

# Spaces of graphs and surfaces - on the work of Søren Galatius

Ulrike Tillmann, Oxford University

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$\Sigma_n :=$  permutation group on  $n$  letters

$F_n :=$  free, non-abelian group on  $n$  letters

$\text{Aut}F_n :=$  automorphism group of  $F_n$

$$\Sigma_n \subset \text{Aut}F_n$$

**Galatius' main result:**

For  $0 < * < (n - 1)/2$ ,

$$H_*(\Sigma_n) = H_*(\text{Aut}F_n)$$

and in particular

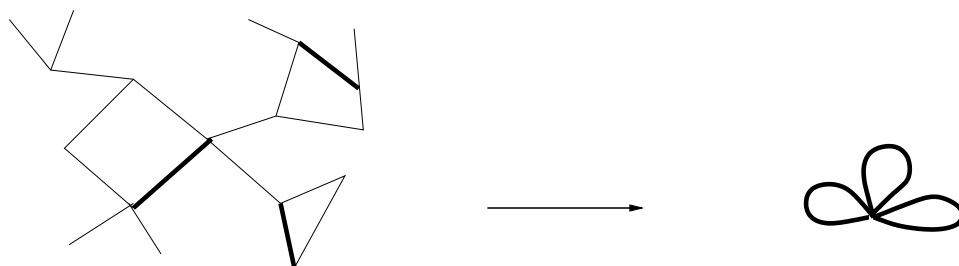
$$H_*(\text{Aut}F_n) \otimes \mathbb{Q} = 0$$

$F_n \longrightarrow \mathbf{Z}^n$  induces a surjection

$$\text{Aut}F_n \longrightarrow \text{GL}_n\mathbf{Z}$$

$G_n :=$  a connected, compact graph with Euler characteristic  $1 - n$ . Then

$$\pi_1(G_n) = F_n$$



$\text{HE}(G_n; *) :=$  monoid of homotopy equivalences of  $G_n$  that fix a base point

$$\text{Aut}F_n = \pi_0(\text{HE}(G_n; *)) \simeq \text{HE}(G_n; *)$$

$S_{g,1} :=$  oriented surface of genus  $g$  with 1 boundary component

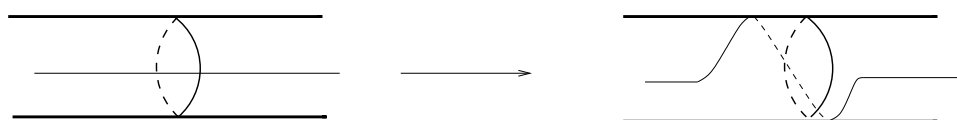


$\text{Diff}^+(S_{g,1}; \partial) :=$  group of diffeomorphisms that preserve the orientation and fix the boundary point wise

Associated *mapping class group*

$$\Gamma_{g,1} := \pi_0(\text{Diff}^+(S_{g,1}; \partial)) \simeq \text{Diff}^+(S_{g,1}; \partial)$$

generated by Dehn twists:



The action on

$H_1(S_{g,1}) = \mathbf{Z}^{2g}$  induces a surjection

$$\Gamma_{g,1} \longrightarrow \mathrm{Sp}_{2g}\mathbf{Z}$$

The action on

$\pi_1(S_{g,1}) = F_{2g}$  induces an injection

$$\Gamma_{g,1} \hookrightarrow \mathrm{Aut}F_{2g}$$

which corresponds to the natural map

$$\mathrm{Diff}^+(S_{g,1}; \partial) \hookrightarrow \mathrm{HE}(S_{g,1}; *)$$

## Homology

Algebraic description:

$$H_0(G) = \mathbf{Z} \text{ and } H_1(G) = G/[G, G]$$

For  $n > 4$  and  $g > 4$ ,

$$H_1(\Sigma_n) = H_2(\Sigma_n) = \mathbf{Z}/2\mathbf{Z}$$

$$H_1(\text{Aut}F_n) = H_2(\text{Aut}F_n) = \mathbf{Z}/2\mathbf{Z}$$

$$H_1(\Gamma_{g,1}) = 0; \quad H_2(\Gamma_{g,1}) = \mathbf{Z}$$

Topological description:

$H_*(G) := H_*(BG)$  where  $BG = EG/G$  is the orbit space of a contractible space  $EG$  with a good, free  $G$  action.

