Equivariant elliptic cohomology and $K$-theory

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1. Background

Elliptic cohomology $Ell$ is a functor assigning to every elliptic curve a complex-oriented cohomology theory, namely the theory based on the formal group law of that curve. Away from the prime 2, the universal elliptic cohomology theory was constructed by the Conner-Floyd method [CF] using the elliptic genera; see [La] for an overview. For the nodal elliptic curve, $\mathbb{P}^1$ with one self-crossing, one thus recovers $K$-theory, $KO$-theory or $L$-theory (away from the prime 2), depending on the exact choice of bordism theory and elliptic genus. (The theories $KO$ and $L$ arise from real bordism and are isomorphic away from 2, save for different choices of universal Thom class.) This is consistent with the fact that the corresponding formal group is the multiplicative group.

(1.1) The Tate curve. When specialising to the Tate elliptic curve, the formal universal deformation of the nodal elliptic curve, one sees the $K$-theory of the loop space of $X$, equivariant for the loop rotation action and suitably completed, so that the ground ring is $\mathbb{Z}[[q^\pm]]$, rather than the representation ring $\mathbb{Z}[q^\pm]$ of the circle group. This is made precise, for instance, in [KM], and is a mathematical expression of Witten’s insight that elliptic genera of a manifold $X$ ought to represent rotation-equivariant indices of Dirac-style operators on the free loop space of $X$ [W].

The variable $q$, standing for the standard representation in circle-equivariant $K$-theory, represents the modulus of the elliptic curve and the standard Taylor expansion variable for modular forms, which form the coefficient ring of the universal theory. ‘Fixing’ the prime 2 required much more effort by Hopkins, Miller et. al, leading to the construction of $tmf$. The difficulty stemming from the fact that the moduli of elliptic curves is not an affine scheme, but a stack; so constructions from its ring of global functions cannot work.

(1.2) The equivariant case. Equivariant cohomology theory for compact Lie group actions on a space — that is, something like $Ell_G(X)$, rather than the elliptic cohomology $Ell(X_{hG})$ of the homotopy quotient — is not quite satisfactory, the nearest construction being the outline [Lu2]. When specialising to the Tate curve, the loop space philosophy predicts that one should discover the representation theory of loop groups; on the other hand, the $K$-theory perspective leads us to the $L_G$-equivariant $K$-theory, which in the case of a point we can interpret as $K_G(G)$ for the conjugation action of $G$ on itself. These two pictures should be matched by the FHT theorem which relates loop group representations to $K$-theory, but the naive attempt to do so fails, and the solution is provided by a modified construction of the circle-equivariant version of that same theorem. In this setting, the level of the loop group representation stems from a twisting of elliptic cohomology, which transgresses to the $K$-theory twisting.

2. Some $K$-theoretic group laws

Recall the Hirzebruch genera considered in the previous lecture:

(i) $Q(x) = \text{Td}(x) = \frac{x}{1 - \exp(-x)}$, leading to the Todd genus

(ii) $Q(x) = \hat{A}(x) = \frac{x/2}{\sinh(x/2)}$, leading to the $\hat{A}$-genus, computing the index of the Dirac operator on manifolds which admit a Spin structure

(iii) $Q(x) = L(x) = \frac{x}{\tanh(x/2)}$ gives the $L$-genus, computing the signature of oriented manifolds.

These examples are related to $K$-theory, and one should really land in $\mathbb{Q}[[\beta]]$ instead of $\mathbb{Q}$, with the Bott element of degree 2, and use $\beta^{-1}x$ instead of $x$. (The $\hat{A}$-genus need not be integer-valued on manifolds which do not have a spin structure, but it always lands in $\mathbb{Z}[1/2]$. The other genera
are integer-valued.) By the index theorem, we can see that the $L$-genus is the index of the Dirac
operator on the total Spin bundle $S := S^+ \oplus S^-$ of a manifold:

$$L(M) = \int_M \prod x_i \tanh(x_i/2) = \int_M \prod x_i \sinh(x_i/2) \cdot 2 \cosh(x_i/2)$$

and the additional factor $\prod 2 \cosh x_i/2$ is precisely $\text{ch}(S)$. Of course, $S^+$ only exist as projective
bundles, if the manifold does not have a spin structure, but the total bundle on which the Dirac
operator acts is the super vector bundle $(S^+ \oplus S^-) \otimes S$; the symbol $\otimes$ indicates that $S^-$ is in odd
degree. This bundle is well-defined on oriented manifolds, and can be identified with the bundle of

