The Cascading Haar Wavelet algorithm for computing the Walsh-Hadamard Transform

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The Walsh-Hadamard Transform (WHT)

- The WHT is a staple of the signal processing world.
- It is widely used...
  - ...in coding in wireless communications
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- Given $x \in \mathbb{R}^n$ where $n = 2^m$, its WHT (with coefficients in dyadic/Paley order) is equal to $H_m x$, where H_m is a Walsh-Hadamard matrix defined by the recursion

$$H_0 := 1; \quad H_{r+1} = \frac{1}{\sqrt{2}} \left[\begin{array}{c} H_r \otimes \left(\begin{array}{cc} 1 & 1 \end{array} \right) \\ H_r \otimes \left(\begin{array}{cc} 1 & -1 \end{array} \right) \end{array} \right] \text{ for } r \geq 0.$$

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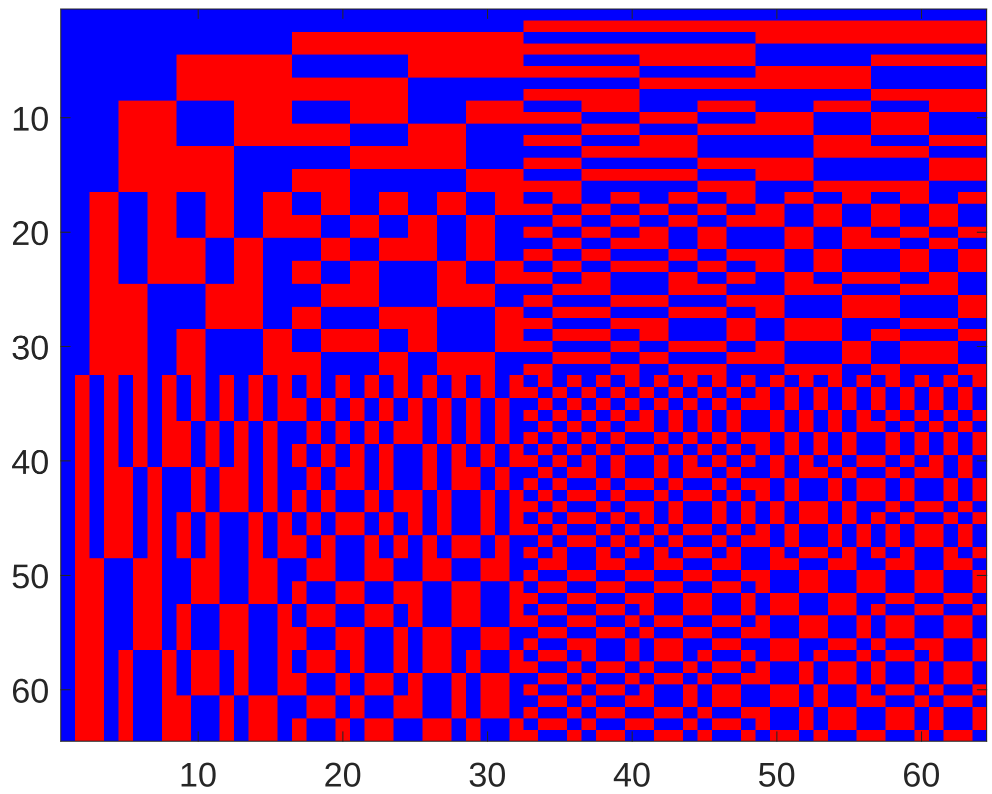
- Divide-and-conquer algorithms (e.g. Cooley-Tukey) exist which require exactly $n \log_2 n$ operations.

The Walsh-Hadamard Transform (WHT)

H_6: blue$= +$, red$= -$.



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- It requires $2n - 2$ operations.

