Computational Algebraic Topology Topic B: Sheaf cohomology and applications to quantum non-locality and contextuality
Lecture 1

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Background Material

See the syllabus and reading material at

http://people.maths.ox.ac.uk/tillmann/CAT.html

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Papers (all available on arxiv):

1. Essential
   1. S. Abramsky and A. Brandenburger. The sheaf-theoretic structure of non-locality and contextuality. *New Journal of Physics*, 13(2011):113036, 2011.
   2. S. Abramsky, R. S. Barbosa and S. Mansfield, The Cohomology of Non-Locality and Contextuality, in *Proceedings of QPL 2011*, EPTCS 2011.
   3. S. Abramsky, R. S. Barbosa, K. Kishida, R. Lal and S. Mansfield, Contextuality, Cohomology and Paradox (submitted).
2. Useful additional reading
   - S. Abramsky and L. Hardy. Logical Bell Inequalities. *Phys. Rev. A* 85, 062114 (2012).
   - S. Abramsky, G. Gottlob and P. Kolaitis, Robust Constraint Satisfaction and Local Hidden Variables in Quantum Mechanics, Proceedings IJCAI 2013.
   - S. Abramsky, Relational Databases and Bell's Theorem, In *In Search of Elegance in the Theory and Practice of Computation: Essays Dedicated to Peter Buneman*, Springer 2013.

Beginnings . . .

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It is no longer safe to assume this!

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- There is a fascinating two-way interplay developing between Computer Science and Physics, extending to the foundations of both, as well as to more practical matters. Quantum technology — "hacking matter" — will be a huge feature of 21st Century science and engineering, and a lot of it will be to do with information.

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- This is an exciting emerging area, attracting students with backgrounds in CS, Physics, Mathematics, Philosophy, . . .

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- They also have profound consequences for our understanding of the very nature of physical reality.
- We shall describe recent work in which tools from Computer Science are used to shed new light on these phenomena.
- There are also striking and unexpected connections with a number of topics in **classical** computer science, including relational databases and constraint satisfaction.

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- Cohomology. Sheaf theory provides the natural mathematical setting for our analysis, since it is directly concerned with the passage from local to global. In this setting, it is furthermore natural to use **sheaf cohomology** to characterise contextuality. Cohomology is one of the major tools of modern mathematics, which has until now largely been conspicuous by its **absence**, in logic, theoretical computer science, and quantum information.

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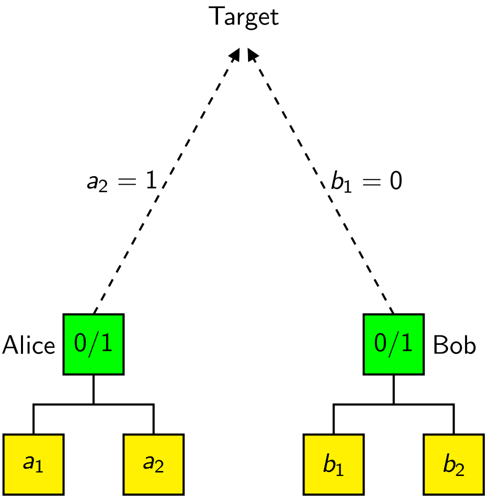
- Our results show that **cohomological obstructions to the extension of local sections to global ones** witness a large class of contextuality arguments.

Alice and Bob look at bits



A Probabilistic Model Of An Experiment

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Example: The Bell Model

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a_1	b_1	$1/2$	0	0	$1/2$
a_1	b_2	$3/8$	$1/8$	$1/8$	$3/8$
a_2	b_1	$3/8$	$1/8$	$1/8$	$3/8$
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The entry in row 2 column 3 says:

If Alice looks at a_1 *and Bob looks at* b_2*, then* $1/8$*th of the time, Alice sees a* 0 *and Bob sees a* 1*.*

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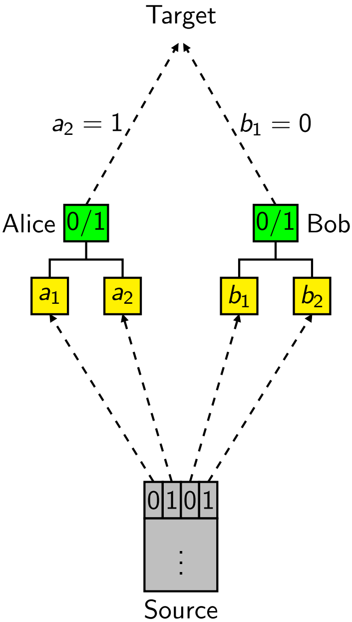
How can we explain this behaviour?

