Computational Algebraic Topology

Hilary Term 2012

1. Simplicial complexes

1.1. Definitions

An abstract, finite *simplicial complex* K is a collection of non-empty subsets of a finite set of vertices K_0 that satisfy the condition

$$\alpha \in K,\, \beta \subset \alpha \Longrightarrow \beta \in K.$$

$\beta \subset \alpha$ is called a *face* of α.

The *dimension of* K is one less than the cardinality of the largest element of K:

$$\dim(K) := \max_{\alpha \in K} \{\#\alpha - 1\}.$$

The subset $K_p \subset K$ of sets of size $p+1$ are the p-*simplices*.

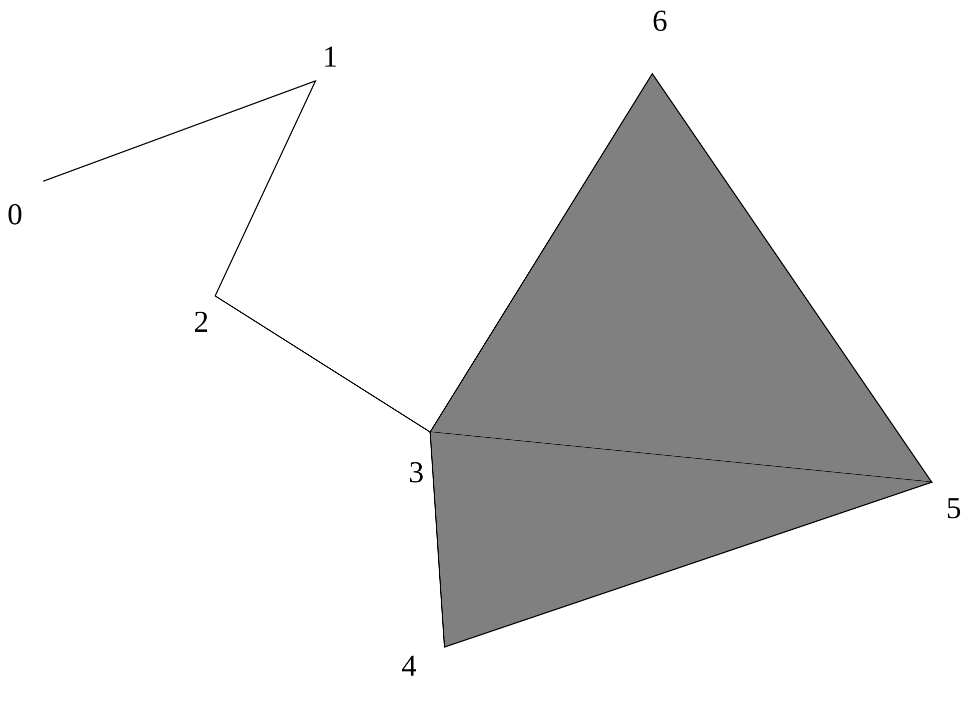
The union K^p of all subsets of size $p+1$ or less is a subcomplex of K called the p-*skeleton*.

The *Euler characteristic* of K is defined by

$$\chi(K) := \Sigma_p (-1)^p \# K_p.$$

Example: $K_0 = \{0, 1, 2, 3, 4, 5, 6\}$,
$K_1 = \{\{0, 1\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 3\}, \{3, 5\}\}$,
$K_2 = \{\{3, 4, 5\}, \{3, 5, 6\}\}$

$\chi(K) = 7 - 8 + 2 = 1$



1.2. Geometric realization

The *simplex spanned* by $\{v_0, \ldots, v_p\} \subset \mathbf{R}^n$ is the set of points

$$\{\Sigma_{i=0}^{p} t_i v_i \,|\, 0 \leq t_i \in \mathbf{R},\, \Sigma_i t_i = 1\}.$$

$\{v_0, \ldots, v_p\}$ are *affinely independent* if they span a p-simplex, or equivalently, if $\{v_1 - v_0, \ldots, v_n - v_0\}$ are linearly independent.

Assume we have an assignment of the vertex set K_0 to points in $\mathbf{R}^n$ such that the images of the elements for every $\sigma \in K$ are affinely independent. Let $|\sigma|$ be the simplex spanned by the images of the elements in σ. The set

$$|K| := \bigcup_{\sigma \in K} |\sigma|$$

is a *geometric realization* of K.