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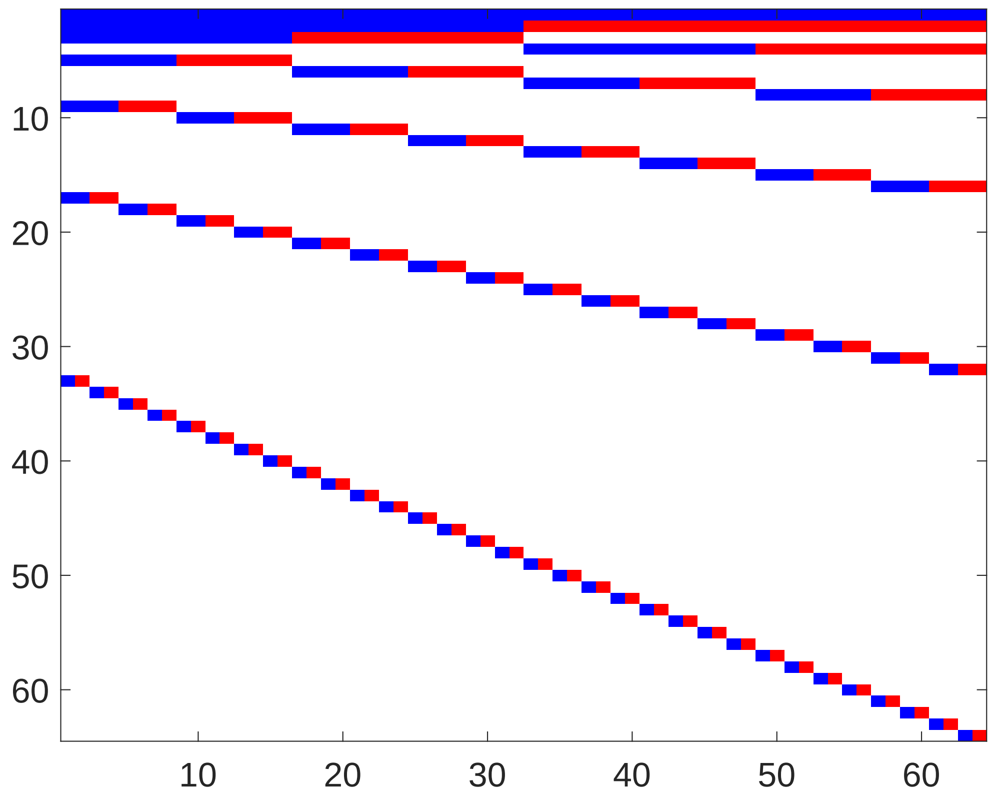

Known connections between WHT and Haar

- Computing the WHT of x is equivalent to computing mini-WHTs of the coefficients in each scale of the Haar wavelet transform of x (Fino 1972, Falkowski/Rahardja 1996).

$$H_m \Psi_m^T = \begin{bmatrix} 1 & & & & \\ & H_0 & & & \\ & & H_1 & & \\ & & & \ddots & \\ & & & & H_{m-1} \end{bmatrix}$$

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- The result also describes the *asymptotic mutual coherence* of the Walsh-Hadamard and Haar bases.

A new decomposition formula

Theorem (Thompson 2017):

$$H_m = \left\{ \prod_{r=1}^{m-1} I_{r-1} \otimes \begin{bmatrix} I_{m-r} & 0 \\ 0 & \Psi_{m-r} \end{bmatrix} \right\} \Psi_m, \text{ for } m \geq 1.$$

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- The WHT can be computed by first computing the Haar wavelet transform, and then employing a divide-and-conquer approach also consisting of Haar wavelet transforms.