Classical Correlations



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Using elementary probability theory, we can calculate:

$$p_N \;\leq\; \mathrm{Prob}(\bigvee_{i=1}^{N-1} \neg\phi_i) \;\leq\; \sum_{i=1}^{N-1} \mathrm{Prob}(\neg\phi_i) \;=\; \sum_{i=1}^{N-1}(1 - p_i) \;=\; (N-1) - \sum_{i=1}^{N-1} p_i.$$

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Hence we obtain the inequality

$$\sum_{i=1}^{N} p_i \;\leq\; N - 1.$$

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	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
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$$\begin{array}{ccccccccc}
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The violation of the logical Bell inequality is $1/4$.

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Alice and Bob's choices are now of **measurement setting** (e.g. which direction to measure spin) rather than "which register to load".

The Quantum Case: Spin Measurements

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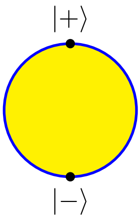
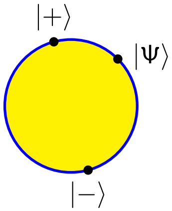
States of the system can be described by complex unit vectors in $\mathbb{C}^2$. These can be visualized as points on the unit 2-sphere:





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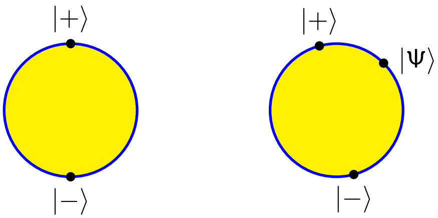

Spin can be measured in any direction; so there are a continuum of possible measurements. There are **two possible outcomes** for each such measurement; spin in the specified direction, or in the opposite direction. These two directions are represented by a pair of orthogonal vectors. They are represented on the sphere as a pair of **antipodal points**.

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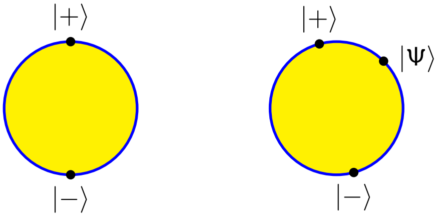

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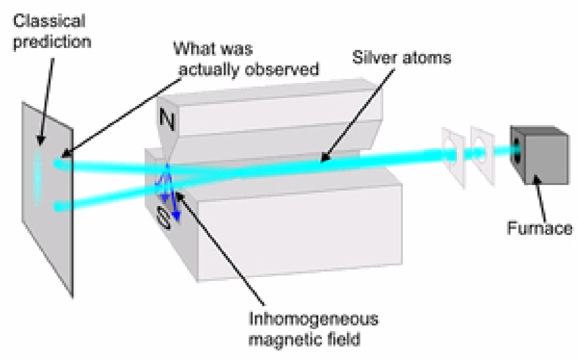
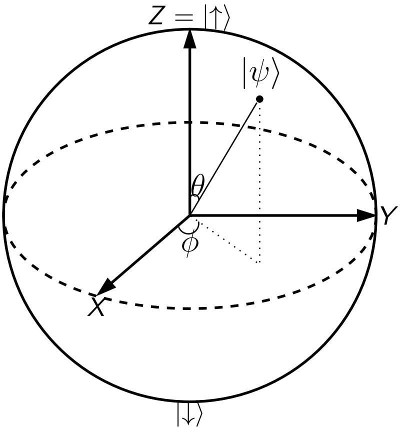
Note the appearance of **quantization** here: there are not a continuum of possible outcomes for each measurement, but only two!

The Stern-Gerlach Experiment



The Bloch sphere representation of qubits



Properties of the Qubit

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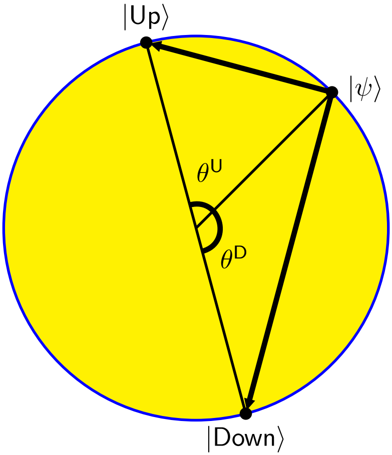
- When we subject a qubit to a measurement $(\mathsf{Up}, \mathsf{Down})$, the state of the qubit determines a probability distribution on the two possible outcomes. The probabilities are determined by the **angles** between the qubit state $|\psi\rangle$ and the points $(|\mathsf{Up}\rangle, |\mathsf{Down}\rangle)$ which specify the measurement. In algebraic terms, $|\psi\rangle$, $|\mathsf{Up}\rangle$ and $|\mathsf{Down}\rangle$ are unit vectors in the complex vector space $\mathbb{C}^2$, and the probability of observing Up when in state $|\psi\rangle$ is given by the square modulus of the inner product:

$$|\langle\psi|\mathsf{Up}\rangle|^2.$$

  This is known as the **Born rule**. It gives the basic predictive content of quantum mechanics.

