Homotopy theory for geometric groups

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Groups with geometry

$\Sigma_n :=$ permutation group

$$< a_1, \ldots, a_{n-1} | \, a_i a_j = a_j a_i \text{ for } |i-j| > 2,$$
$$a_i a_j a_i = a_j a_i a_j \text{ for } |i-j| = 2,$$
$$a_i^2 = e >$$

$\beta_n :=$ braid group

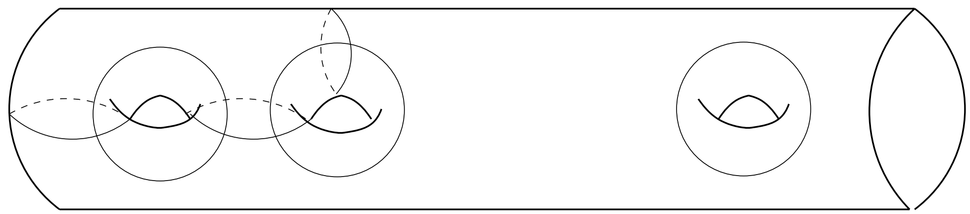
$$< a_1, \ldots, a_{n-1} | \, a_i a_j = a_j a_i \text{ for } |i-j| > 2,$$
$$a_i a_j a_i = a_j a_i a_j \text{ for } |i-j| = 2 >$$

$\text{Aut} F_n :=$ automorphisms of free group

with $F_n = < x_1, \ldots, x_n >$

$$\omega : \beta_n \longrightarrow \Sigma_n \subset \text{Aut} F_n$$

$S_{g,k}$:= oriented surface of genus g with k ∂-components

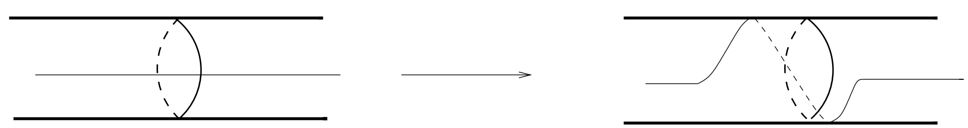


$\mathrm{Diff}(S_{g,k};\partial)$:= group of diffeomorphisms fixing ∂

$\Gamma_{g,k}$:= mapping class group

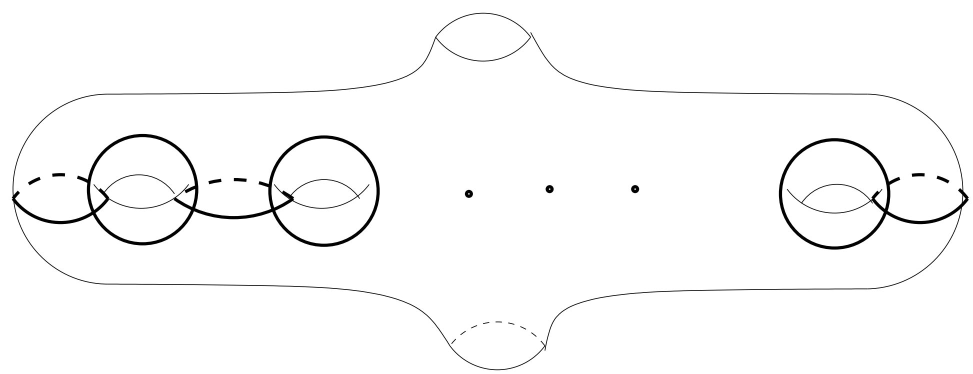
$$= \pi_0(\mathrm{Diff}(S_{g,k};\partial)) \xleftarrow{\simeq} \mathrm{Diff}(S_{g,k};\partial) \text{ if } \chi < 0$$

generated by Dehn twists:



$$\Gamma_{g,k} =< \text{ Dehn twists } D_a |\, D_aD_b = D_bD_a \text{ if } |a\cap b| = 0;$$
$$D_aD_bD_a = D_bD_aD_b \text{ if } |a\cap b| = 1;$$
$$\text{other relations} >$$

$$\mathcal{N}_{g,k} = \pi_0 \mathrm{Diff}(N_{g,k};\partial) \xleftarrow{\simeq} \mathrm{Diff}(N_{g,k};\partial) \text{ if } \chi < 0$$
$$=< \text{ Dehn twists } D_a, \text{ cross cap slide } \sigma |\text{ relations } >$$