$$\prod \text{ and the additional factor } \phi \text{Series expansion would compute the values of the associated genera } KO \text{ from } (\text{Thom class}). \text{ Just like } \hat L \text{ would give the complex } K \text{-theory we recall that } \hat L^2 \text{, unless we restrict to vector bundles with Spin structure. In case that } \text{of orientation, the Atiyah-Bott-Shapiro Thom class } [ABS], \text{ which does not extend over the prime } 2. \text{ Inverting } 2, \text{ we have } MSO^* = (MU^*)^{2/2}, \text{ the fixed-point ring of complex conjugation on } MU^*, \text{ and using Landweber exactness away from } 2 \text{ defines (for finite cell complexes } X) \text{ the cohomology theories}

$$KO^*[1/2](X) = MSO^*(X) \otimes_{MSO} \mathbb{Z}[1/2], \quad L^*[1/2](X) = MSO^*(X) \otimes_{MSO} \mathbb{Z}[1/2]$$

As a side note, one can correctly define $KO^*$ integrally by the Conner-Floyd formula if we start form
the Spin bordism theory $MSpin$, on which the $\hat A$-genus takes integral values. Using $MU^*$ instead
would give the complex $K$-theory $K^*[1/2]$ instead of the real one $KO$, but with a different choice
of orientation, the Atiyah-Bott-Shapiro Thom class [ABS], which does not extend over the prime
2, unless we restrict to vector bundles with Spin structure. In case that $L$-theory is unfamiliar,
recall that $\hat A$ and $L$ differ by an invertible factor when 2 is a unit, so that $L^*$ and $KO^*$ represent
isomorphic cohomology theories away from the prime 2, albeit with different choices of orientation
(Thom class). Just like $KO$-theory, $L$-theory has a natural extension over the prime 2, but it differs
from $KO$ there: it is in fact ordinary cohomology with coefficients. (An indication of that comes
from the fact that its group law, unlike that of $KO$, is equivalent to the additive formal group law
at the prime 2: all denominators appearing in $L(x)$ are odd, so the exponential $x/L(x)$ of the group
law is integral in $\mathbb{Z}_2$.)

3. The LSO and Witten elliptic genera

The original Landweber-Stong-Ochanine construction of elliptic cohomology over $R^* := \mathbb{Z}[1/2][\delta, \varepsilon]$ with deg $\delta = 4, \text{deg } \varepsilon = 8$ uses the group law of the elliptic curve in Jacobi quartic form,

$$w^2 = R(z) = 1 - 2\delta z^2 + \varepsilon z^4$$

with origin at $z = 0, w = 1$ and formal coordinate $z$. The logarithm of the genus is the linear
coordinate on the elliptic curve,

$$g(z) = \int_0^z \frac{dt}{w} = \int_0^z \frac{dt}{\sqrt{R(t)}}$$

Series expansion would compute the values of the associated genera $\varphi_{LSO}(\mathbb{C}P^n)$. Evenness of $g(z)$
ensures the vanishing for odd projective spaces, but in any case the answer turns out to be cleaner
when expressed in terms of the rational generators $\mathbb{C}P^2, \mathbb{H}P^{n+1}$ of the oriented bordism ring:

$$\varphi_{LSO}(\mathbb{C}P^2) = \delta, \quad \varphi_{LSO}(\mathbb{H}P^n) = \begin{cases} \varepsilon^{n/2}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$
(3.1) Special values. At $\delta = \varepsilon = 1$, we get by integration $g(z) = \tanh^{-1}(z)$. The exponential of the group law is therefore $\tanh(x)$ and the associated Hirzebruch $Q$-series is $x/\tanh(x)$, the $L$-genus. At $\delta = 1/8, \varepsilon = 0$ we get $g(z) = 2\sinh^{-1}(z/2)$, so the Hirzebruch series is $x/(2\sinh(x/2))$, giving the $A$-genus.