Example:  $G = \mathbf{Z}$ ,  $EG = \mathbf{R}$

$$\implies BG = \mathbf{R}/\mathbf{Z} \simeq S^1$$

$$\implies H_*(\mathbf{Z}) = H_*(S^1)$$

## Homology stability

Let  $\{G_n\}_{n>0}$  be a family of nested groups with  $G_n \subset G_{n+1}$ . Define the *stable group*

$$G_\infty := \lim_{n \rightarrow \infty} G_n$$

How is the homology of  $G_n$  related to that of  $G_{n+1}$  and  $G_\infty$ ?

$$H_*(\Sigma_n) \simeq H_*(\Sigma_{n+1}) \text{ for } * < (n+1)/2$$

$$H_*(\Gamma_{g,1}) \simeq H_*(\Gamma_{g+1,1}) \text{ for } * < 2g/3$$

$$H_*(\text{Aut}F_n) \simeq H_*(\text{Aut}F_{n+1}) \text{ for } * < (n-1)/2$$

These are difficult theorems due to Nakaoka, Harer (Ivanov, Boldsen, Randal-Williams), and Hatcher-Vogtmann-Wahl.

Strategy: determine the homology of  $G_\infty$ .

Miller/Morita:

$$H^*(B\Gamma_\infty) \otimes \mathbf{Q} \supset \mathbf{Q}[\kappa_i], \quad \deg(\kappa_i) = 2i$$

Mumford conj./Madsen-Weiss th.:

$$H^*(B\Gamma_\infty) \otimes \mathbf{Q} = \mathbf{Q}[\kappa_i], \quad \deg(\kappa_i) = 2i$$

Hatcher:

$$H^*(\text{Aut}F_\infty) \supset H^*(\Sigma_\infty)$$

Hatcher-Vogtmann conj./Galatius th.:

$$H^*(\text{Aut}F_\infty) = H^*(\Sigma_\infty)$$

and

$$H^*(\text{Aut}F_\infty) \otimes \mathbf{Q} = \mathbf{Q}$$



Product maps

$$G_n \times G_m \longrightarrow G_{n+m}$$

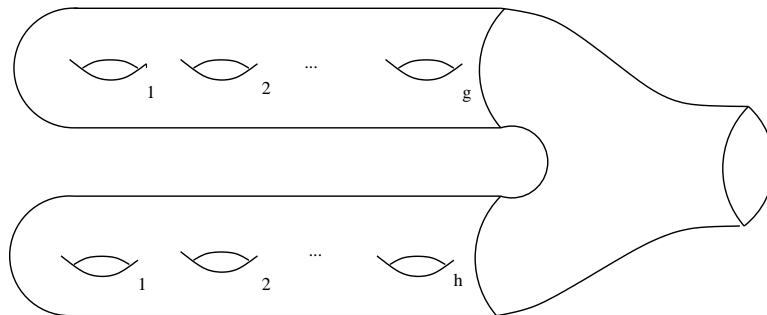
induce a monoid structure on

$$M := \bigsqcup_{n \geq 0} BG_n$$

$$\Sigma_n \times \Sigma_m \longrightarrow \Sigma_{n+m}$$

$$\text{Aut}F_n \times \text{Aut}F_m \longrightarrow \text{Aut}F_{n+m}$$

$$\Gamma_{g,1} \times \Gamma_{h,1} \longrightarrow \Gamma_{g+h,1}$$



all three products are commutative upto  
 conjugation by an element in  $G_{n+m}$   
 $\implies$  products are commutative in  $H_*$

## Group completion

Algebraic:

$M \longrightarrow \mathcal{G}(M)$ , the Grothendieck group of  $M$

Example:  $\mathbf{N} \longrightarrow \mathcal{G}(\mathbf{N}) = \mathbf{Z}$

Homotopy theoretic:

*group*  $\sim$  *loop space*:  $\Omega Y = \text{map}_*(S^1, Y)$

$$M \longrightarrow \Omega BM$$

Examples: •  $M$  a group  $\implies \Omega BM \simeq M$   
•  $M$  discrete  $\implies \Omega BM \simeq \mathcal{G}(M)$   
•  $M$  arbitrary  $\implies \pi_0(\Omega BM) = \mathcal{G}(\pi_0(M))$   
 $\implies$  for  $M = \bigsqcup_{n \geq 0} M_n$

$$\Omega BM = \mathbf{Z} \times \Omega_0 BM.$$

Group Completion Theorem:

Let  $M = \sqcup_{n \geq 0} M_n$  be a topological monoid such that the multiplication on  $H_*(M)$  is commutative. Then

$$H_*(\Omega_0 BM) = \lim_{n \rightarrow \infty} H_*(M_n) = H_*(M_\infty)$$

Barratt-Priddy-Quillen 1972

$$\Omega B\left(\bigsqcup_{n \geq 0} B\Sigma_n\right) \simeq \lim_{n \rightarrow \infty} \text{map}_*(S^n, S^n) =: \Omega^\infty S^\infty$$

Madsen-Weiss 2007

$$\Omega B\left(\bigsqcup_{g \geq 0} B\Gamma_{g,1}\right) \simeq \Omega^\infty \mathbf{MTSO}(2)$$

Galatius 2011?

$$\Omega B\left(\bigsqcup_{n \geq 0} B\text{Aut}F_n\right) \simeq \lim_{n \rightarrow \infty} \text{map}_*(S^n, S^n) =: \Omega^\infty S^\infty$$

Moduli space

$G$  acts on a compact geometric object  $W$   
 $\implies BG = \mathcal{M}^{top}(W)$ , a *topological moduli space*:

(i)  $W' \in \mathcal{M}^{top}(W)$  with  $W' \simeq W$

(ii)  $W$ -bundle  $E \rightarrow X \iff f_E : X \rightarrow \mathcal{M}^{top}(W)$

$W$  a manifold:

$\mathcal{M}^{top}(W) :=$  space of embedded  $W' \subset \mathbf{R}^N$  for  
some  $N \gg 0$

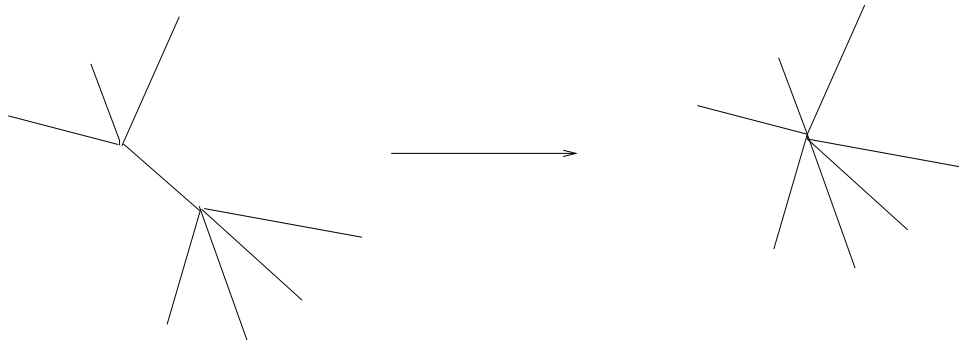
... for  $W = n$  points, this is the configuration  
space  $\mathcal{C}_n(\mathbf{R}^\infty)$  of  $n$  unordered points in  $\mathbf{R}^N$ ,  
 $N \gg 0$

... for  $W = S_{g,1}$ , this is homotopic to the cor-  
responding moduli space of Riemann surfaces

$$\mathcal{M}_{g,1} \simeq \mathcal{M}^{top}(S_{g,1}) \simeq B\Gamma_{g,1}$$

$W$  a graph:

$\mathcal{M}^{top}(W) :=$  space of embedded finite graphs  $W'$  (with base point) in  $\mathbf{R}^N$ ,  $N \gg 0$ , topologized such that edge collapses are continuous:



... for  $W = G_n$ ,

$$\mathcal{M}^{top}(G_n) = B\text{Aut}F_n$$

Galatius uses Culler-Vogtmann's Outer space and ideas of Igusa

$W$ -characteristic classes of  $E$ :

$$W\text{-bundle } E \rightarrow X \quad \longleftrightarrow \quad f_E : X \rightarrow \mathcal{M}^{top}(W)$$

$$\text{Example: } E = X \times W \quad \longleftrightarrow \quad f_E \simeq \text{const}_W$$

