Sparse near-equiangular tight frames with applications in full duplex wireless communication

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Rapid on-off division duplex

- We consider ad hoc communication networks in which each node can either transmit or receive transmissions from other nodes, but not at the same time.

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- In the *rapid on-off division duplex* (RODD) framework, (Guo/Zhang, 2010) each node transmits according to a unique on-off mask, so that the node can receive signals from other nodes during its off slots.
- All nodes can transmit and receive information within a given frame.
- Virtual full-duplex communication is achieved at the frame scale using half-duplex hardware.

System model

- Suppose we have a network of n users, each assigned a codeword of length $m < n$ and write

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

for the full $m \times n$ codebook matrix.

- Assume a synchronous random access model: only $k \ll n$ users are active. The codebook is known to the users, but the active users are not known.

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- Assume a synchronous random access model: only $k \ll n$ users are active. The codebook is known to the users, but the active users are not known.
- Each user receives a truncated version of other users' codewords (corresponding to its off slots).
- For user i, write X^i for the $\tilde{m} \times \tilde{n}$ submatrix of X with column i and the rows corresponding to users i's on slots removed. Model the signal received by user i as

$$y^i = X^i s + z,$$

  where z is white Gaussian noise, $z_j \sim N(0, \sigma^2)$.

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Deterministic ($m = n$):

- Choir codes and LASSO (l_1-based) decoding (Applebaum et al. 2011)

Introducing sparse Kerdock matrices

- Let $u_1, u_2, a_1, a_2, b \in \mathbb{Z}_2^r$ and define the $2^{2r} \times 2^{3r}$ **sparse Kerdock matrix** $\mathcal{S}^r$ by

$$\mathcal{S}^r_{(u_1,u_2),(a_1,a_2,b)} = \begin{cases} \frac{1}{2^{r/2}}(-1)^{u_1^T a_1} & P_b u_1 + u_2 = a_2 \\ 0 & \text{otherwise.} \end{cases}$$

- The set of $r \times r$ binary symmetric matrices $\{P_b \ : \ b \in \mathbb{Z}_2^r\}$ is a **Kerdock set**: $P_{b_1} + P_{b_2}$ is full rank for any two distinct P_{b_1}, P_{b_2}.

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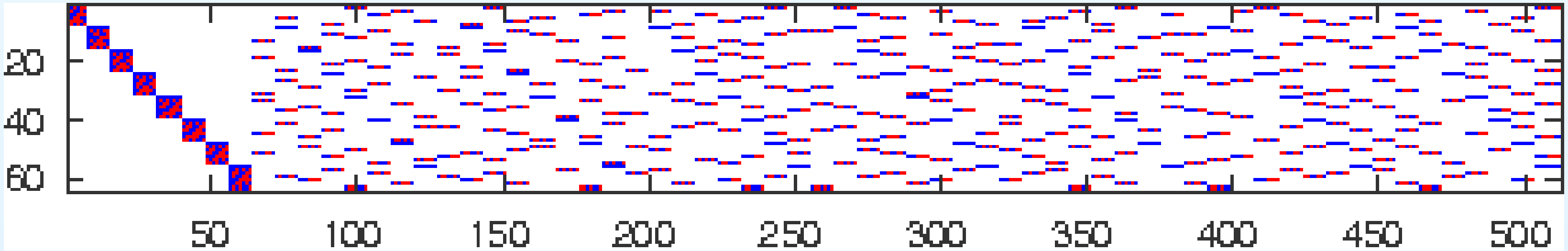
Here is the 64×512 matrix $\mathcal{S}^3$ (+=blue,$-$=red):



- $\mathcal{S}^r$ is extremely sparse: only $2^r = \sqrt{m}$ nonzero entries per column.

Relation to Delsarte-Goethals frames

- Define I_{2^r} to be the $2^r \times 2^r$ identity matrix and define H_{2^r} to be the $2^r \times 2^r$ Hadamard matrix (of Sylvester type).

Theorem (Calderbank/Thompson 2017): $\mathcal{S}^r = (H_{2^r} \otimes I_{2^r})\mathcal{K}^r$ where $\otimes$ denotes Kronecker product and $\mathcal{K}^r$ consists of 2^{3r} out of 2^{4r} columns from a Delsarte Goethals (DG) frame $DG(r, 0)$.

