CAT L4: Quantum Non-Locality and Contextuality

Samson Abramsky

Department of Computer Science, University of Oxford

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Direct path from sheaf theory to computing global sections using Mathematica$^{\text{TM}}$!

The Basic Scenario

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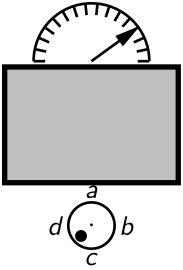
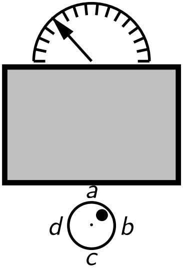


Alice



Bob

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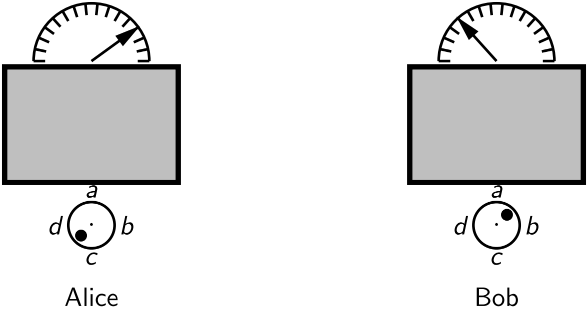

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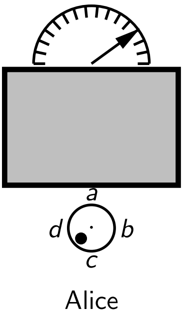

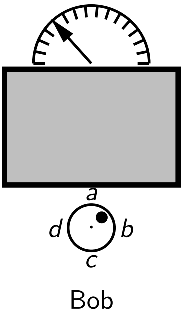

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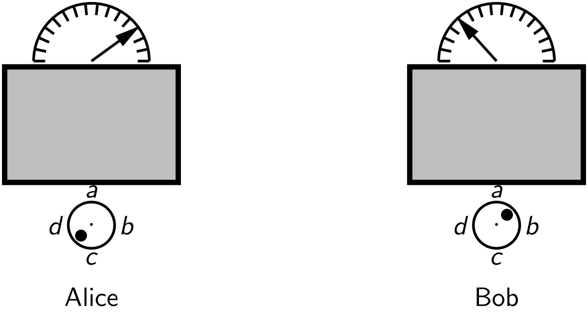

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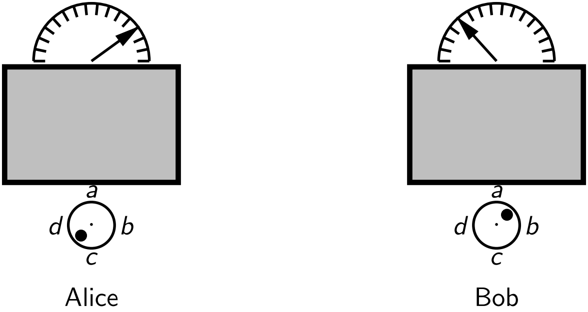

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Can we tell from this observational history if there is interference/dependence between different parts of the system?

The Quantum Case: Spin Measurements

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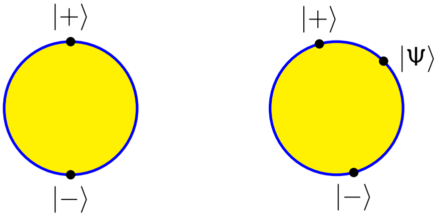

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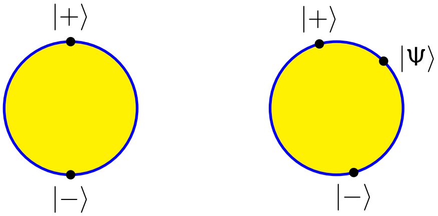

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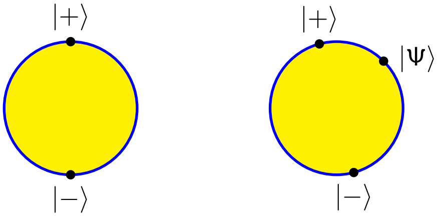

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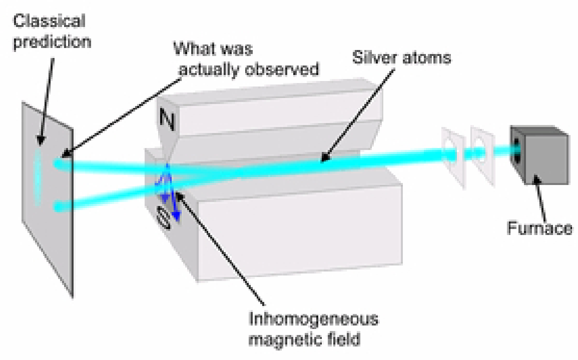
Note the appearance of **quantization** here: there are not a continuum of possible outcomes for each measurement, but only two!