Lemma: Let K be a simplicial complex and $N = \# K_0$. Then K has a geometric realisation in $\mathbf{R}^N$.

Proof: Map the elements in K_0 to the standard basis elements in $\mathbf{R}^N$. $|K|$ is the union of all simplices spanned by the images of the elements in K. $\diamondsuit$

A *simplicial map* $f : K \to L$ is a map of vertices $f : K_0 \to L_0$ such that $f(\sigma) \in L$ for all $\sigma \in K$.

It induces a continuous map on realizations by setting

$$f(\Sigma_{i=0}^{p} t_i v_i) = \Sigma_{i=0}^{p} t_i f(v_i),$$

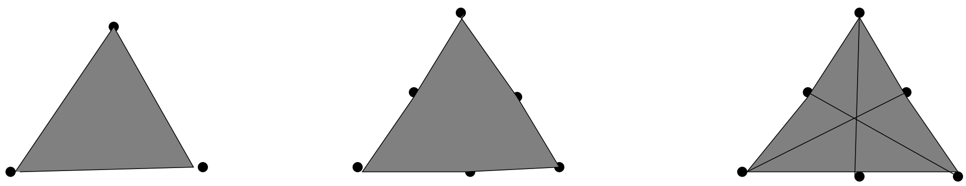
where v_i is the realisation of the i-th vertex of a p-simplex in K and $f(v_i)$ is the realization of the image of this vertex in $|L|$.

Exercise: Show that any two geometric realizations are homeomorphic.

1.3. Subdivision and approximation

The *barycenter* of the realization of p-simplex is the point corresponding to $t_0 = \cdots = t_p = 1/(p+1)$.

The *barycentric subdivision* $Sd(K)$ of a simplicial complex K is constructed by induction: at the p-th stage, for every p-simplex σ, add the barycenter of σ and all the $(p-1)$-simplicies (and their faces) containing the barycenter and any other vertices and barycenters of any face of σ.



Simplicial Approximation Theorem: For any continuous map $g : |K| \to |L|$ there is an n and a simplicial map $f : Sd^n(K) \to L$ such that f and g are homotopic.

2. Homology

2.1. Basic definitions

A *chain complex* (C, d) over a field k is a sequence of k-vector spaces and maps

$$\dots C_{n+1} \stackrel{d_{n+1}}{\longrightarrow} C_n \stackrel{d_n}{\longrightarrow} C_{n-1} \stackrel{d_{n-1}}{\longrightarrow} \dots C_0 \longrightarrow 0$$

such the $d_n \circ d_{n+1} = 0$. A *map of chain complexes* $F : (C, d) \to (C', d')$ is a collection of k-linear maps $F_n : C_n \to C'_n$ such that $F_{n-1} \circ d_n = d'_n \circ F_n$.

n-th *boundary map*: d_n
n-*chains*: C_n
n-*cycles*: $Z_n := Ker(d_n)$
n-*boundaries*: $B_n := Im(d_{n+1})$
n-th *homology group* of C:

$$H_n(C) := Z_n/B_n = Ker(d_n)/Im(d_{n+1})$$

Exercise: A map of chain complexes induces a map of homology groups.

2.2. F_2-homology of a simplicial complex

Let K be a simplicial complex and let

$$C_n(K) := \mathbf{F}_2[K_n]$$

be the $\mathbf{F}_2$-vector space with basis K_n. Define on basis elements

$$d_n(\alpha) := \Sigma_{\beta\subset\alpha,\,\#\beta=n}\ \beta$$

and extend d_n linearly to all chains.

Fundamental Lemma: $d_n \circ d_{n+1} = 0$.

Proof: It is enough to check this on basis elements. Let $\alpha = \{v_0, \ldots, v_{n+1}\}$ be an $(n+1)$-simplex. Then

$$d_n(d_{n+1}(\alpha)) = d_n(\Sigma_{i=0}^{n+1}\{v_0, \ldots, \widehat{v}_i, \ldots v_{n+1}\})$$

$$= 2\,\Sigma_{j\neq i}\{v_0, \ldots, \widehat{v}_j, \ldots, \widehat{v}_i, \ldots v_{n+1}\} = 0$$

$\diamondsuit$

Caveat: When the characteristic of k is not 2, one has to take in the definition of the boundary map d_n the alternating sum over the $n+1$ faces. Otherwise the Fundamental Lemma does not hold.