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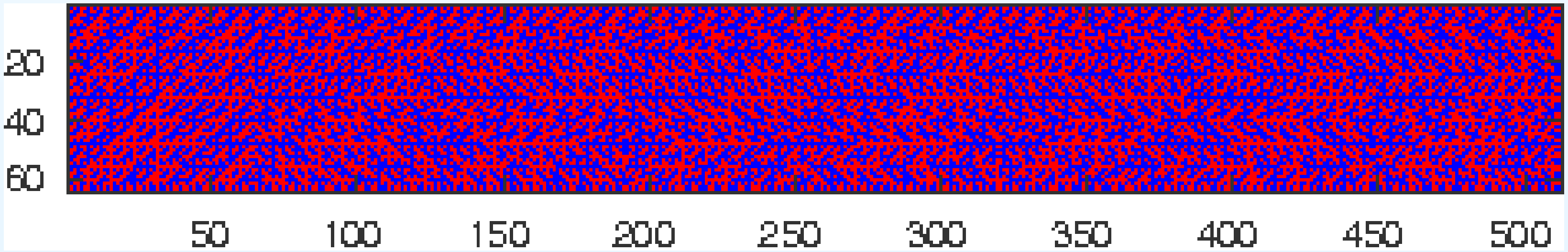

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Why is this significant?

- $DG(r,0)$ frames are near-equiangular tight frames $\longrightarrow$ known to perform well for the Gaussian multiple access channel.
- They can also be decoding using the chirp reconstruction algorithm (Howard/Calderbank/Searle 2008) with complexity sublinear in n.

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- They can also be decoding using the chirp reconstruction algorithm (Howard/Calderbank/Searle 2008) with complexity sublinear in n.
- $\mathcal{S}^r$ can be obtained from $\mathcal{K}^r$ via a unitary transformation, which means...
  - $\mathcal{S}^r$ is also a near-equiangular tight frame;
  - The same chirp reconstruction algorithm can be used to decode.
  - And it is also extremely sparse!

Other sparse equiangular tight frames?

Welch bound: For any $X \in \mathbb{C}^{m \times n}$,

$$\min_{i \neq j} \frac{|\langle x_i, x_j \rangle|}{\|x_i\|_2 \|x_j\|_2} \geq \sqrt{\frac{n-m}{m(n-1)}} \approx \frac{1}{\sqrt{m}},$$

and this bound is achieved when X is an equiangular tight frame (ETF)

$\Rightarrow$ an optimal spread of points on the hypersphere.

Other sparse equiangular tight frames?

Specification:

Suppose we want *infinite families of* $m \times n$ *sparse ETFs (*$\approx \sqrt{m}$ *nonzeros per column) with as many columns as possible (*$n \approx m^{3/2}$*)...*

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Candidates: Steiner ETFs (Fickus et al 2012):

- Small block designs ($k = 2, 3, 4, 5$): $N \approx kn$
- Unitals: $n \approx m^{5/4}$
- Affine/Projective: $n \approx m^{3/2}$
- Denniston: $n \approx \frac{1}{2}m^{3/2}$.

$\longrightarrow$ Steiner ETFs of Affine, Projective and Denniston type have similar specifications...

Some advantages of sparse Kerdock frames

- **Real-valued:** Sparse Kerdock matrices are real, whereas Affine/Projective/Denniston ETFs cannot be real.

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- **Matrix-vector products:** We can perform matrix-vector products with the fast Walsh-Hadamard transform in $\mathcal{O}(n \log n)$, whereas the lack of fast transform means that the complexity is $\mathcal{O}(n^{4/3})$ for Steiner ETFs.
- **Sublinear decoding algorithm:** No equivalent of the chirp reconstruction algorithm has been developed for Steiner ETFs.

Experiments

A simple, universal decoding algorithm: One Step Thresholding (OST)

Inputs: $y^i \in \mathbb{R}^{\tilde{m}}$, $X^i \in \mathbb{R}^{\tilde{m} \times \tilde{n}}$, k.

1. $g = (X^i)^T y^i$.
2. $\Gamma := \{j$ corresponding to the k largest $|g_j|\}$.

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- $n = 2^{15}$ nodes and codewords of length $m = 2^{10}$.
- We vary the number of active users k.
- Let $\Lambda \subseteq \{1, \ldots, n\}$ be the true active users:

$$\text{Error rate} = \frac{|\Lambda \setminus \Gamma|}{k}$$

averaged over 200 different users.