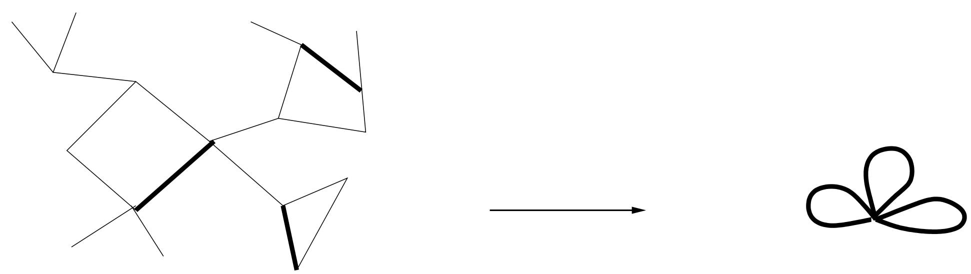


$$\phi : \beta_{2g+2} \hookrightarrow \Gamma_{g,2}$$

X_n := any space of homotopy type a wedge of n circles

...• a connected, compact graph with $\chi = 1 - n$
...• an n-punctured disc
...• a surface with boundary and $\chi = 1 - n$

$$\pi_1(X_n) = F_n$$



HE(X_n; $*$) := monoid of homotopy equivalences of X_n that fix a base point

$$\mathrm{Aut}F_n = \pi_0(\mathrm{HE}(X_n;*)) \xleftarrow{\simeq} \mathrm{HE}(X_n;*)$$

The action on of the mapping class group on the fundamental group of the surface induces injections

$$\alpha : \beta_n \hookrightarrow \mathsf{Aut} F_n$$

$$\pi^+ : \Gamma_{g,1} \hookrightarrow \mathsf{Aut} F_{2g}$$

$$\pi : \mathcal{N}_{g,1} \hookrightarrow \mathsf{Aut} F_g$$

Question:
What is the (co)homology of these groups, and what are the induced maps?

Homology

Algebraic description:

$H_0(G) = \mathbb{Z}$

$H_1(G) = G/[G, G]$

$H_2(G) = R \cap [F, F]/[F, R]$ where $G = F/R$

$H_*(G) = \mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}, -)$ = left derived functor of $\otimes_G \mathbb{Z}$

Thus, for $n \geq 4$ and $g \geq 7$,

$$H_1(\Sigma_n) = H_2(\Sigma_n) = \mathbb{Z}/2\mathbb{Z}$$

$$H_1(\beta_n) = \mathbb{Z}$$

$$H_1(\mathrm{Aut}F_n) = H_2(\mathrm{Aut}F_n) = \mathbb{Z}/2\mathbb{Z}$$

$$H_1(\Gamma_{g,1}) = 0; \qquad H_2(\Gamma_{g,1}) = \mathbb{Z}$$

$$H_1(\mathcal{N}_{g,1}) = \mathbb{Z}/2\mathbb{Z}$$

Topological description:

$H_*(BG)$ where $BG = EG/G$ is the orbit space of a contractible space EG with a good, free G action. The classifying space BG is defined up to homotopy.

Example: $G = \mathbb{Z}$, $EG = \mathbb{R}$
$\Longrightarrow BG = \mathbb{R}/\mathbb{Z} \simeq S^1$
$\Longrightarrow H_*^{EM}(\mathbb{Z}) = H_*(S^1)$

Functorial construction: $BG = |B_\bullet G|$ where $B_q G = G^q$ and

$$\partial_i(g_1, \ldots, g_q) = \begin{cases} (g_2, \ldots, g_q) \text{ if } i = 0 \\ (g_1, \ldots, g_i g_{i-1}, \ldots g_q) \text{ if } 0 < i < q \\ (g_1, \ldots, g_{q-1}) \text{ if } i = q \end{cases}$$

Homology stability

$\{G_n\}_{n>0}$ a family of nested groups with $G_n \subset G_{n+1}$.
The associated stable group

$$G_\infty := \lim_{n \to \infty} G_n$$

Question:
How does $H_*(G_n)$ relate to $H_*(G_{n+1})$ and $H_*(G_\infty)$?

