On Support Sizes of Restricted Isometry Constants

Andrew Thompson

University of Edinburgh

Joint with Jeff Blanchard (Grinnell)

Compressed Sensing

Let $x \in \mathbb{R}^N$ be a given **signal**.

Suppose we obtain a vector $b = Ax \in \mathbb{R}^n$ of **linear measurements** where $A \in \mathbb{R}^{n \times N}$ is the **measurement matrix**.

Compressed Sensing

Let $x \in \mathbb{R}^N$ be a given **signal**.

Suppose we obtain a vector $b = Ax \in \mathbb{R}^n$ of **linear measurements** where $A \in \mathbb{R}^{n \times N}$ is the **measurement matrix**.

We assume:

- $n < N \quad \Rightarrow$ underdetermined system
- x sparse with $k < n$ non-zeros

Compressed Sensing

Let $x \in \mathbb{R}^N$ be a given **signal**.

Suppose we obtain a vector $b = Ax \in \mathbb{R}^n$ of **linear measurements** where $A \in \mathbb{R}^{n \times N}$ is the **measurement matrix**.

We assume:

- $n < N \quad \Rightarrow$ underdetermined system
- x sparse with $k < n$ non-zeros

Algorithm for the sparsest solution:

$$\min \|x\|_0 \text{ subject to } Ax = b.$$

l_1-Minimization and the RIP

Suppose we solve instead the l_1**-minimization**:

$$\min \|x\|_1 \text{ subject to } Ax = b.$$

When does l_1 recover l_0?

l_1-Minimization and the RIP

Suppose we solve instead the l_1**-minimization**:

$$\min \|x\|_1 \text{ subject to } Ax = b.$$

When does l_1 recover l_0?

Restricted Isometry Constants:

$$L_k := \min_{c \geq 0} c \;\text{ subject to }\; (1-c)\|x\|_2^2 \leq \|Ax\|_2^2 \text{ for all } k\text{-sparse } x$$

$$U_k := \min_{c \geq 0} c \;\text{ subject to }\; (1+c)\|x\|_2^2 \geq \|Ax\|_2^2 \text{ for all } k\text{-sparse } x$$

$$R_k := \max\{L_k, U_k\}$$

Conditions for l_1 to recover l_0

Chartrand (2007) $$bR_{(b+1)k} + R_{bk} < b - 1; \quad b > 2$$

$$bL_{(b+1)k} + U_{bk} < b - 1; \quad b > 2$$

Conditions for l_1 to recover l_0

Chartrand (2007) $bR_{(b+1)k} + R_{bk} < b - 1; \quad b > 2$

$$\frac{b}{b-1} L_{(b+1)k} + \frac{1}{b-1} U_{bk} < 1; \quad b > 2$$

Conditions for l_1 to recover l_0

Chartrand (2007) $$bR_{(b+1)k} + R_{bk} < b - 1; \quad b > 2$$

$$bL_{(b+1)k} + U_{bk} < b - 1; \quad b > 2$$

Candès (2008) $$R_{2k} < \sqrt{2} - 1 \approx 0.4142$$

$$(1 + \sqrt{2})L_{2k} + U_{2k} < \sqrt{2}$$

Conditions for l_1 to recover l_0

Chartrand (2007)

$$bR_{(b+1)k} + R_{bk} < b - 1; \quad b > 2$$

$$bL_{(b+1)k} + U_{bk} < b - 1; \quad b > 2$$

Candès (2008)

$$R_{2k} < \sqrt{2} - 1 \approx 0.4142$$

$$(1 + \sqrt{2})L_{2k} + U_{2k} < \sqrt{2}$$

Foucart, Lai (2009)

$$R_{2k} < \frac{2}{3+\sqrt{2}} \approx 0.4531$$

$$\frac{1+U_{2k}}{1-L_{2k}} < 4\sqrt{2} - 3$$

Conditions for l_1 to recover l_0

Chartrand (2007)

$$bR_{(b+1)k} + R_{bk} < b - 1; \quad b > 2$$

$$bL_{(b+1)k} + U_{bk} < b - 1; \quad b > 2$$

Candès (2008)

$$R_{2k} < \sqrt{2} - 1 \approx 0.4142$$

$$(1 + \sqrt{2})L_{2k} + U_{2k} < \sqrt{2}$$

Foucart, Lai (2009)

$$R_{2k} < \frac{2}{3+\sqrt{2}} \approx 0.4531$$

$$\frac{1+U_{2k}}{1-L_{2k}} < 4\sqrt{2} - 3$$

What do these results mean quantitatively? Which is better?