(3.2) Explicit formulae. Let $(\omega_1, 2\omega_2) = (2\pi i, 4\pi i\tau)$ be the periods of this elliptic curve; we can uniformise it in terms of standard Weierstraß functions with periods $(2\pi i, 2\pi i\tau)$ as

$$(z, w) = (s(x), s'(x)), \quad s(x) := \frac{1}{\sqrt{\wp'(x) - e_1}}$$

with the half-period value $e_1 = \wp(1/2)$. This singles out one of the 2-torsion points on the elliptic curve, so the symmetry group in $\tau$ is a congruence subgroup of index 3 in $\text{PSL}(2; \mathbb{Z})$. The function $s$ is odd, has simple zeroes at 0, $\tau$ and simple poles at $1/2, \tau + 1/2$, with residues $\pm 1$. The Weierstraß product expansion method leads to a formula, with $u = e^z, q = e^s$:

$$s(x) = \frac{u^{1/2} - u^{-1/2}}{u^{1/2} + u^{-1/2}} \prod_{n>0} \frac{(1 - q^n u)(1 - q^n u^{-1})(1 + q^n)^2}{(1 + q^n u)(1 + q^n u^{-1})(1 - q^n)^2}$$

In terms of the Weierstraß $\sigma$-function

$$\sigma(x) := \left(u^{1/2} - u^{-1/2}\right) \prod_{n>0} \frac{(1 - q^n u)(1 - q^n u^{-1})}{(1 - q^n)^2}, \quad s(x) = \frac{\sigma(x)}{\sigma(x + \frac{1}{2})} \cdot N(q).$$

The normalisation factor $N(q) = \prod(1+q^n)^2(1-q^n)^{-2}$ ensures that $s(x) = x + O(x^2)$. Its topological meaning will be clear below: it normalises the tangent bundle of the manifold by subtracting a trivial bundle of the same dimension.

The function $s(x)$ is the exponential of the group law; the relevant Hirzebruch $Q$-series is therefore $\varphi_{LSO}(x) := x/s(x)$. Since $\varphi_{LSO}(x)$ is even, the associated genus on $\text{MU}^*$ factors through $\text{MSO}^*$. This suggests a theory $\text{Ell}_{LSO}(X) := \text{MSO}^*(X) \otimes_{\text{MSO}^*} R^*$. As a matter of detail, one must invert either $\varepsilon, (\delta^2 - \varepsilon)$ or both, as in the discriminant $\Delta = 8\varepsilon(\delta^2 - \varepsilon)^2$ before Landweber exactness can be checked; but it is shown in [La] by an alternate argument that $\text{Ell}_{LSO}$ really can be defined over $R^*$.

(3.3) The Dirac index on the free loop space. The LSO elliptic genus can be interpreted as a circle-equivariant $L$-genus, or signature, of the loop space. (There is no obvious topological interpretation of this, as the ‘middle dimension’ of loop space would be infinite, and there is no cup product to the ‘top dimension’: the interpretation comes from a formal application of the fixed-point theorem.)

Already, the first factor $\tanh(x/2)$ in $s(x)$ suggests the relation to the signature, but we spell out the argument for the slightly simpler Witten genus $\varphi_W(x) := x/\sigma(x)$, defined as a ‘multiplicative half’ of $\varphi_{LSO}$ from the $\sigma$-function above. The first factor $x/2 \cdot \sinh(x/2)^{-1}$ suggests the close relation to the $A$ genus and therefore a Dirac index. While the interpretation of the Witten genus as a Dirac index is more straightforward, it has weaker modularity properties than $\varphi_{LSO}$, and may not land in the ring $R^*$ if the manifold $X$ has non-vanishing first Pontryagin class $p_1$. Computationally, the obstruction to good modularity properties is related to the periodicity factors in the Weierstraß $\sigma$-function, which couple to the quadratic expression $\sum x_k^2$, in terms of the Chern roots of $T^*_C X$. Geometrically, the existence of a Spin structure on the free loop space of $X$ is obstructed by $w_2(X)$ and the first Pontryagin class.²

On the oriented manifold $X$, consider the formal series $\mathcal{N}$ of complex vector bundles built from the complexified tangent bundle $T^*_C X$:

$$\mathcal{N} := \bigoplus_{n>0} q^n T^*_C X.$$  

If $\pm x_k$ are the Chern root pairs of $T^*_C X$, then the Chern character of the symmetric power of $\mathcal{N}$ is

$$ch(\text{Sym} \mathcal{N}) = \prod_{n>0} ch(\text{Sym} q^n T^*_C X) = \prod_{n>0} (1 - q^n e^{2x_k})^{-1} (1 - q^n e^{-2x_k})^{-1}.$$  

¹My convention for $\sigma$ is leaving out a Gaussian factor, which however cancels out in $s$.