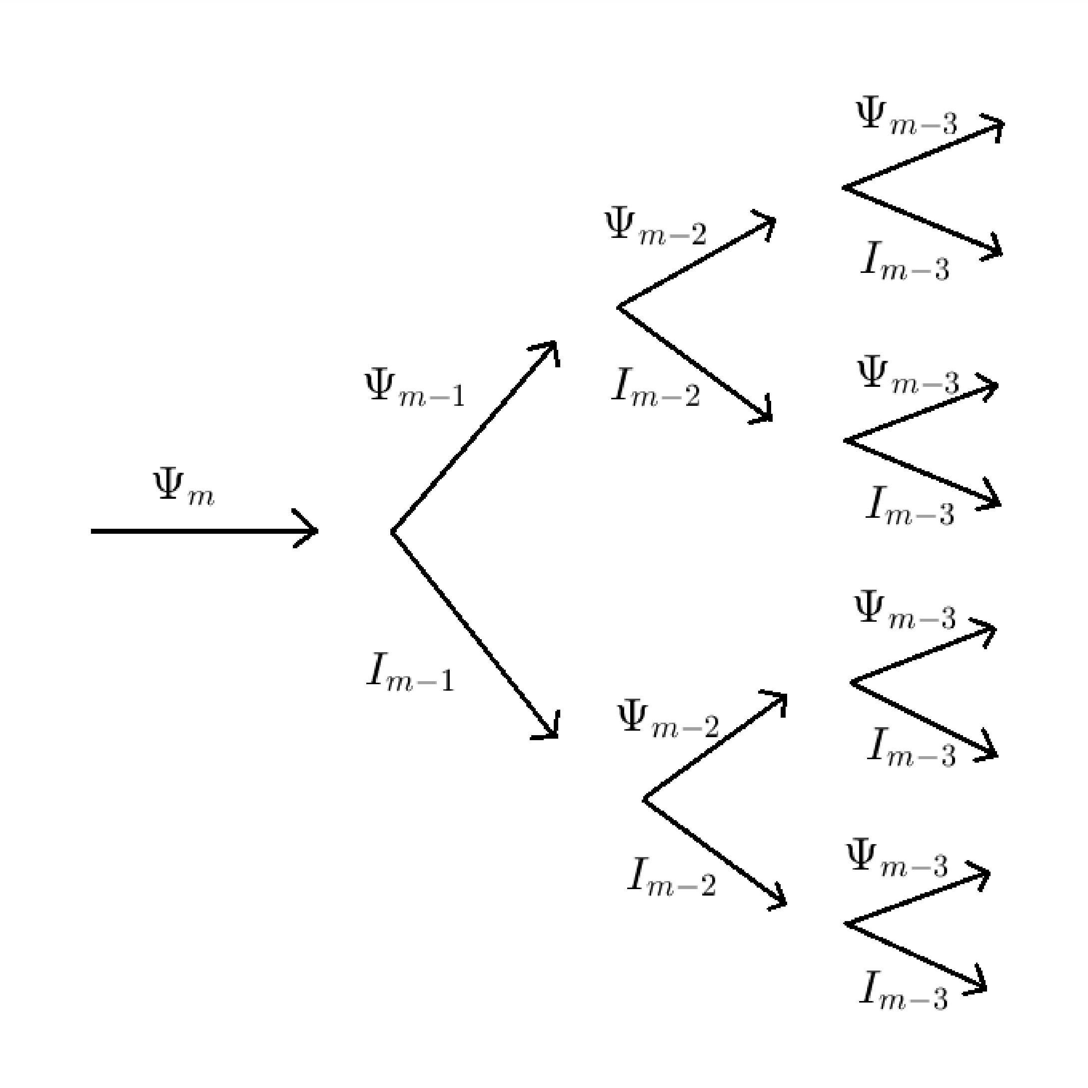
A new decomposition formula

Expanding the product:

$$H_m = \left\{ \begin{bmatrix} I_1 & & & & & & \\ & \Psi_1 & & & & & \\ & & I_1 & & & & \\ & & & \Psi_1 & & & \\ & & & & \ddots & & \\ & & & & & I_1 & \\ & & & & & & \Psi_1 \end{bmatrix} \cdots \right.$$

$$\left. \cdots \begin{bmatrix} I_{m-2} & & & \\ & \Psi_{m-2} & & \\ & & I_{m-2} & \\ & & & \Psi_{m-2} \end{bmatrix} \begin{bmatrix} I_{m-1} & \\ & \Psi_{m-1} \end{bmatrix} \right\} \Psi_m,$$

The Cascading Haar Wavelet (CHW) algorithm



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Theorem (Thompson 2017): The CHW algorithm can be implemented in $n \log_2 n$ operations.