Nakaoka:

$H_*(\Sigma_n) \simeq H_*(\Sigma_{n+1})$ for $* < n/2$

Harer/Ivanonv/Boldsen:

$H_*(\Gamma_{g,1}) \simeq H_*(\Gamma_{g+1,1})$ for $* < 2g/3$

Hatcher-Vogtmann/Wahl:

$H_*(\mathrm{Aut}F_n) \simeq H_*(\mathrm{Aut}F_{n+1})$ for $* < n/2$

Wahl/Randal-Williams:

$H_*(\mathcal{N}_{g,1}) \simeq H_*(\mathcal{N}_{g+1,1})$ for $* < g/3$

Consider the stable groups G_∞!

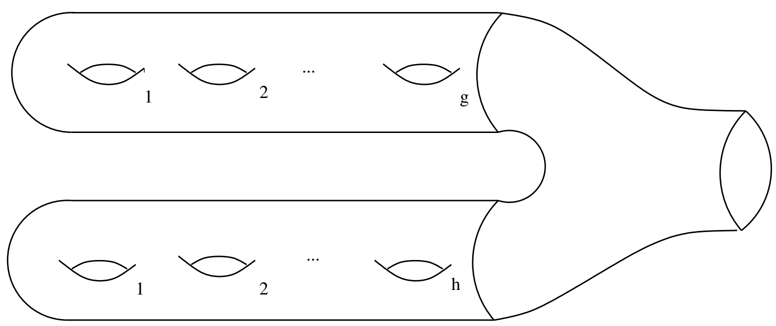
Multiplicative structure

$$\Sigma_n \times \Sigma_m \longrightarrow \Sigma_{n+m}$$

$$\beta_n \times \beta_m \longrightarrow \beta_{n+m}$$

$$\mathrm{Aut}F_n \times \mathrm{Aut}F_m \longrightarrow \mathrm{Aut}F_{n+m}$$

$$\Gamma_{g,1} \times \Gamma_{h,1} \longrightarrow \Gamma_{g+h,1}$$



Products

$$G_n \times G_m \longrightarrow G_{n+m}$$

induce a monoid structure on

$$M := \coprod_{n \geq 0} BG_n$$

If products are commutative upto conjugation by an element in G_{n+1},

then the induced product on $H_*(M)$ is commutative

Group completion

Algebraic: $M \longrightarrow \mathcal{G}(M) =$ Grothendieck group of M
Example: $\mathbb{N} \longrightarrow \mathcal{G}(\mathbb{N}) = \mathbb{Z}$

Homotopy theoretic:
$M \longrightarrow \Omega BM = \mathrm{map}_*(S^1, BM) =$ loop space of BM

- $M = G$ a group $\Longrightarrow \Omega BG \simeq G$
- M discrete $\Longrightarrow \Omega BM \simeq \mathcal{G}(M)$

Group Completion Theorem:
Let $M = \bigsqcup_{n \geq 0} M_n$ be a topological monoid such that the multiplication on $H_*(M)$ is commutative. Then

$$H_*(\Omega BM) = \mathbb{Z} \times \lim_{n \to \infty} H_*(M_n) = \mathbb{Z} \times H_*(M_\infty)$$

Barratt-Priddy-Quillen 1972

$\Omega B(\coprod_{n\geq 0} B\Sigma_n) \simeq \lim_{n\to\infty} \mathrm{map}_*(S^n, S^n) =: \Omega^\infty S^\infty$

$H_*(\Omega^\infty S^\infty)\otimes\mathbb{Q} = H_*(B\Sigma_\infty)\otimes\mathbb{Q} = \mathbb{Q}$ Serre

F. Cohen 1976

$\Omega B(\coprod_{n\geq 0} B\beta_n) \simeq \Omega^2 S^2 = \mathbb{Z}\times\Omega^2 S^2(S^1)$

Madsen-Weiss 2007

$\Omega B(\coprod_{g\geq 0} B\Gamma_{g,1}) \simeq \Omega^\infty \mathbf{MTSO}(2)$ conj Madsen-Tillmann

$H^*(\Gamma_\infty)\otimes\mathbb{Q} = \mathbb{Q}[\kappa_i],\quad \deg(\kappa_i) = 2i$ conj Mumford

Galatius 2011

$\Omega B(\coprod_{n\geq 0} B\mathrm{Aut}F_n) \simeq \lim_{n\to\infty} \mathrm{map}_*(S^n, S^n) =: \Omega^\infty S^\infty$

$H^*(\mathrm{Aut}F_\infty) = H^*(\Sigma_\infty)$

$H^*(\mathrm{Aut}F_\infty)\otimes\mathbb{Q} = \mathbb{Q}$ conj Hatcher-Vogtmann

Maps on stable homology

Song-Tillmann/Segal-Tillmann

$\phi_* : H_*(\beta_\infty) \longrightarrow H_*(\Gamma_\infty)$ is zero for $* > 0$.