The Stern-Gerlach Experiment



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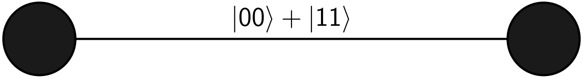

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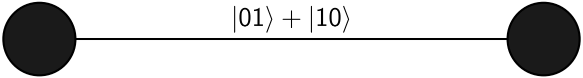

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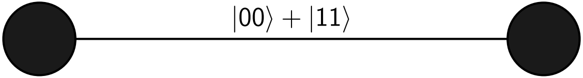

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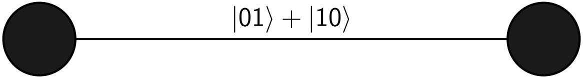

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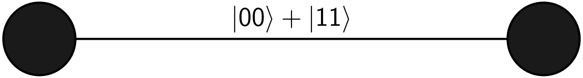

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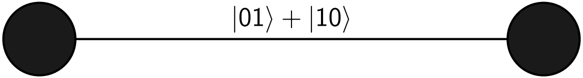

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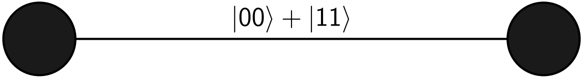

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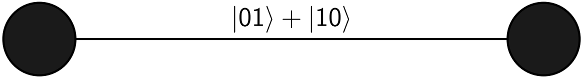

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Bell's theorem: QM is **essentially non-local**.

A Probabilistic Model Of An Experiment

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Example: The Bell Model

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  Is is, famously, **not** satisfied by QM (Bell's theorem).

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Moreover, as we shall see, there are probability tables for which, as a mathematical fact, there is **no** consistent extension to a joint distribution on outcomes; so we must consider certain combinations of measurements as not jointly performable **in principle**, under any physical theory whatever.

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Each row of the table specifies a **probability distribution** on events O^C for a given choice of measurements C.

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(ii) We could vary R.

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The sets $C \in \mathcal{M}$ are the **measurement contexts**; the sets of measurements which can be performed together.

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Thus each e_C is a probability distribution on the row indexed by C; it specifies a probability for the events corresponding to the observation of an outcome for each measurement in C.

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This says that the probability for Alice to get the outcome $s_0(m_a)$ is the same, whether we marginalize over the possible outcomes for Bob with measurement m_b, or with m'_b.

In other words, Bob's choice of measurement cannot influence Alice's outcome.

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If d is a global section for the model $\{e_C\}$, we recover the predictions of the model by **averaging over the values of these hidden variables**:

$$e_C(s) \;=\; d|C(s) \;=\; \sum_{s' \in \mathcal{E}(X), s'|C = s} d(s') \;=\; \sum_{s' \in \mathcal{E}(X)} \delta_{s'|C}(s) \cdot d(s').$$

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existence of a local hidden-variable model for a given empirical model
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Hence:

No such h.v. model exists (the empirical model is **non-local**/**contextual**)
IFF
there is an **obstruction to the existence of a global section**

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Incidence matrix for $(2, 2, 2)$ is 16×16.

The $(2, 2, 2)$ Incidence Matrix

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$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
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This matrix has rank 9.

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Hence solutions correspond exactly to global sections — which as we have seen, correspond exactly to local hidden-variable realizations!

The Bell Model

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	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
(a, b)	0	$1/2$	$1/2$	0
(a', b)	$3/8$	$1/8$	$1/8$	$3/8$
(a, b')	$3/8$	$1/8$	$1/8$	$3/8$
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Solutions in the non-negative reals: this corresponds to solving the linear system over $\mathbb{R}$, subject to the constraint that $\mathbf{X} \geq \mathbf{0}$ (linear programming problem).

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Proof We focus on 4 out of the 16 equations, corresponding to rows 3, 7, 11 and 14 of the incidence matrix. We write X_i rather than $\mathbf{X}[i]$.

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X_9 & + & X_{10} & + & X_{11} & + & X_{12} & = & 1/2 \\
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Adding the last three equations yields

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Since all these numbers must be non-negative, the left-hand side of this equation must be greater than or equal to the left-hand side of the first equation, yielding the required contradiction. □

The Hardy Model

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We consider the possibilistic version of the Hardy model, specified by the following table.