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- The CHW algorithm requires a single Haar wavelet transform of size 2^m, and 2^{m-1-r} Haar wavelet transforms of size 2^r, for $r = 1, 2, \ldots, m-1$.

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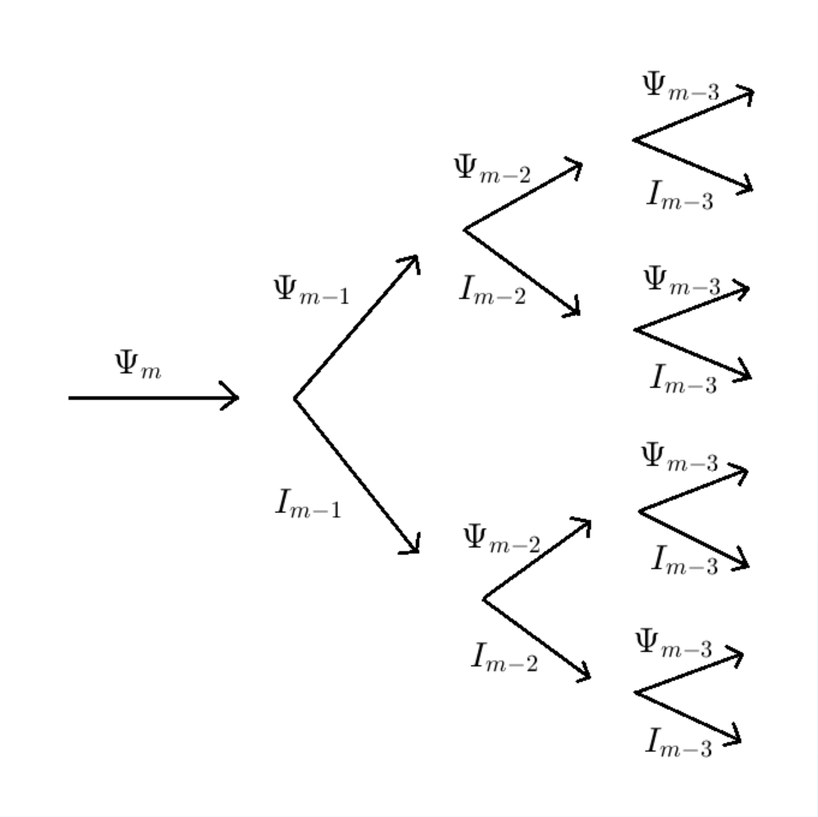
Proof:

- The CHW algorithm requires a single Haar wavelet transform of size 2^m, and 2^{m-1-r} Haar wavelet transforms of size 2^r, for $r = 1, 2, \ldots, m-1$.
- The total number of operations is therefore

$$\begin{aligned} & 2(2^m - 1) + \sum_{r=1}^{m-1} \left\{2^{m-1-r} \cdot 2(2^r - 1)\right\} \\ = \quad & 2^{m+1} - 2 + \sum_{r=1}^{m-1} 2^m - \sum_{r=1}^{m-1} 2^{m-r} \\ = \quad & 2^{m+1} - 2 + 2^m(m-1) - 2(2^{m-1} - 1), \end{aligned}$$

which simplifies to $m \cdot 2^m = n \log_2 n$.

Proposal for a parallel implementation



- Collapse by removing the identity transformations
  $\longrightarrow$ a cascade of Haar wavelet transforms...