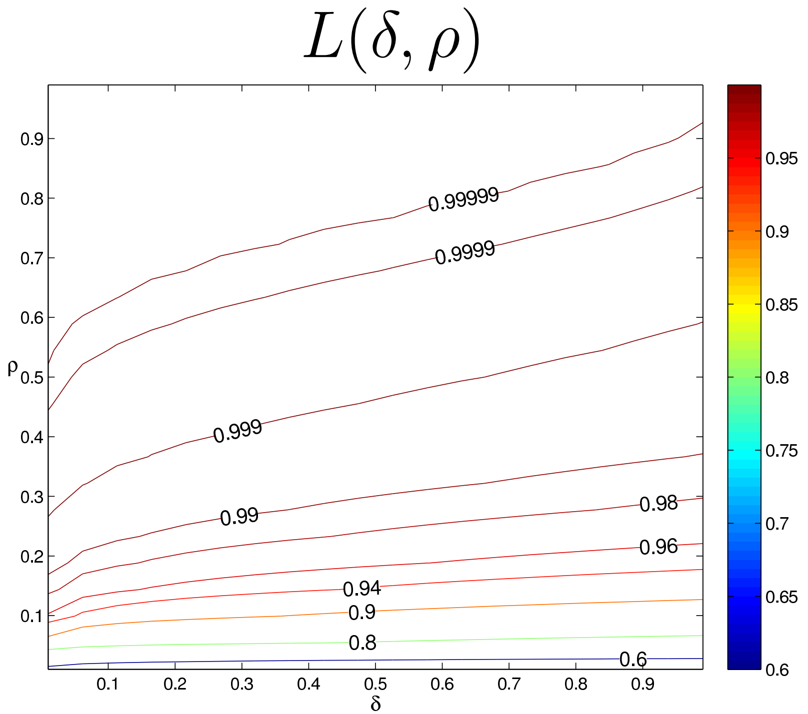
Gaussian RIP Upper Bounds

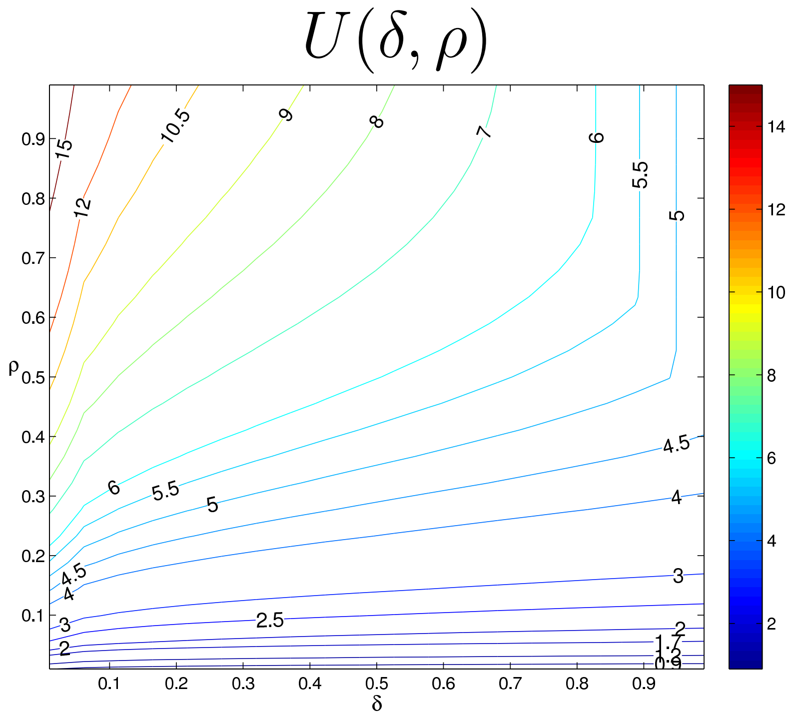
(Blanchard, Cartis and Tanner, 2009)

Theorem: Let A be a matrix of size $n \times N$ whose entries are drawn i.i.d. from $\mathcal{N}(0, \frac{1}{n})$.
Let $(k, n, N) \to \infty$ with $\frac{k}{n} \to \rho$ and $\frac{n}{N} \to \delta$.
Then there exist numerically computable functions $L(\delta, \rho)$ and $U(\delta, \rho)$ such that, for any $\epsilon > 0$,

$$\mathbb{P}\{L(k, n, N) < L(\delta, \rho) + \epsilon\} \to 1,$$
$$\mathbb{P}\{U(k, n, N) < U(\delta, \rho) + \epsilon\} \to 1.$$