	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
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Now we are interested in solutions over the **boolean semiring**, *i.e.* a boolean satisfiability problem. E.g. the equation specified by the first row of the incidence matrix gives the clause

$$X_1 \;\vee\; X_2 \;\vee\; X_3 \;\vee\; X_4$$

while the fifth yields the formula

$$\neg X_1 \;\wedge\; \neg X_3 \;\wedge\; \neg X_5 \;\wedge\; \neg X_7.$$

The 'Hardy paradox'

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A solution is an assignment of boolean values to the variables which simultaneously satisfies all these formulas. Again, it is easy to see by a direct argument that no such assignment exists.

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Since every disjunct in the first formula appears as a negated conjunct in one of the other three formulas, there is no satisfying assignment. □

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Conclusion: Bell $<$ Hardy.

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To compute the tensor product of vectors:

$$\sum_i a_i |i\rangle \otimes \sum_j b_j |j\rangle = \sum_{i,j} a_i b_j |ij\rangle.$$

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The possibilistic collapse of this model is thus a Hardy model.

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Thus negative probabilities characterize the no-signalling rather than the quantum realm.

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As a special case, we derive a formula for the dimension for Bell-type (n, k, l)-scenarios:

$$D = (k \cdot (l-1) + 1)^n.$$

Reasons

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In the case of $(n, 2, 2,)$ systems, this result can be visualized in terms of elegant self-similarity properties of the inductively defined incidence matrices $\mathbf{M}(n)$:

$$\mathbf{M}(1) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{M}(n+1) = \begin{bmatrix} \mathbf{M}(n) & \mathbf{M}(n) & 0 & 0 \\ 0 & 0 & \mathbf{M}(n) & \mathbf{M}(n) \\ \mathbf{M}(n) & 0 & \mathbf{M}(n) & 0 \\ 0 & \mathbf{M}(n) & 0 & \mathbf{M}(n) \end{bmatrix}$$

and of the probability vectors $\mathbf{V}$ corresponding to no-signalling models, from which it follows that

$$\mathrm{rank}(\mathbf{M}(n)) = \mathrm{rank}([\mathbf{M}(n)|\mathbf{V}]) = 3^n.$$

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The 'Popescu-Rohrlich box':

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This vector can be taken as giving a **local hidden-variable realization of the PR box using negative probabilities**. Similar explicit realizations can be given for the other PR boxes.

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We shall now show that the well-known GHZ models, of type $(n, 2, 2)$ for all $n > 2$, are strongly contextual. This will establish a strict hierarchy

$$\text{Bell} < \text{Hardy} < \text{GHZ}$$

of increasing strengths of obstructions to non-contextual behaviour for these salient models.

Spin Measurements

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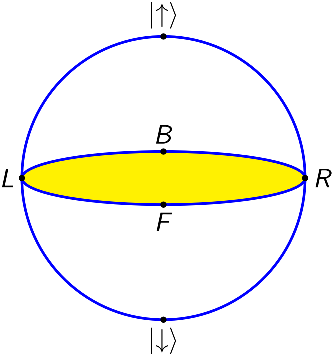

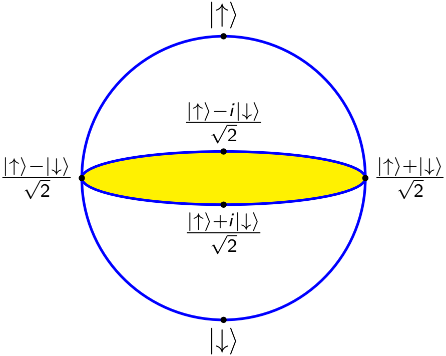

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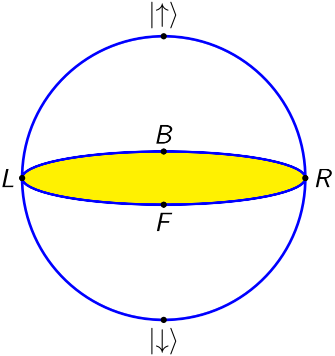

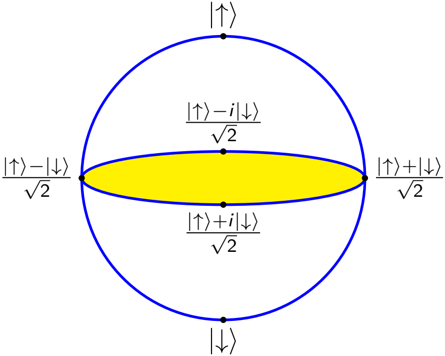

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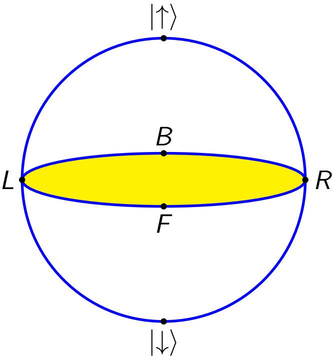