Truth makes an angle with reality



Qubits vs. Bits

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The sense in which the qubit generalises the classical bit is that, for each question we can ask — *i.e.* for each measurement — there are just two possible answers. We can view the states of the qubit as superpositions of the classical states 0 and 1, so that we have a probability of getting each of the answers for any given state.

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Note in addition that a measurement has an **effect** on the state, which will no longer be the original state $|\psi\rangle$, but rather one of the states Up or Down, in accordance with the measured value.

Quantum Entanglement

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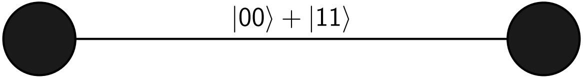
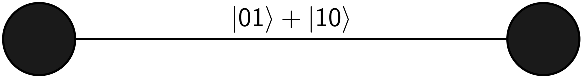
Bell state:



EPR state:



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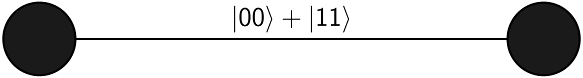

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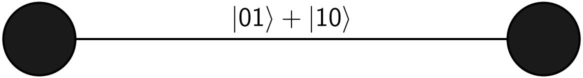

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$$\sum_i \lambda_i \cdot \phi_i \otimes \psi_i$$

Superposition encodes **correlation**.

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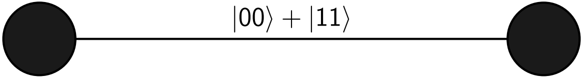

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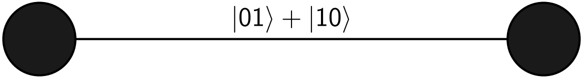

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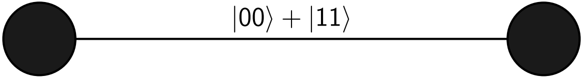

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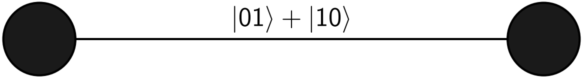

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Bell's theorem: QM is **essentially non-local**.

A Probabilistic Model Of An Experiment

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Example: The Bell Model

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	$1/2$	0	0	$1/2$
a_1	b_2	$3/8$	$1/8$	$1/8$	$3/8$
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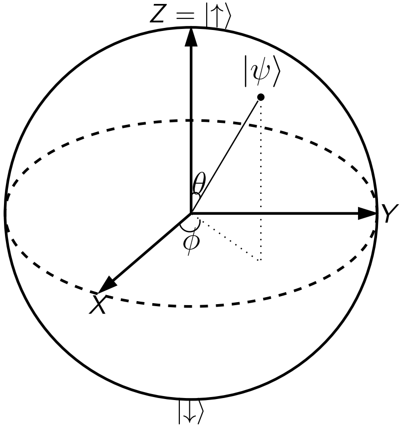
Extensively tested experimentally.

Computing the Bell table



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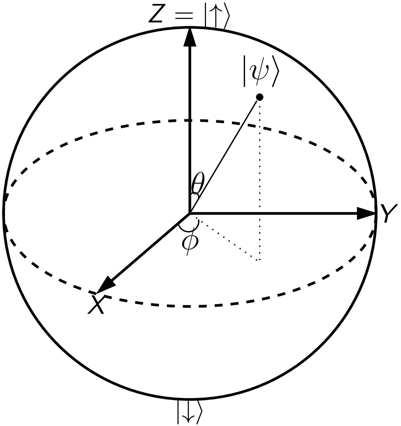

Spin measurements lying in the equatorial plane of the Bloch sphere
Spin Up: $(|\uparrow\rangle + e^{i\phi}|\downarrow\rangle)/\sqrt{2}$, Spin Down: $(|\uparrow\rangle + e^{i(\phi+\pi)}|\downarrow\rangle)/\sqrt{2}$

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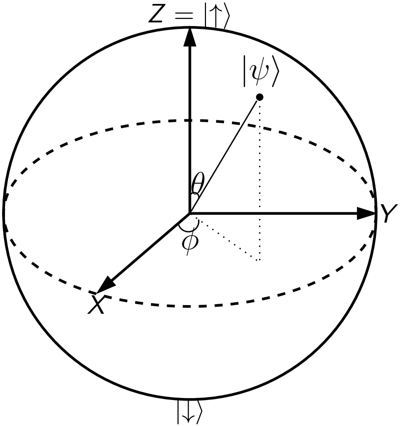

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X itself, $\phi = 0$:
Spin Up $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ and Spin Down $(|\uparrow\rangle - |\downarrow\rangle)/\sqrt{2}$.

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Probability of this event M when measuring (a, b') on $B = (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)/\sqrt{2}$ is given by Born rule:

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The other entries can be computed similarly.