Galatius

$incl_* : H_*(\Sigma_\infty) \xrightarrow{\simeq} H_*(\mathrm{Aut}F_\infty)$

Tillmann

$\pi_* : H_*(\mathcal{N}_\infty) \longrightarrow H_*(\mathrm{Aut}F_\infty)$ is split surjective.

$\pi_*^+ : H_*(\Gamma_\infty) \otimes \mathbb{Z}[1/2] \longrightarrow H_*(\mathrm{Aut}F_\infty) \otimes \mathbb{Z}[1/2]$

. is split surjective.

$\alpha_* = \omega_* : H_*(\beta_\infty) \longrightarrow H_*(\mathrm{Aut}F_\infty)$

Operad structure

$\mathcal{O} = \{\mathcal{O}_k\}_{k\geq 0}$ is an operad :

$$\gamma : \mathcal{O}_n \times (\mathcal{O}_{k_1} \times \cdots \times \ldots \mathcal{O}_{k_n}) \longrightarrow \mathcal{O}_{\Sigma_i k_i}$$

$M = \{M_k\}_{k\geq 0}$ is an $\mathcal{O}$-algebra:

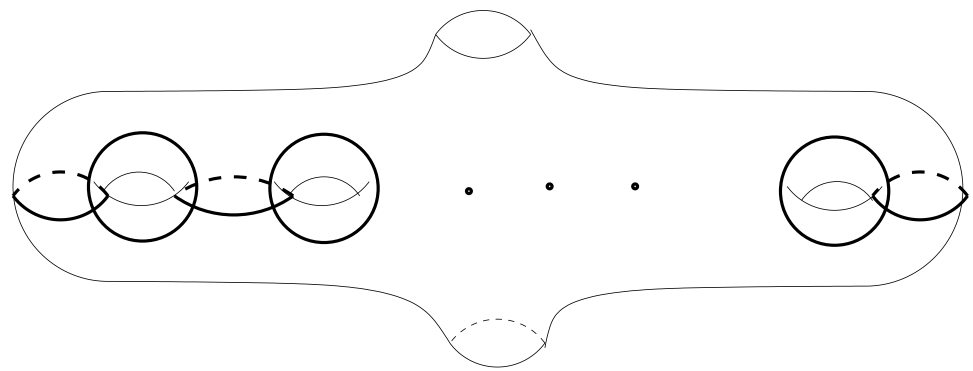
$$\theta : \mathcal{O}_n \times (M_{k_1} \times \cdots \times M_{k_n}) \longrightarrow M_{\Sigma_i k_i}$$

F. Cohen: $\mathcal{B}$ with $\mathcal{B}_k = BP\beta_k$
M a $\mathcal{B}$-algebra $\Longrightarrow \Omega B(M)$ an Ω^2-space.

Wahl: $\mathcal{RB}$ with $\mathcal{RB}_k = B\Gamma_{0,k+1}$
M a $\mathcal{RB}$-algebra $\Longrightarrow \Omega B(M)$ Ω^2-space with S^1-action.

Tillmann: $\mathcal{S}$ with $\mathcal{S}_k = \coprod_{g\geq 0} B\Gamma_{g,k+1}$
M a $\mathcal{S}$-algebra $\Longrightarrow \Omega B(M)$ an Ω^∞-space.

$$\mathcal{B} \longrightarrow \mathcal{RB} \longrightarrow \mathcal{S}$$



Sketch of Proof: $\phi : \beta_{2g+2} \to \Gamma_{g,2}$ induces a map of $\mathcal{B}$-algebras and hence an Ω^2-map

$$\Phi : \Omega_0^2 S^2 = \Omega^2 S^2(S^1) \longrightarrow \Omega_0 B(\coprod_{g\geq 2} B\Gamma_{g,2})$$

As $\Omega^2 S^2(S^1)$ is the free Ω^2-space on S^1, the map Φ is determined upto homotopy by its restriction to S^1. But $H_1(\Gamma_\infty) = 0$ and so

$$\Phi|_{S^1} = * \quad \text{and} \quad \phi_* = 0 \text{ for } * > 0$$

Sketch of Proof:

$$\pi : \mathcal{N}_{g,1} \to \mathsf{Aut}F_g$$

induces a map of $\mathcal{S}$-algebras. Hence, using Galatius' theorem, an Ω^∞-map

$$\Pi : \Omega B(\coprod_{g \geq 0} B\mathcal{N}_{g,1}) \longrightarrow \Omega^\infty S^\infty$$

Construct a splitting of spaces as follows:
Map S^0 by sending the non-basepoint to $* \in B\mathcal{N}_{1,1}$.
Extend to an Ω^∞-map from $\Omega^\infty S^\infty = \Omega^\infty S^\infty(S^0)$.
Compose with Π.