Gaussian RIP Upper Bounds





Quantitative Sufficient Conditions

Chartrand (2007):

$$bL([b+1]k, n, N) + U(bk, n, N) < b - 1; \quad b > 2$$

Candès (2008):

$$(1+\sqrt{2})L(2k, n, N) + U(2k, n, N) < \sqrt{2}$$

Foucart, Lai (2009):

$$\frac{1 + U(2k, n, N)}{1 - L(2k, n, N)} < 4\sqrt{2} - 3$$

Quantitative Sufficient Conditions

Chartrand (2007):

$$bL(\delta, [b+1]\rho) + U(\delta, b\rho) < b - 1; \quad b > 2$$

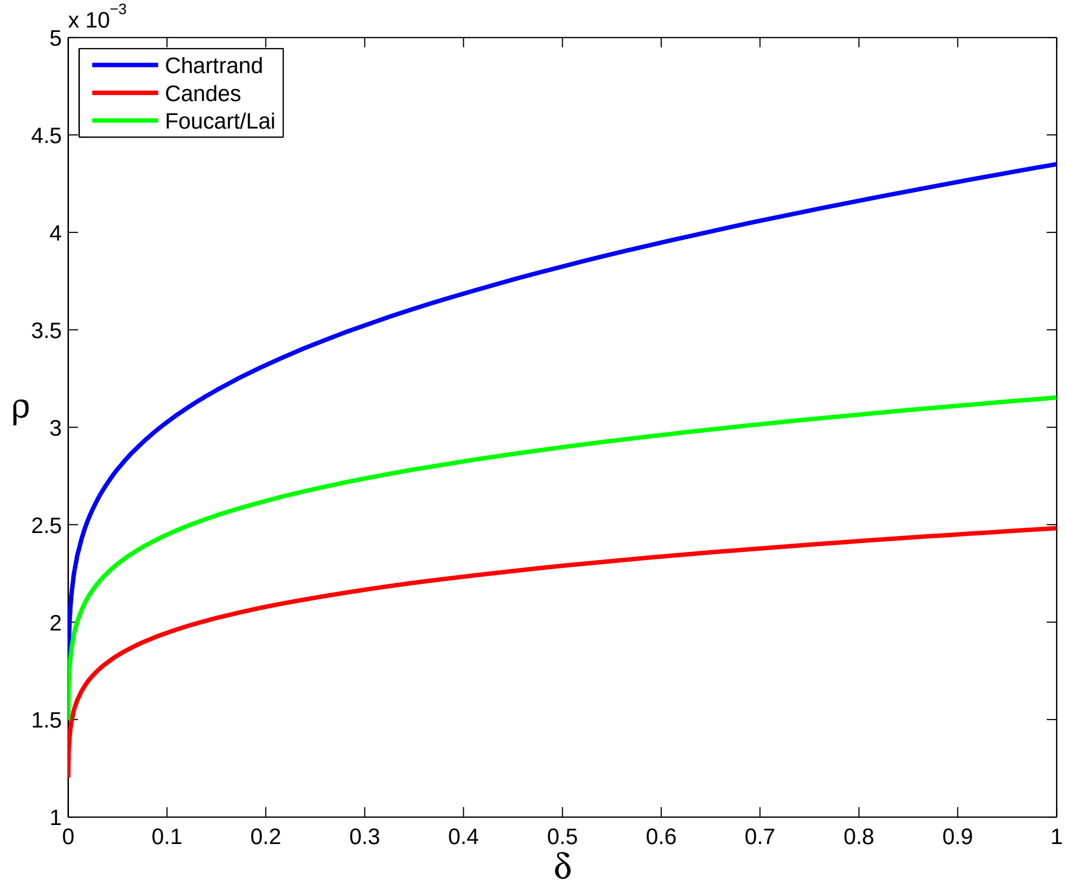
Candès (2008):

$$(1+\sqrt{2})L(\delta, 2\rho) + U(\delta, 2\rho) < \sqrt{2}$$

Foucart, Lai (2009):

$$\frac{1 + U(\delta, 2\rho)}{1 - L(\delta, 2\rho)} < 4\sqrt{2} - 3$$

Comparison of l_1 Phase Transitions



The highest phase transitions are obtained by taking $b \approx 11$ in the result by Chartrand: $11L(12k, n, N) + U(11k, n, N) < 10$.

Recent Results of Cai/Wang/Xu

$$R_{k+a} + \sqrt{\frac{k}{b}} R_{k+a+b} < 1$$

$$L_{k+a} + \frac{1}{2}\sqrt{\frac{k}{b}} (L_{k+a+b} + U_{k+a+b}) < 1$$

where $2a \leq b \leq 4a$.