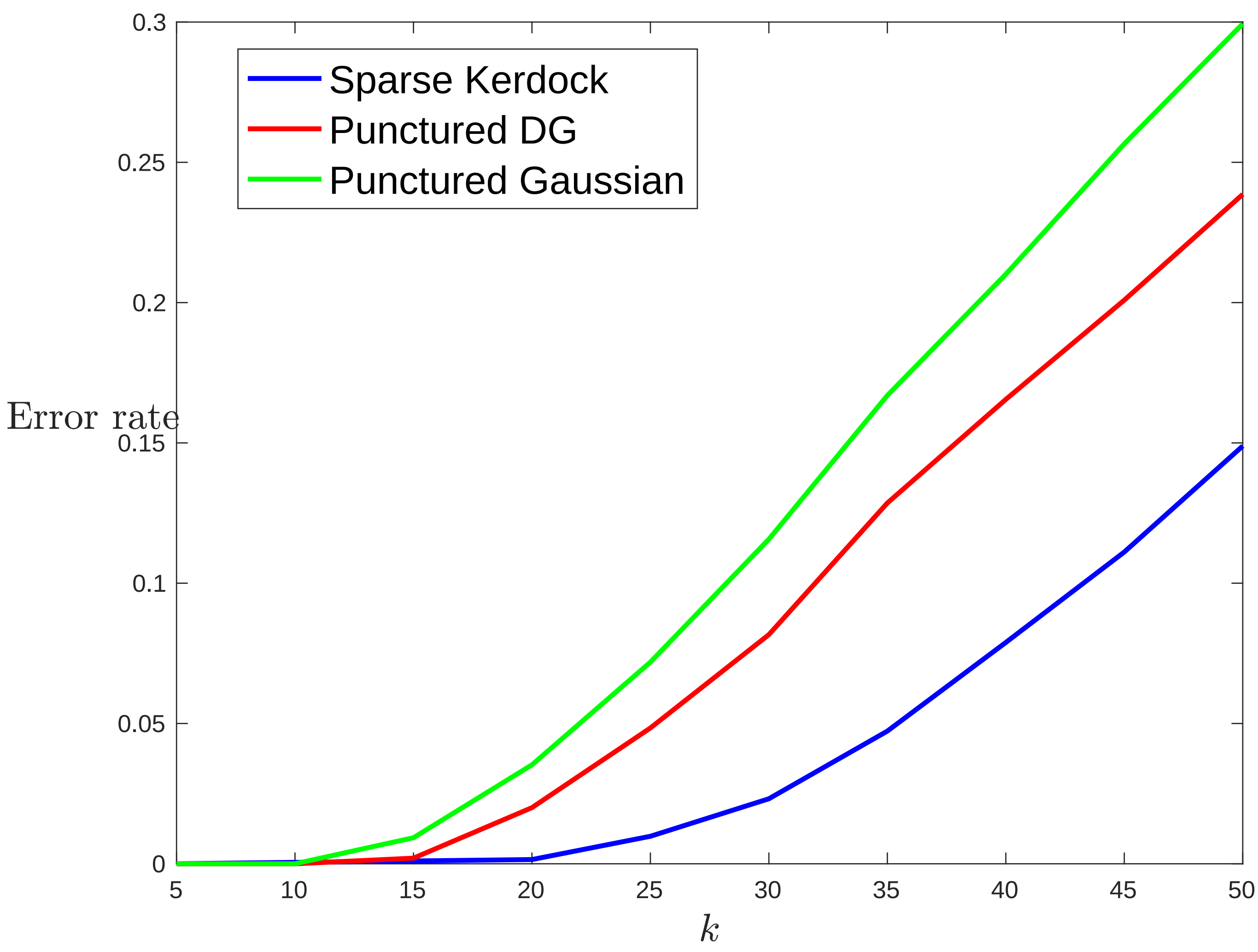
Experiments

We compare three different $2^{10} \times 2^{15}$ codebook matrices:

- Sparse Kerdock matrices.
- Truncated $DG(10, 0)$ frames with i.i.d. Bernoulli erasures with 'on' probability $1/2^r$ (optimized over r).
- Random Gaussian matrices with i.i.d. Bernoulli erasures with 'on' probability $1/2^r$ (optimized over r).