$$c \in H^*(\mathcal{M}^{top}(W)) \longrightarrow f_E^*(c) \in H^*(X)$$

$$\text{Example: } W = \mathbf{C}^n, \quad G = \text{GL}_n \mathbf{C}$$

$$\implies \mathcal{M}^{top}(\mathbf{C}^n) = \text{Gr}^{\mathbf{C}}(n) \text{ } \mathbf{C}\text{-Grassmannian}$$

$$H^*(\text{Gr}^{\mathbf{C}}(n)) = \mathbf{Z}[c_i], \quad \deg(c_i) = 2i$$

$$\text{Example: } W = S^1, \quad G = \text{Diff}^+(S^1) \simeq S^1$$

$$\implies \mathcal{M}^{top}(S^1) = \text{Gr}^{\mathbf{C}}(1) = \mathbf{C}P^\infty$$

$$H^*(\mathbf{C}P^\infty) = \mathbf{Z}[c_1]$$

What are the characteristic classes for more general  $W$ ?

Madsen-Weiss: stable classes for surfaces

Galatius: stable classes for graphs

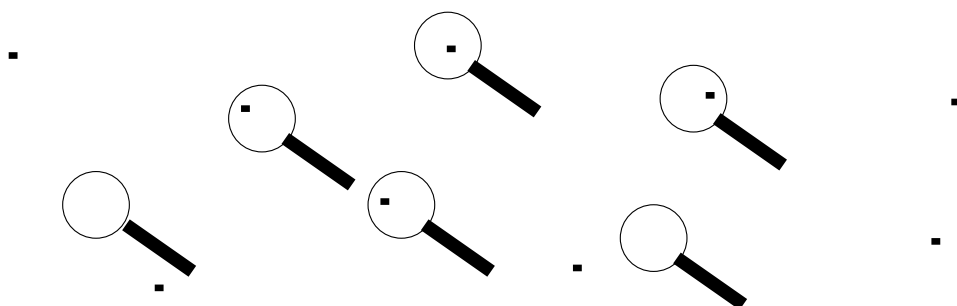
Scanning map

$$\alpha : \mathcal{M}^{top}(W)^N \longrightarrow \Omega^N(L^N) := \text{map}_*((\mathbf{R}^N)^c, L^N)$$

$\mathcal{M}^{top}(W)^N :=$  space of  $W$ 's in  $\mathbf{R}^N$

$L^N :=$  spaces of local data

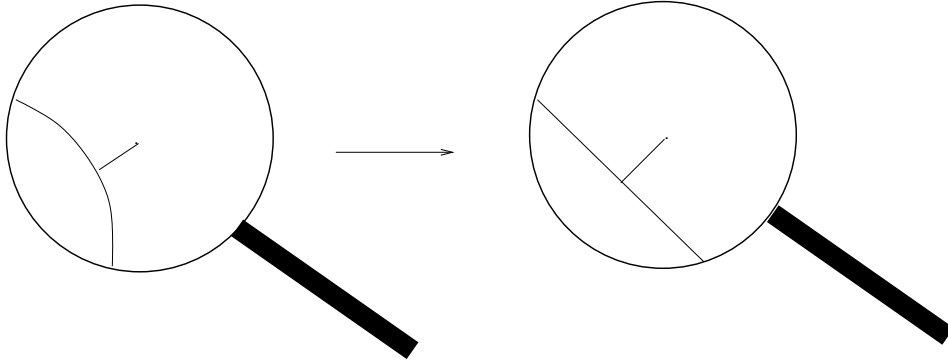
$W = n$  points:



$$L^N = (\mathbf{R}^N)^c = S^N$$

$$\alpha : \bigsqcup_{n \geq 0} \mathcal{C}_n(\mathbf{R}^N) \longrightarrow \Omega^N S^N$$

$W =$  a closed, oriented  $d$  dimensional manifold:



$$L^N \simeq (\gamma_{d,N}^\perp)^c = Th(\gamma_{d,N}^\perp)$$

$\gamma_{d,N} \rightarrow \text{Gr}(d, N) :=$  canonical bundle over the Grassmannian of real, oriented  $d$ -planes in  $\mathbf{R}^N$

