CAT 2015-16

Problem Sheet (3)

Categorification, stability, and spectral sequences.

(1) Let $X = X_N \supset \cdots \supset X_0 = \emptyset$ be a finite filtered complex and $f : X \to \mathbb{R}$ be a discrete Morse function compatible with the filtration, i.e. $f(\sigma) < f(\tau)$ for all $\sigma \in X_s$ and $\tau \in X \setminus X_s$.

Prove that the associated Morse complex has a natural filtration such that the associated persistent homology is the same as that of the original filtration on X.

(2) For two $(\mathbb{R}, \leq)$-diagrams F and G in $\mathcal{D}$ and any functor $H : \mathcal{D} \to \mathcal{E}$ show that the interleaving distance satisfies:

$$d(HF, HG) \leq d(F, G).$$

(3) For the interval $I \subset R$ let χ_I be the functor from $(\mathbb{R}, \leq)$ to the category of finite vector spaces be the characteristic diagram defined in lectures.

(i) Show that if χ_I has no critical value than it is constant.

(ii) Let I and J be two intervals. Find the interleaving distance from χ_I to χ_J. Treat the cases when I or J are empty, or when one of them is infinite separately.

(4) For partial matchings θ' between A and B, and θ'' between B and C, show that the penalty of the induced partial matching $\theta = \theta'' \circ \theta'$ is less or equal to the sum of the penalties for θ' and θ'':

$$P(\theta) \leq P(\theta'') + P(\theta').$$

(5) Given an exact sequence of vector spaces

$$V_{-2} \xrightarrow{i} V_{-1} \xrightarrow{p} E \xrightarrow{\partial} V_1 \xrightarrow{i'} V_2$$

prove that

$$\dim(E) = [\dim(V_{-1}) - \dim(\mathrm{Im}(i))] + [\dim(V_1) - \dim(\mathrm{Im}(i'))].$$

(6) Let (E, D, i, p, ∂) be an exact couple. Show that the derived couple is again exact.

(7) A pair of simplicial complexes (X, A) may be considered as a filtered simplicial complex. Compute the spectral sequence $\{E^{(r)}, d^{(r)}\}_{r>0}$.

Explicitly compute the spectral sequence of the pair $(\mathbb{R}P^2, \mathbb{R}P^1)$.