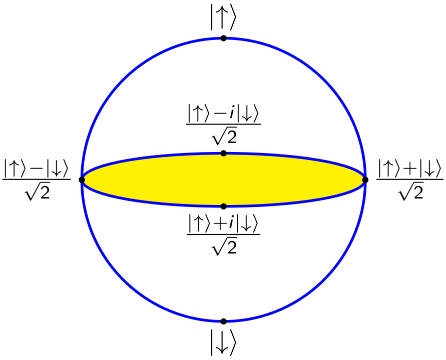

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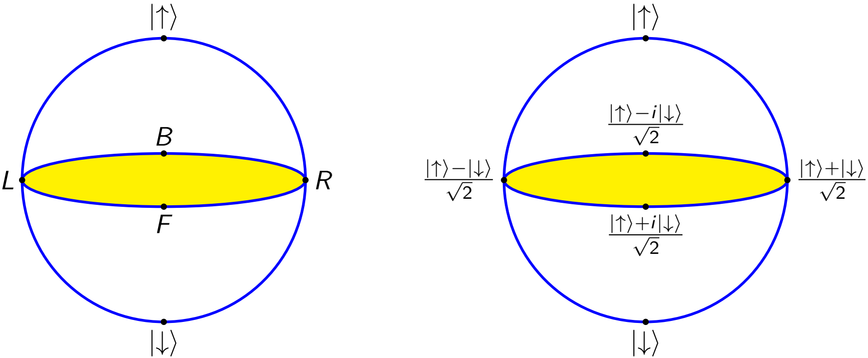
Spin Measurements



- Spin Right or Left along the x-axis.
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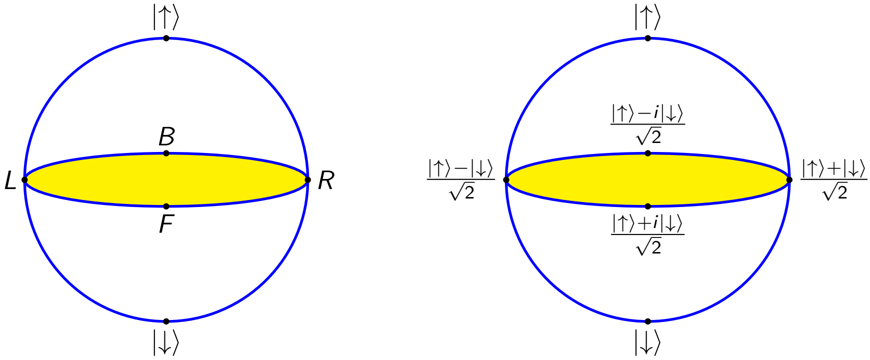

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Note that X and Y do not commute; hence according to quantum mechanics, they are **incompatible**; they cannot be measured together.

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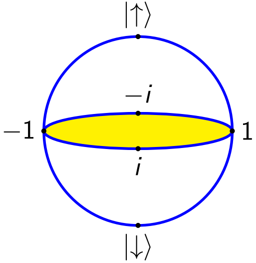
This computation is controlled by the product of the $|\downarrow\rangle$-coefficients of the basis vectors: cyclic group generated by $i \cong \mathbb{Z}_4$.



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NB: a model with these properties can be realized in quantum mechanics.

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Thus for any $Y^{(i)}$, $Y^{(j)}$ assigned the **same** value, if we substitute X's in those positions they must receive **different** values. Similarly, for any $Y^{(i)}$, $Y^{(j)}$ assigned different values, the corresponding $X^{(i)}$, $X^{(j)}$ must receive the same value.

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Suppose not all $Y^{(i)}$ are assigned the same value. Then for some i, j, k, $Y^{(i)}$ is assigned the same value as $Y^{(j)}$, and $Y^{(j)}$ is assigned a different value to $Y^{(k)}$. Thus $Y^{(i)}$ is also assigned a different value to $Y^{(k)}$. Then $X^{(i)}$ is assigned the same value as $X^{(k)}$, and $X^{(j)}$ is assigned the same value as $X^{(k)}$. By transitivity, $X^{(i)}$ is assigned the same value as $X^{(j)}$, yielding a contradiction.

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The remaining cases are where all Y's receive the same value. Then any pair of X's must receive different values. But taking any 3 X's, this yields a contradiction, since there are only two values, so some pair must receive the same value.

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S. Abramsky and A. Brandenburger, The Sheaf-Theoretic Structure of Non-Locality and Contextuality. Available at `arXiv:1102.0264`.