²Or rather, its half; see Witten’s discussion.
Let us now ‘normalise’ $T_{\mathbb{C}}X$ to dimension 0 by subtracting a trivial complex bundle, and call the corresponding series $\tilde{N}$. The Chern character for $\text{Sym} \tilde{N}$ acquires an additional factor of $\prod_n (1 - q^n)^{2\dim X}$. If $X$ has a Spin structure, then the index of the Dirac operator on $X$ coupled to $\text{Sym} \tilde{N}$ is given by the Atiyah-Singer index formula

$$\text{Index}(X; \text{Sym} \tilde{N}) = \int_X \tilde{A}(X) \cdot \text{ch} \left( \text{Sym} \tilde{N} \right) = \int_X \prod_k \frac{x_k}{\sigma(x_k)} = \varphi_W(X). \quad (3.4)$$

Now observe that the loop rotation action defines a complex structure on the normal bundle to $X$ within its free loop space $LX$, by means of the Fourier decomposition; therewith, the normal bundle can be identified with $\tilde{N}^*$, with the powers of $q$ tracking the character of the circle action. We choose negative Fourier modes to define the complex structure to get positive modes on the bundle $\text{Sym} \tilde{N}$, which describes the germs of fiberwise holomorphic functions on $\tilde{N}^*$. Holomorphic functions are solutions of the Dolbeault equation $\bar{\partial}f = 0$, and on a vector space they can be thought of as solutions of the Dirac equation (twisted by the square root of the canonical bundle). We see that, if generously conceived, the index in (3.4) represents the index of the Dirac operator on a neighbourhood of the constant loops $X$ in $LX$. This can actually be made precise, with the requisite analysis: see Taubes [T].

Now we invoke, on a heuristic basis, the localisation theorem in equivariant $K$-theory for the loop rotation action on $LX$. One version of the theorem, for a compact manifold $Y$ with circle action, says that the circle-equivariant Dirac index, as a character of $S^1$, may be computed from the neighbourhoods of the fixed-point-sets: specifically, as the index over the fixed-points of the Dirac operator coupled to the fiber-wise germs of solutions of the Dirac operator on their normal bundles. Invoking this in our setting teaches us that the index computed by (3.4) is in fact the circle-equivariant index of the Dirac operator on $LX$. (Making this precise, save in very special settings, is well outside the bounds of current techniques.) At any rate, we learn that the Witten genus is the $\tilde{A}$-genus of the free loop space, and so Witten elliptic cohomology is the circle-equivariant $K$-theory of the free loop space.

Part of this heuristics can be made precise: specifically, if we specialise to the formal neighbourhood of $q = 0$, which parametrises the Tate elliptic curve, then it is shown in [KM] that a suitably completed circle-equivariant $K$-theory $\tilde{\mathcal{K}}_{S^1}(LX)$ is a cohomology theory in $X$, and agrees with the Tate completion of a version of elliptic cohomology (depending exactly on which genus we use in defining the Thom class).

4. The equivariant theory

For a compact group $G$ acting on a space $X$, we would like to define an equivariant version of elliptic cohomology, $Ell^*_G(X)$. There is no good equivariant version of complex or real bordism that would allow us to recover true equivariant theories by a Conner-Floyd construction; in the case of $K$-theory, we can construct the Borel completed theory $K^*(X_{hG})$ but not the genuine $K^*_G(X)$ in this way. The problem of finding geometric cocycles for elliptic cohomology that could be made a geometrically equivariant is open, and the subject of current research interest. In [Lu2], an approach is described based on the observation that the completion in the equivariant C.-F. construction stems from the use of formal groups, rather than genuine group schemes. More precisely, the ring of functions of formal group of the complex-oriented cohomology theory $E^*$ is $E^*(\mathbb{C}P^\infty)$. Remembering that $\mathbb{C}P^\infty = BS^1$, the geometric realisation of the classifying stack, leads one to guess that $E^*_S(pt)$ is the ring of functions on the un-completed group. To extend this to an equivariant cohomology theory for spaces $X$ with $S^1$-action, the uncompleted group scheme must be ‘lifted’ from the coefficient ring $E^*(pt)$ to the world of spectra. In the case of elliptic cohomology, the formal group law of the elliptic curve has a natural ‘un-completion’ to the elliptic curve itself. Lurie’s work accomplishes precisely the lifting elliptic curves form $\mathbb{Z}$ to the sphere spectrum, and this leads to a natural construction of equivariant elliptic cohomology for compact abelian groups. In [Lu2], an extension of this theory to non-abelian groups is also characterised, but explicit construction seems more difficult: it seems to involve a lift, over the sphere spectrum, of the moduli space of $G$-bundles over the universal elliptic curve.
(4.1) The Tate curve and expected $K$-theory connection. It seems easy to describe the Tate curve completion of equivariant elliptic cohomology, in light of the expected relation to $K$-theory of loop spaces. I will specialise to the case when $X$ is a point: this has the advantage of relating to the positive energy representations of the loop group $LG$, which are expected to be co-cycles for equivariant elliptic cohomology. Actually, we need to include the loop rotation action in our representations, with the modular expansion variable $q$ playing the role of the standard representation of $S^1$. Let $G$ be a compact Lie group and denote by $G:G$ the quotient stack of $G$ by its own conjugation action. It carries the natural ‘loop rotation’ action of the circle group $S^1$, which is visible geometrically in the presentation $\mathcal{M}/LG$ of the space of $\mathfrak{g}$-valued connections on the circle modulo gauge transformations.\(^3\) Denote also by $h\text{PER}(S^1 \ltimes LG)$ the category of positive energy representations at level $h$; it is a free module over the category of $S^1$-representations, with basis labelled by irreducible PERs of $LG$. Let also $c \in H^2_{S^1}(G;G;\mathbb{Z})$ denote the $K$-theory twisting transgressed from $H^4(BG;\mathbb{Z})$, pulled back from the generator of $BSO$ by the adjoint representation. Choose a level $h$ so that $h + c > 0$. We then have