The composition is an Ω^∞-map and hence is determined by its restriction to S^0.

Barratt-Priddy-Quillen 1972

$\Omega B(\coprod_{n\geq 0} B\Sigma_n) \simeq \lim_{n\to\infty} \mathrm{map}_*(S^n, S^n) =: \Omega^\infty S^\infty$

$H_*(\Omega^\infty S^\infty) \otimes \mathbb{Q} = \mathbb{Q}$ Serre

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$\Omega B(\coprod_{n\geq 0} B\mathrm{Aut}F_n) \simeq \lim_{n\to\infty} \mathrm{map}_*(S^n, S^n) =: \Omega^\infty S^\infty$

$H^*(\mathrm{Aut}F_\infty) = H^*(\Sigma_\infty)$

$H^*(\mathrm{Aut}F_\infty) \otimes \mathbb{Q} = \mathbb{Q}$ conj Hatcher-Vogtmann

Topological moduli space

W **a manifold**, $G = \text{Diff}(W)$:

$$\mathcal{M}^{top}(W) := \text{Emb}(W, \mathbb{R}^\infty)/\text{Diff}(W)$$

the space of embedded $W' \subset \mathbb{R}^\infty$ with $W \simeq W'$

... for $W = n$ points, this is the configuration space of n unordered points in $\mathbb{R}^\infty$

$$\mathcal{M}^{top}(n \text{ pts}) = \mathcal{C}_n(\mathbf{R}^\infty) = B\Sigma_n$$

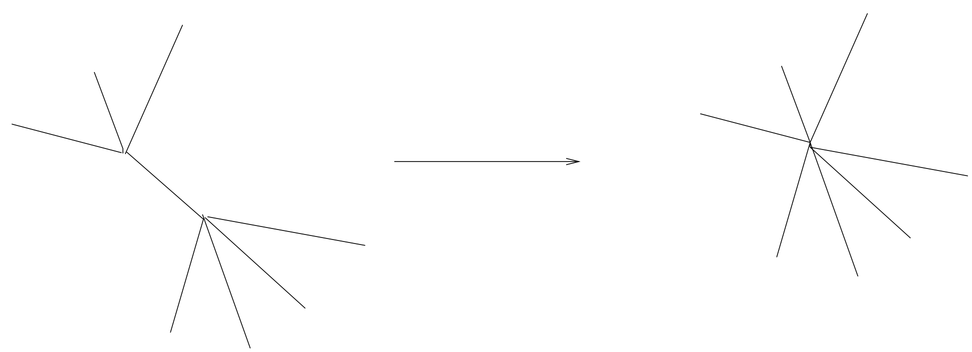
... for $W = S_{g,1}$, this is homotopic to the corresponding moduli space of Riemann surfaces

$$\mathcal{M}_{g,1} \simeq \mathcal{M}^{top}(S_{g,1}) \simeq B\Gamma_{g,1}$$

W **a pointed, finite graph**, $G = \mathsf{HE}(W, *)$:

$\mathcal{M}^{top}(W) :=$ space of homotopic graphs $W' \subset \mathbb{R}^\infty$

topologized such that edge collapses are continuous:



... for $W = X_n$, this is a pointed version of Culler-Vogtmann's Outer space

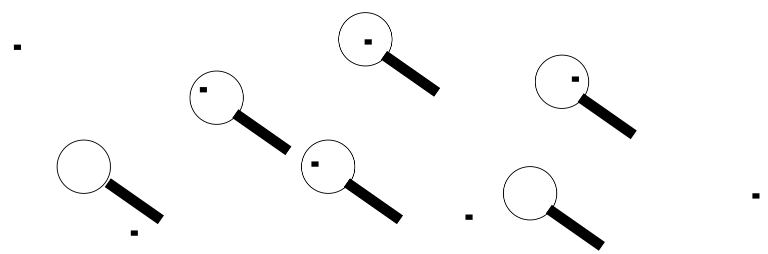
$$\mathcal{M}^{top}(X_n) = B\mathsf{Aut}F_n$$