Recent Results of Cai/Wang/Xu

$$R_{k+a} + \sqrt{\tfrac{k}{b}} R_{k+a+b} < 1$$

$$L_{k+a} + \tfrac{1}{2}\sqrt{\tfrac{k}{b}}(L_{k+a+b} + U_{k+a+b}) < 1$$

where $2a \le b \le 4a$.

$$R_a + tR_{a+b} < 1$$

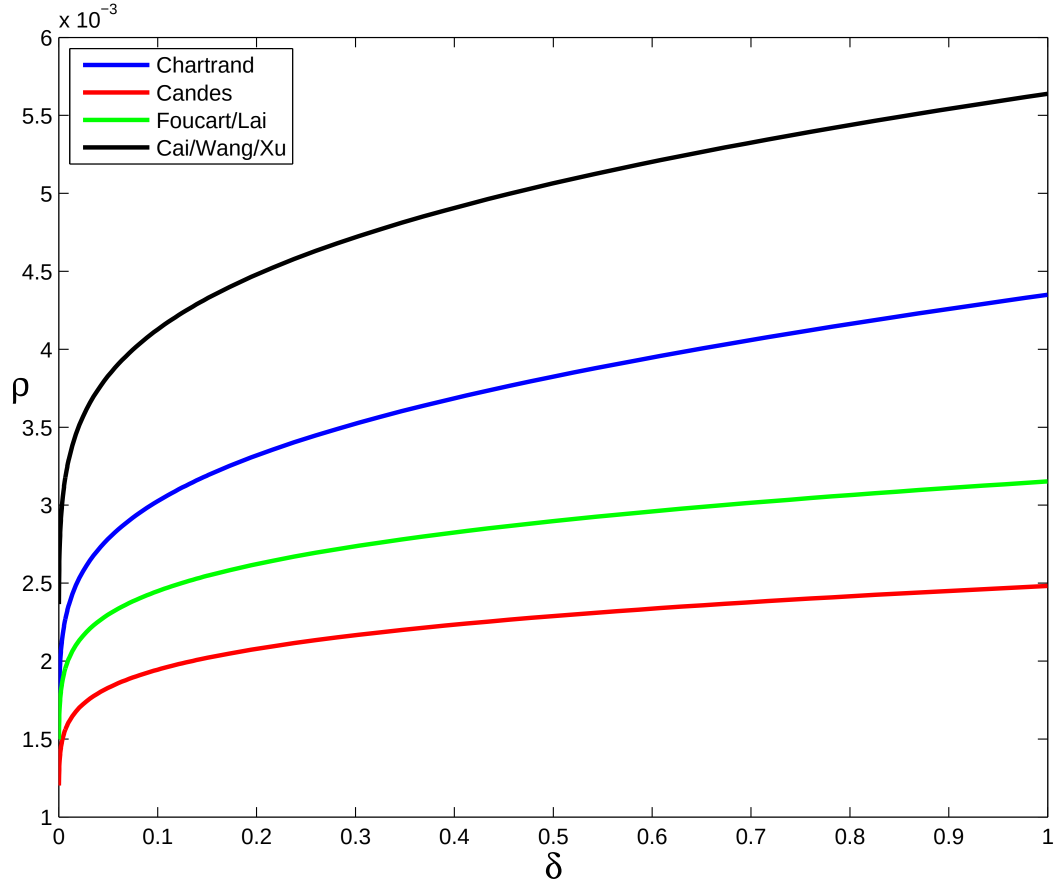
$$L_a + \tfrac{1}{2}t(L_{a+b} + U_{a+b}) < 1$$

where $a \ge k$, $8(a-k) \le b$, and

$$t = \sqrt{\tfrac{a}{b}} + \tfrac{1}{4}\sqrt{\tfrac{b}{a}} - \tfrac{2(a-k)}{\sqrt{ab}}.$$

$$\left[\Rightarrow \quad R_k < \tfrac{1}{1+\sqrt{5}} \approx 0.3090\right]$$

Comparison of l_1 Phase Transitions



Higher phase transitions can be obtained from the new results by again choosing large support sizes:

$$4L(2k,n,N) + L(6k,n,N) + U(6k,n,N) < 4.$$

Conclusions

- It is important to understand what RIP conditions mean quantitatively: the phase transition framework combined with RIP bounds for Gaussian matrices is a useful tool to investigate this.

Conclusions

- It is important to understand what RIP conditions mean quantitatively: the phase transition framework combined with RIP bounds for Gaussian matrices is a useful tool to investigate this.
- We observe that a results by Cai *et al.* with surprisingly high support sizes give the highest phase transitions for l_1 minimization.

Conclusions

- It is important to understand what RIP conditions mean quantitatively: the phase transition framework combined with RIP bounds for Gaussian matrices is a useful tool to investigate this.
- We observe that a results by Cai *et al.* with surprisingly high support sizes give the highest phase transitions for l_1 minimization.
- It is not always quantitatively beneficial to have RIP results with the smallest possible support sizes.

Thank you for your attention!