assemble to a $(Z(\langle g \rangle)$-equivariant) flat line bundle $L(g, \tau)$ to $X^g$. The trace in (3.3) is valued in this flat line bundle, and the (twisted) global Chern character takes values in $\tau$-twisted cohomology with coefficients in $L(g, \tau)$.

4.4 Theorem. [FHT] For a finite group $G$ acting on $X$ and a twisting $\tau \in H^3_G(X; \mathbb{Z})$,

$$\tau K^*_G(X) \otimes \mathbb{C} \cong \bigoplus_{g \in Q} \tau H^*_Z(g)(X^g; L(g, \tau)),$$

where $g$ is a representative of the conjugacy class $g$. The $g$-component of a vector bundle $V \rightarrow X$ is the sum of twisted Chern characters of its eigen-bundles under the (projective) action of $g$, weighted by eigenvalue.

To prove the theorem, one need not repeat the full argument of [AS2]. It suffices instead to construct the map from left to right, along the lines indicated above. Once this is done, we can
pass to a $G$-cover of $X$, where the twisting can be trivialised,\(^2\) and the original result shows the map to be an isomorphism there. The usual Mayer-Vietoris argument then shows it to be a global isomorphism.

(4.5) The case of compact groups. Recall the notation of §3.4. We will define a line bundle $\mathcal{L}(g, \tau)$ over the fixed-point set $X^g$ from the twisting $\tau$, as before; namely, we get a central extension of the topologically cyclic group $\langle g \rangle \subset G$ from the fibre-wise action of that group on the projective Hilbert bundle defining the twisting. This is even clearer in the bundle gerbe definition, which is a $U(1)$-central extension of the quotient groupoid $X^g : Z(g)$, and $\langle g \rangle$ is contained in the stabiliser of each point. Central extensions of compact groups are discrete, forming an affine space (‘torsor’) over the group of characters $\text{Hom}(\langle g \rangle; \mathbb{C}^\times)$, and as a result the fibres of these extension over the group element $g \subset \langle g \rangle_C$ assemble to a flat, $Z(g)$-equivariant line bundle over $X^g$.

It will be useful to know the isomorphism class of the line bundle. Because $\langle g \rangle$ acts trivially, the homotopy quotient of $X^g$ by $Z(g)$ fibers with fibre $B(g)$ over the homotopy quotient of $X^g$ by $Z(g)/\langle g \rangle$. The restricted class $[\tau] \in H^3_{Z(g)}(X^g; \mathbb{Z})$ has a leading term, for the Leray spectral sequence of this fibration, in $H^1_{Z(g)/\langle g \rangle}(X^g; H^2(B(g); \mathbb{Z}))$. Now the element $g$ gives a homomorphism

$$H^2(B(g); \mathbb{Z}) \to \mathbb{C}^\times$$

by evaluating a complexified character of $\langle g \rangle$ at $g$, and inserting it in the leading term of $[\tau]$ produces a class in $H^1_{Z(g)/\langle g \rangle}(X^g; \mathbb{C}^\times)$, describing the monodromy of $\mathcal{L}(g, \tau)$.

4.6 Theorem. [FHT] For a compact Lie group $G$ acting on $X$ and a twisting $\tau \in H^3_G(X; \mathbb{Z})$, we have an identification of the formal completion of twisted $K$-theory with the twisted equivariant cohomology of the fixed-point set

$$\tau K^*_q(X; \mathbb{C})_{\langle g \rangle}^\wedge \cong \tau H^*_Z(g)(X^g; \mathcal{L}(g, \tau)).$$

Decomposing $V = \bigoplus \alpha \, V_\alpha$ into $g$-eigenbundles, the map is given by

$$V \mapsto \sum \alpha \cdot \tau \text{ch}_{Z(g)}(X^g; V_\alpha)$$

where we use a (cocycle-level) model for the equivariant $\tau$-Chern character, weighted by the eigenvalues $\alpha$, which are valued in $\mathcal{L}(g, \tau)$, to obtain a cohomology class with line bundle coefficients.

References


\(^2\)We may need to pass to finite central extensions of $G$ also.