A tree projection algorithm for wavelet-based sparse approximation

Andrew Thompson

Duke University, North Carolina, USA

joint with Coralia Cartis (University of Edinburgh)

Wavelet trees

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- $\Rightarrow$ **Why the DWT is so effective:** it provides sparse representations for piecewise smooth signals
- ...also, we expect it to be **tree-sparse**:
  the coefficients form a rooted subtree.

Tree projection

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- **Tree projection:** Find the k-tree sparse z which is closest in Euclidean distance to x:

$$\mathcal{P}_k(x) := \arg\min_{z \in \mathcal{T}_k} \|z - x\|_2.$$

Applications of tree projections

- Classification and regression trees (CART)

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- Tree-based Compressed Sensing for wavelets.

  (Baraniuk et al.,2010)

Existing algorithms for wavelets

- **Greedy tree approximation** (GTA)
  Choose $l < k$, take the largest l coefficients, fill out to a rooted tree, repeat until the rooted tree is of size k.
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- **Condensing sort and select algorithm** (CSSA)

  (Baraniuk/Jones,1994)

Optimization formulation

- $\mathcal{P}_k$ preserves the value of selected coefficients.

$$\{\mathcal{P}_k(x)\}_i := \begin{cases} x_i & i \in \Gamma \\ 0 & i \notin \Gamma \end{cases} \qquad \text{where} \qquad \Gamma := \operatorname{supp}\{\mathcal{P}_k(x)\}\,.$$

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- $\Rightarrow$ we can formulate as an Integer Program (IP).

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$$\max_{\tau \in \mathbb{Z}^N} \sum_{i=1}^{N} x_i^2 \tau_i \quad \text{subject to} \quad \begin{aligned} &\tau \geq 0 \\ &\{\tau_i\} \text{ tree-nonincreasing} \\ &\sum \tau_i = k \\ &\tau_1 = 1. \end{aligned} \qquad \text{(IP)}$$

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- $\Rightarrow$ neither guaranteed to exactly find $\mathcal{P}_k$.

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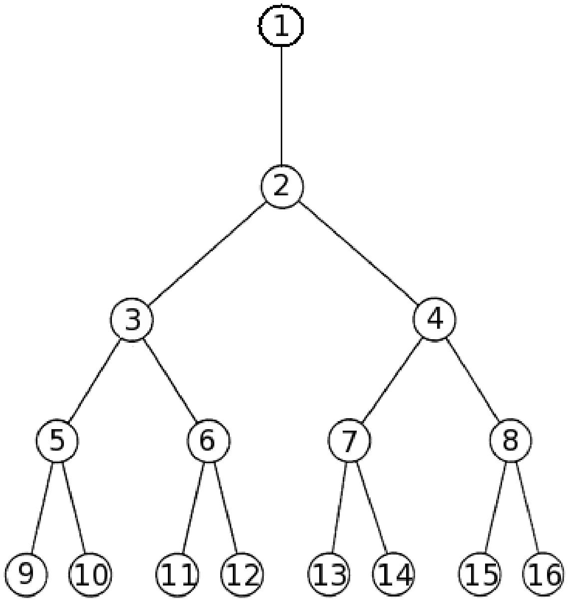

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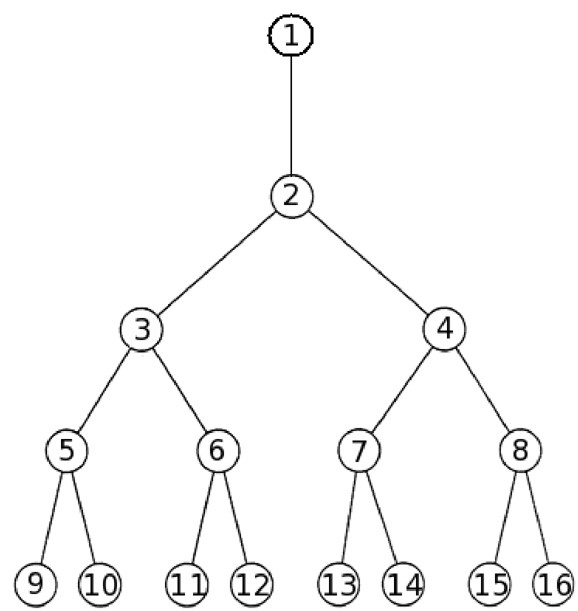

- A similar algorithm was used for CART (Bohanec/Bratko,1994).

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- Trace back precedences to identify solution.

Complexity of ETP

- **Theorem:** ETP requires at most $(3d^2Nk + N)$ additions/comparisons to calculate $\mathcal{P}_k$.
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- **Comparison with approximate algorithms:** both CPSS and CSSA are $\mathcal{O}(N \log N)$

- $\Rightarrow$ if $\log N \ll k$, ETP guarantees exact optimality at the expense of a worse order of complexity.

An application of ETP

- **Compressed sensing:** Recover $x \in \mathbb{R}^n$ from $b = Ax \in \mathbb{R}^m$ where $m < n$

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- **Gradient projection algorithm:**

  (Blumensath/Davies,2009;Baraniuk et al.,2010)

$$x^{i+1} := \mathcal{P}_k \left\{ x^i + \alpha A^T (b - Ax^i) \right\}$$

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Theorem: Let

$$n \to \infty \text{ and } \frac{k}{n} \to \rho.$$

Then

$$\mathbb{P}(\text{recovery of } x) \to 1$$

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$\Rightarrow \quad \mathcal{P}_k$ must be exact for result to hold.

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- **Paper:** *An exact tree projection algorithm for wavelets*
- **Code:** both available at `www.math.duke.edu/~thompson`

References

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