The n-th *homology group of a simplicial complex* K is

$$H_n(K)(= H_n(K, \mathbf{F}_2)) := H_n(C(K))$$

The p-th *Betti number* of K is

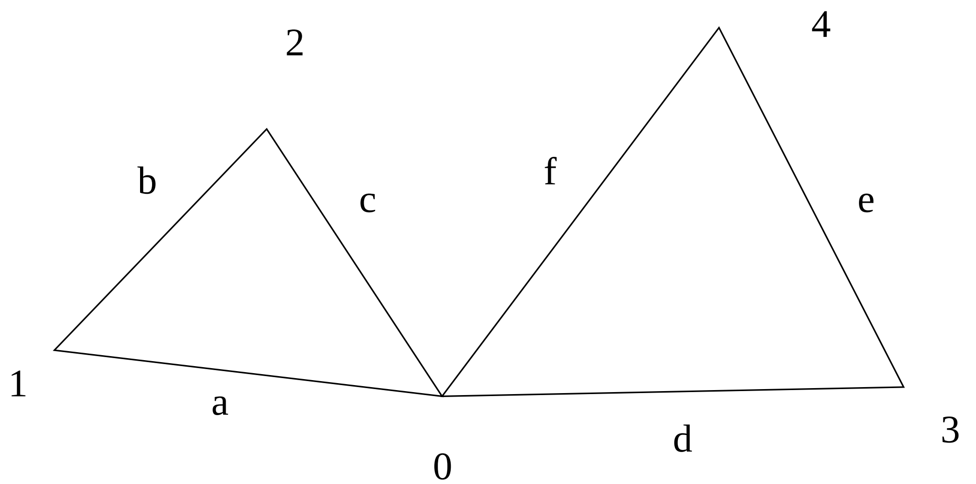
$$b_p := \dim H_p(K).$$

Proposition: $\chi(K) = \Sigma_p \, (-1)^p \, b_p$.

Proof: Use rank-nullity formula. $\diamondsuit$

Example 1: K a one-point union of the boundary of two triangles. Then $C_1(K) =< a, b, c, d, e, f >$ and $C_0(K) =< v_0, v_1, v_2, v_3, v_4 >$, and d_1 has matrix

1	0	1	1	0	1
1	1	0	0	0	0
0	1	1	0	0	0
0	0	0	1	1	0
0	0	0	0	1	1



Using column and row operations this can be transformed to

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Hence by the rank-nullity theorem $H_0(K) = \mathbf{F_2}$ and $H_1(K) = \mathbf{F}_2 \oplus \mathbf{F}_2$. All other homology groups are zero.

Example 2: K an n-simplex (i.e. an n-ball). Then $H_p(K) = \mathbf{F}_2$ when $p = 0$ and is zero otherwise.

Example 3: K is the boundary of an n-simplex (i.e. an $n-1$ sphere). Then $H_p(K) = \mathbf{F}_2$ when $p = n-1, 0$ and is zero otherwise.

2.2. Relative homology

Let $A \subset C$ be a sub chain complex of C (i.e. $A_n \subset C_n$ and $d_n(A_n) \subset A_{n-1}$ for all n). Then C/A is the quotient chain complex with n-chains C_n/A_n and boundary map induced by d_n.

Theorem: A short exact sequence of chain complexes $0 \longrightarrow A \longrightarrow C \longrightarrow C/A \longrightarrow 0$ induces a long exact sequence on homology groups

$$\ldots H_{n+1}(C/A) \stackrel{\delta}{\to} H_n(A) \to H_n(C) \to$$

$$H_n(C/A) \stackrel{\delta}{\to} H_{n-1}(A) \ldots H_0(C/A) \to 0$$

The connecting homomorphims δ takes a cycle in C_n/A_n which is represented by a chain $c \in C_n$ to $d_n(c) \in A_{n-1}$.

Example: n-simplex B^n and its boundary S^{n-1}. Then $C(S^{n-1})$ is a sub-complex of $C(B^n)$, and the long exact sequence yields $H_p(B^n, S^{n-1}) = \mathbf{F}_2$ if $p = n$ and zero otherwise.