4.2 Theorem. [FHT3] There is a $K_{S^1}(pt)$-module isomorphism

$$h+cK^\dim_{S^1}(G:G) \cong K^0(h\text{PER}(S^1 \ltimes LG)),$$

in particular, the left-hand side is a free module over $K_{S^1}(pt) \cong \mathbb{Z}[q^\pm]$ whose fiber over $q = 1$ is identified with $h+cK^\dim_{S^1}(G:G)$.

Assembling the heuristics so far, we would like to see a commutative triangle of isomorphisms, in which $\hat{\text{Ell}}_G$ represents Tate-completed equivariant elliptic cohomology:

$$\begin{array}{ccc}
\hat{\text{Ell}}_G^*(pt) & \xrightarrow{h+c} & K^*(h\text{PER}(S^1 \ltimes LG)) \\
& & \\
& & \hat{\text{Ell}}_G^*(G:G)
\end{array}$$

(4.3)

An obstacle is the appearance of a level $h$ in the $K$-theories and the role it plays in $\text{Ell}_G^*$; but there turns out to be a clear place for it. Namely, elliptic cohomologies $\text{Ell}(X)$ may be twisted by classes in $H^4(X;\mathbb{Z})$. It is natural to expect $\text{Ell}_G^*(X)$ to have twisted versions, classified by $H^1_{S^1}(X;\mathbb{Z})$, so the pre-transgression in $H^4(BG;\mathbb{Z})$ of the level $h$ should appear as a twisting of $\text{Ell}$.

(4.4) Difficulties. After adding the twisting to $\hat{\text{Ell}}_G$, and overcoming early enthusiasm, several, increasingly severe, inconsistencies present themselves in the diagram (4.3):

(i) The degrees $*$ do not all match, since they differ on the $K$-theory sides: $* = 0$ for representations but $* = \dim G$ for the topological $K$-theory.

(ii) The twistings do not match when $G$ is non-abelian, because of the shift by $c$. Which level should go with $\text{Ell}_G$?

(iii) There is a cup-product on elliptic cohomology; for the twisted version, the twistings add. On representations, the tensor product meets these properties. But in $K_{S^1}(G:G)$, the cup-product is zero, because the $K$-classes are volume forms on $G$ (they are pushed forward from the identity of $G$) and thus square to zero.

At any rate, because of the $c$-shift, the product lands in a wrongly twisted $K$-group. So (4.3) cannot be compatible with products.

(iv) The ground rings do not match. On either $K$-side, the obvious ground ring is $\mathbb{Z}[q^\pm]$; but there is no specialisation from modular forms to Laurent polynomials, so $\hat{\text{Ell}}^*$ lives over the Tate base $\mathbb{Z}[[q]]$ and does not arise by extensions of scalars from $\mathbb{Z}[q^\pm]$.