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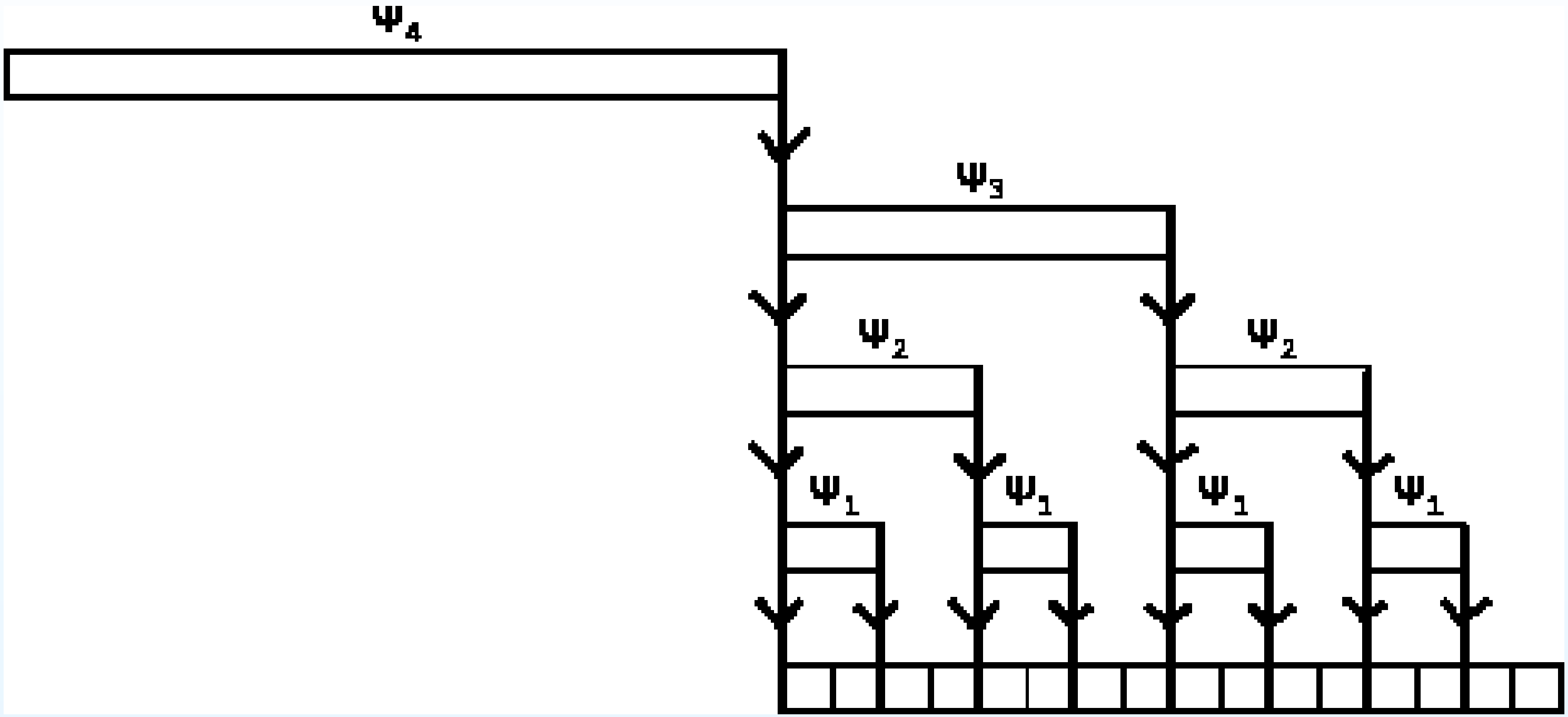

- There is a natural parallelization in which each of $m-1$ nodes is devoted to the task of performing Haar wavelet transforms of a certain size.
- Here illustrated for $m=4$.

Features of the proposed parallelization

- **Fixed tasks:** Each node only needs to be programmed once to do a single fixed task.

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- **Distributed memory:** Nodes pass on their output to other nodes by predetermined rules, with no need for shared memory.
- **Asynchronous:** Synchronization occurs automatically, when each node has received all of its inputs. It could therefore be implemented by a circuit which is not governed by a global clock.

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- A novel algorithm is proposed for computing the WHT which involves a cascade of Haar wavelet transforms.
- Its serial complexity is identical to the classical Cooley-Tukey algorithm.
- There is a natural way to parallelize the algorithm which has a number of potentially beneficial features.

References

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