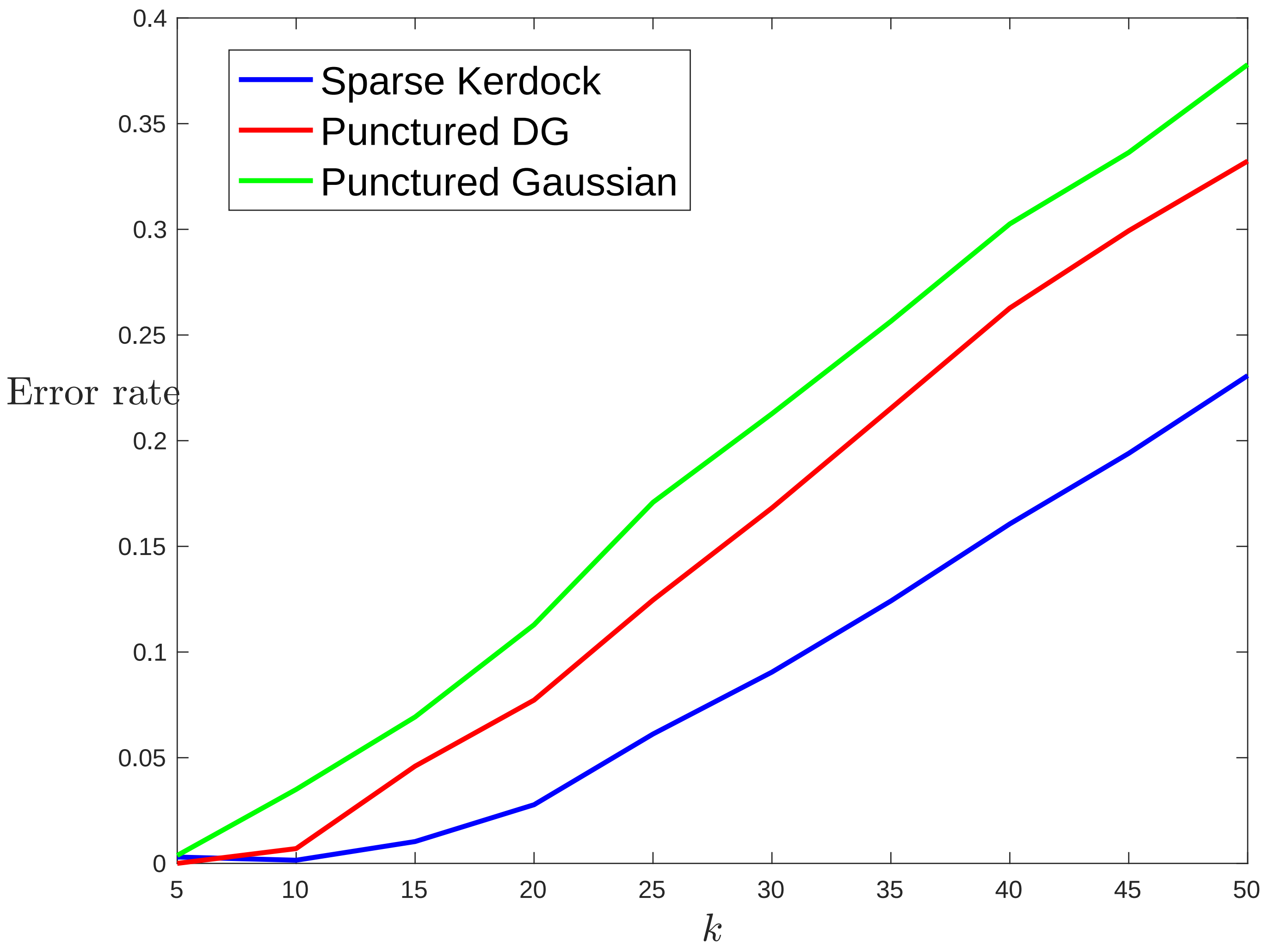
Experiments

$\sigma = 0$ (noiseless)



Experiments

$\sigma = 0.1$ (AWGN)



Summary

- We have proposed **sparse Kerdock matrices**: a new family of extremely sparse near-equiangular tight frames.
- They are related to Delsarte-Goethals frames by unitary transformation.
- They outperform codebooks based on random erasures in the context of a Gaussian multiple access channel (GMAC) model for synchronous RODD when the decoder is One Step Thresholding (OST).

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Future directions:

- Extension of the system model to incorporate
  - Rayleigh fading noise models.
  - The effect of network topology (neighbour discovery).
  - Asynchronous networks.
- Extension of the numerical comparison to the sub-linear chirp reconstruction decoding.

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