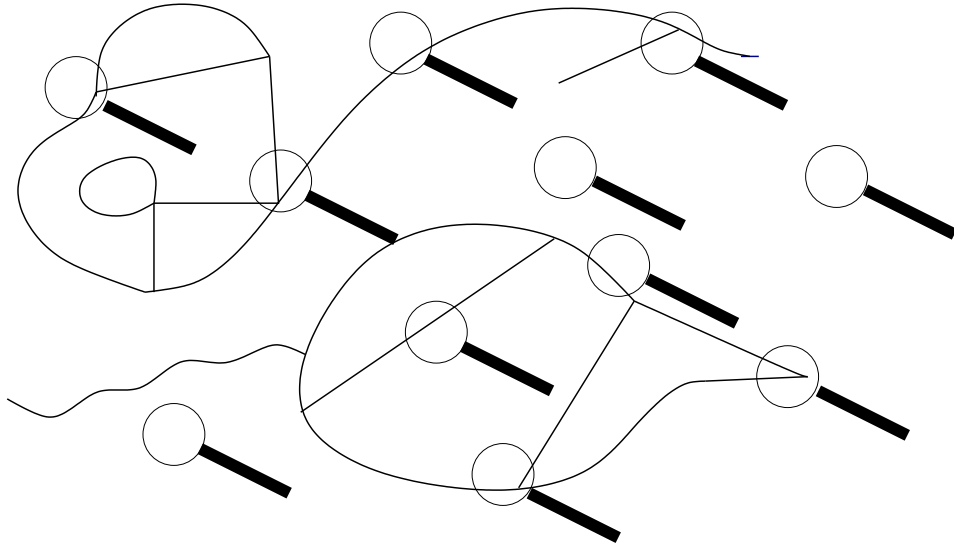
$$\alpha : \bigsqcup_W \mathcal{M}^{top}(W)^N \longrightarrow \Omega^N Th(\gamma_{d,N}^\perp)$$

$$\Omega^\infty \text{MTSO}(d) := \lim_{N \rightarrow \infty} \Omega^N Th(\gamma_{d,N}^\perp)$$

$$H^*(\Omega^\infty \text{MTSO}(d)) \otimes \mathbf{Q} = \mathbf{Q}[H^{*>d}(\text{Gr}(d))[-d] \otimes \mathbf{Q}]$$



$W =$  a finite graph:



$$\alpha : \bigsqcup_{n \geq 0} \mathcal{M}^{top}(G_n)^N \longrightarrow \Omega^N L^N$$

$$L^N \sim S^N \quad (2N - c) \text{ -- connected}$$

$$\lim_{N \rightarrow \infty} \Omega^N L^N \sim \lim_{N \rightarrow \infty} \Omega^N S^N = \Omega^\infty S^\infty$$

Outline of proof (for graphs):

(the argument also works for  $d$ -manifolds  
 $\implies$  main result of Galatius-Madsen-T-Weiss)

$$\Phi^{N,N} \supset \dots \supset \Phi^{N,1} \supset \Phi^{N,0} \simeq \bigsqcup_{n \geq 0} \mathcal{M}(G_n)^N$$

$\Phi^{N,i} :=$  all graphs in  $\mathbf{R}^i \times (0, 1)^{N-i}$ ,  
possibly not connected or infinite;  
the empty graph is the basepoint