Addressing these problems starts with the more serious one, fixing the coefficients (iv), which leads to a different, Laurent series version of circle-equivariant $K$-theory, compatible with the one

\(^3\)When $G$ is disconnected, several spaces $\mathcal{M}$ must be considered, going with different $G$-bundles over the circle.
considered by Kitchloo and Morava in [KM]. For a space with circle action, this allows $K$-classes with infinite-dimensional fibers over the fixed-point sets, provided the circle action breaks them into finite eigenspaces with spectrum bounded below. The definition is more subtle for a stack such as $G:G$, when a formal convergence condition must be imposed, bounding the growth of the highest weights of stabiliser actions on the $q$-Fourier coefficients. We denote this version of $S^1$-equivariant $K$-theory by $\tilde{K}_{S^1}$.

(4.5) **Key example: the circle group.** Let me illustrate how completing $K$-theory with respect to loop rotation changes the answer. To keep notation clean, let $G$ be a torus $T$ (eventually of rank one). The quotient stack $T:T$ is a product $T \times BT$, but the circle action is not trivial; the quotient $(T:T):S^1$ is a gerbe over $T$ with band $S^1$, an extension of the trivial gerbe $T \times BS^1$ by the trivial gerbe $T \times BT$; the extension is detected by its monodromy, where a loop $p \in \pi_1T$ shears $B(T \times S^1)$ into itself using the homomorphism $S^1 \to T$ determined by $p$, $q \mapsto q^p$.

A class $h \in H^2(BT;\mathbb{Z})$ is a quadratic form on $\pi_1T$, and determines a linear map $dh : \pi_1T \to \Lambda := (\pi_1T)^*$. The transgression of $h$ to $H^2_T(T:T;\mathbb{Z})$ determines a gerbe over the stack $(T:T):S^1$ with band $S^1$, where the subscript $c$ indicates the central nature of this extra circle: the total stack can be alternatively described as the quotient $\mathcal{C}$ of the space of $T$-valued connections under the gauge action of the ‘Kac-Moody’ group $\hat{L}T$, the $S^1_c$-central extension of $S^1 \times LT$ at level $h$. (The centre acts trivially and the other $S^1$ acts by loop rotation.) Skipping the calculations, which can be found in [PS], the resulting stack can be presented by lifting from $T = \mathfrak{t}/\pi_1T$ to $\mathfrak{t}$ and descending back, as the fibre-wise classifying stack of the bundle of groups over $S^1$. None of these are Laurent polynomials, giving the (known) $\ell$-adic completion in positive $q$-powers is not well-defined in the gerbe $(T:T):S^1$.
5. Reconciliation

The following result addresses the puzzles of §4 and supplies the correct version of the commutative triangle (4.3).

5.1 Theorem (No reference).

(i) For simple $G$ and $h \geq 0$, or for general $G$ and $h > 0$, there is a natural isomorphism

$$h \hat{K}_{S^1}^0(G : G) \cong K^0\left(h \text{PER}(S^1 \ltimes LG)\right)$$

as modules over $\mathbb{Z}((q))$, matching the cup-product with the tensor product. The isomorphism is realised by viewing each representation as an equivariant projective Hilbert bundle over $G$.

(ii) For $h \geq 0$, there is a natural isomorphism

$$-h-c \hat{K}_{S^1}^{\text{top}}(G : G) \cong K^0\left(-h \text{NER}(S^1 \ltimes LG)\right)$$

realised by the (complex-conjugate of the) Dirac family construction of [FHT3].

(iii) There is a Poincaré duality pairing between the positive and negative twisted $\hat{K}$-theories. When taking the character of representations as sections of line bundles on the elliptic curve, this matches Serre duality on the moduli space of semi-stable $G$-bundles on the elliptic curve.

The main result (i) is rather curious: it establishes an isomorphism

$$h \hat{K}_{S^1}^0(G : G) \cong h+c K_{S^1}^{\text{top}}(G : G) \otimes \mathbb{Z}[(q^\pm)]$$

between two topologically defined equivariant $K$-theories which does not appear to come from any obvious topological operation. (Because classes on the right are volume forms, one would think the map should go from left to right, yet the right side is naturally defined over the smaller Laurent polynomial ring $\mathbb{Z}[q^\pm]$.) A natural isomorphism appears on the dual side, at negative level, where completion of coefficients to the Tate base of Laurent series is ‘inoffensive’ in $K$-theory. The explanation for that is that completion of $q$-series happens on the positive side, whereas the representations which build the $K$-classes have negative energy and negative unbounded $q$-powers in their characters; so they do not represent tempered $K$-classes on their own, and must be coupled to the Spinors in the Lie algebra $Lg$ in order to represent $K$-classes, as in [FHT3].

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