Scanning map

$\mathcal{M}^{top}(W)^N :=$ space of W's in $\mathbb{R}^N$
$L^N :=$ spaces of local data

$$\alpha : \mathcal{M}^{top}(W)^N \longrightarrow \Omega^N(L^N) := \text{map}_*((\mathbb{R}^N)^c, L^N)$$

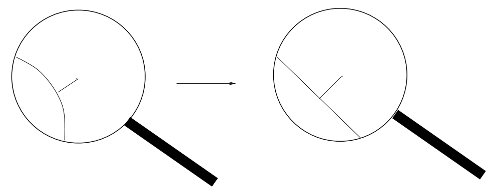
Example: $W = n$ points



$$L^N = (\mathbb{R}^N)^c = S^N \quad \text{and} \quad \alpha : \coprod_{n\geq 0} \mathcal{C}_n(\mathbb{R}^N) \longrightarrow \Omega^N S^N$$

Segal

Example: $W =$ a closed, oriented d-manifold



$$L^N \simeq (\gamma_{d,N}^{\perp})^c = Th(\gamma_{d,N}^{\perp})$$

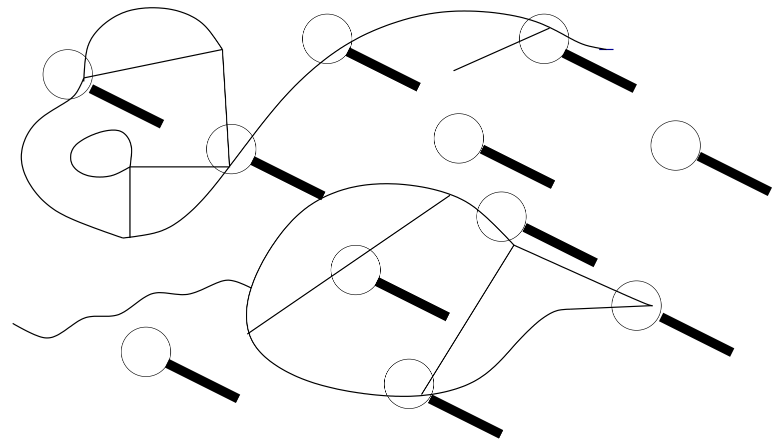
$\gamma_{d,N} \to \mathrm{Gr}(d,N) :=$ canonical bundle over the Grassmannian of real, oriented d-planes in $\mathbb{R}^N$

$$\alpha : \bigsqcup_W \mathcal{M}^{top}(W)^N \longrightarrow \Omega^N Th(\gamma_{d,N}^{\perp})$$

$$\Omega^\infty \mathbf{MTSO(d)} := \lim_{N \to \infty} \Omega^N Th(\gamma_{d,N}^{\perp})$$

Thom-Pontryagin, Madsen-Tillmann

Example: W = a finite graph



$$\alpha : \coprod_{n\geq 0} \mathcal{M}^{top}(G_n)^N \longrightarrow \Omega^N L^N$$

$$L^N \sim S^N \quad (2N - c) - \text{connected}$$

$$\lim_{N\to\infty} \Omega^N L^N \sim \lim_{N\to\infty} \Omega^N S^N = \Omega^\infty S^\infty$$

Galatius

Generalizations

W_n	H_*-stab.	$\Omega B(\sqcup_n \mathcal{M}(W_n))$	
n pts	n/2	$\Omega^\infty S^\infty$	Barratt-Eccles-Priddy
$S_{g,1}$	2g/3	$\Omega^\infty \mathbf{MTSO(2)}$	Madsen-Weiss
$N_{g,1}$	g/3	$\Omega^\infty \mathbf{MTO(2)}$	Madsen-Weiss & Wahl
X_n	n/2	$\Omega^\infty S^\infty$	Galatius
$\#_g(S^1 \times D^2)_1$	(g-1)/2	$\Omega^\infty S^\infty BSO(3)_+$	Hatcher
$\#_g(S^k \times S^k)_1$	(g-4)/2	$\Omega^\infty \mathbf{MTSO(2k)}^{\langle k \rangle}$	Galatius-Randal-Williams
discrete	(g-4)/2	$\Omega^\infty X^{-\gamma}$	Nariman

Happy 150th Birthday to the LMS!