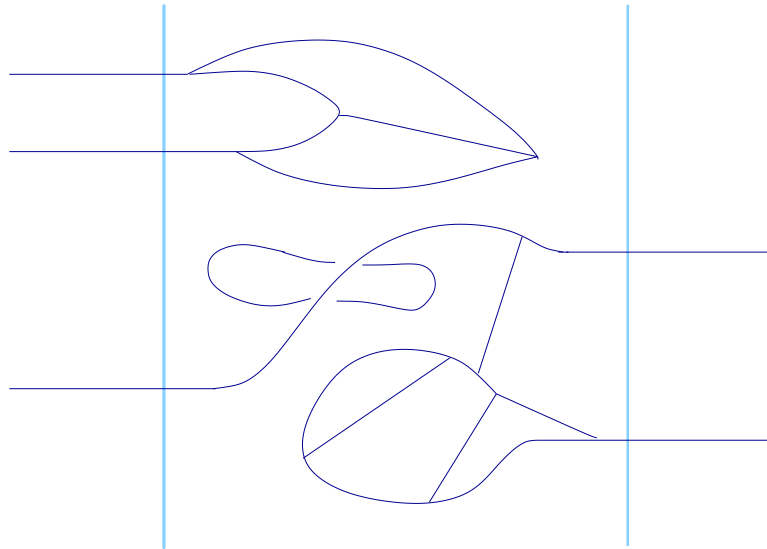
Step 1:  $\Phi^{N,N} \simeq L^N$

*Topology of  $\Phi^{N,N}$  allows to push radially away  
from origin.*

Step 2:  $\Phi^{N,k} \simeq \Omega \Phi^{N,k+1}$  for  $k > 0$

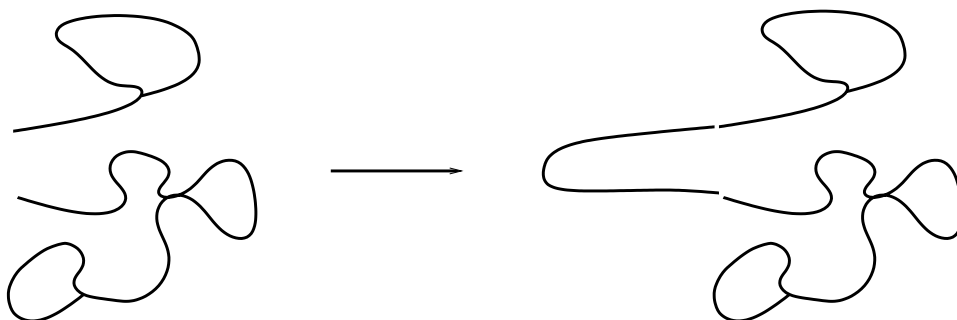
*At time  $t$ ,  $W' \in \Phi^{N,k}$  is translated by  $t$  in the  
direction of  $e_{k+1}$ .*

Step 3:  $\lim_{N \rightarrow \infty} \Phi^{N,1} \simeq B(\text{Cob})$  where Cob is the cobordism category of embedded graphs



Step 4:  $B(\text{Cob}) \simeq B(\bigsqcup_{n \geq 0} \mathcal{M}(G_n))$

*Needs surgery argument.*



## Generalizations

$W_n$	$H_*$ -stab.	$\Omega B(\sqcup_n \mathcal{M}(W_n))$
$n$ pts	$(n+1)/2$	$\Omega^\infty S^\infty$
$S_{g,1}$	$(2g)/3$	$\Omega^\infty \text{MTSO}(2)$
$N_{g,1}$	$(n-3)/3$	$\Omega^\infty \text{MTO}(2)$
$\#_n(S^k \times S^k)_1$	?	$\Omega^\infty \text{MTSO}(2k)^{\langle k \rangle}$
$G_n$	$(n-1)/2$	$\Omega^\infty S^\infty$
$\#_g(S^1 \times D^2)_1$	$(g-1)/2$	$\Omega^\infty S^\infty BSO(3)_+$
$\#_n(S^1 \times S^2)_1$	?	$\Omega^\infty S^\infty BSO(4)_+$

Wahl:  $N_{g,1}$  non-orientable surface of genus  $g$

Galatius & Randal-Williams:

$\#_n(S^k \times S^k)_1$  a  $(k-1)$ -connected manifold

Hatcher:

$\#_g(S^1 \times D^2)_1$  a handlebody of genus  $g$

$\#_n(S^1 \times S^2)_1$  a 3-manifold with one boundary sphere

Questions:

- Does homology stability also for  $n$  connected sums of

$$S^k \times S^k \quad \text{or} \quad S^1 \times S^2?$$

- Are there other manifolds for which similar results can be proved?
- Can we generalise from graphs to higher dimensional simplicial complexes?

Scanning: for any oriented  $d$ -dimensional  $W$

$$\alpha : \mathcal{M}^{top}(W) \rightarrow \Omega^\infty \mathbf{MTSO}(d)$$

Ebert: When  $d$  is even,

for any  $c \in H^*(\Omega^\infty \mathbf{MTSO}(d)) \otimes \mathbf{Q}$

there is a  $W$  with  $\alpha^*(c) \neq 0 \in H^*(W) \otimes \mathbf{Q}$ .

When  $d$  is odd, this no longer holds.

- What is the natural space  $X(d)$  for  $d$  odd whose cohomology contains all rational stable characteristic classes for  $d$  dimensional manifolds (and no more)?

$$X(1) = \Omega^\infty S^\infty(BSO(2))_+$$

$$X(3) = \Omega^\infty S^\infty(BSO(4))_+?$$