2.3. Mayer-Vietoris sequence

Theorem: Let $K = A \cup B$ be the union of two simplicial subcomplexes. Then there is a long exact sequence

$$\ldots H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(K) \xrightarrow{\delta}$$

$$H_{n-1}(A \cap B) \to \cdots \to H_0(K) \to 0.$$

The first two maps are induced by the inclusions of complexes with $c \mapsto (c, c)$ and $(a, b) \mapsto a - b$. To define the connecting homomorphism δ, write a chain $c \in C_n(K)$ as a sum of chains $a \in C_n(A)$ and $b \in C_n(B)$. Then c is taken to the chain $d_n(a) = d_n(b)$ in $A \cap B$.

Proof: The maps induce an exact sequence of chain complexes

$$0 \to C(A \cap B) \to C(A) \oplus C(B) \to C(K) \to 0.$$

$\diamondsuit$

2.4. Homology of $Sd(K)$

Let K be a simplicial complex of dimension d and $Sd(K)$ be its barycentric subdivision. On chain complexes we have a map

$$C(K) \longrightarrow C(Sd(K)), \quad \sigma \mapsto \Sigma_{\mu \subset \sigma}\, \mu$$

that sends a p-simplex $\sigma \in K$ to the sum of all p-simplices $\mu \in Sd(K)$ with $\mu \subset \sigma$. This is a map of chain complexes as

$$d\sigma = \Sigma_{\mu \subset \sigma}\, d\mu$$

Block Lemma: The induced map on homology induces an isomorphism

$$H(K) = H(C(K)) \xrightarrow{\simeq} H(Sd(K)).$$

Proof: Let $X_p \subset Sd(K)$ be the sub-complex of all simplices (and their faces) that are used to describe the image of K_p.
Clearly, $X_0 \subset X_1 \subset \cdots \subset X_d = Sd(K)$.

Claim: $H_q(X_p, X_{p-1}) = C_p(K)$ if $p = q$ and 0 otherwise.

$$\begin{array}{ccccc}
\mathsf{C}_{p+1}(K) & & & & 0{=}\mathsf{H}_{p-1}(X_{p-2}) \\
\downarrow e & & & & \downarrow \\
\mathsf{H}_p(X_p) \overset{f}{\to} & & \mathsf{C}_p(K) \overset{g}{\to} & & \mathsf{H}_{p-1}(X_{p-1}) \\
\downarrow l & & & & \downarrow h \\
\mathsf{H}_p(X_{p+1}) & & & & \mathsf{C}_{p-1}(K) \\
\downarrow & & & & \\
0 = \mathsf{H}_p(X_{p+1}, X_p) & & & &
\end{array}$$

Using the long exact sequence for relative homology one sees that the diagram has exact columns and row. A diagram chase gives

$$H_p(K) = ker(hg)/im(fe) = im(f)/im(fe)$$

$$= H_p(X_p)/im(e) = H_p(X_p)/ker(l)$$

$$= H_p(X_{p+1}) = H_p(Sd(K)).$$

$\diamondsuit$

2.5. Cohomology

The *dual* of a k-vector spaces V is

$$V^* := Hom(V, k).$$

If $T : V \to W$ is a map of vector spaces then its *adjoint*

$$T^* : W^* \to V^*$$

is defined by $T^*(f)(v) := f(T(v))$. If the matrix presentation of T with respect to some bases of V and W is A then matrix presentation of T^* with respect to the dual bases is A^T, the transpose of A.

A cochain complex (C, d) has an associated dual cochain complex (C^*, d^*)

$$\cdots \xleftarrow{d^*_{n+1}} C^n \xleftarrow{d^*_n} C^{n-1} \xleftarrow{d^*_{n-1}} \ldots C^0 \longleftarrow 0$$

where $C^n = (C_n)^* = Hom(C_n,\, k)$.

n-*cocycles*: $Z^n := Ker(d^*_{n+1})$
n-*coboundaries*: $B^n := Im(d^*_n)$
n-th *cohomology group*:

$$H^n(C) := Z^n / B^n$$

For a simplicial complex K,

$$H^n(K) := H^n(C(K))$$

Example 1: Let K be connected. Then the only non-trivial 0-cocycle is the map that assigns 1 to every vertex of K. Hence, $H^0(K) = \mathbf{F}_2$.

Example 2: K the one-point union of the boundary of two triangles as in section 2.2. Then $C^0 = (\mathbf{F}_2)^5$ and $C^1 = (\mathbf{F}_2)^6$. As d^* is the transpose of d, it has rank 4. So $H^0(K) = \mathbf{F}_2$ and $H^1(K) = (\mathbf{F}_2)^2$.

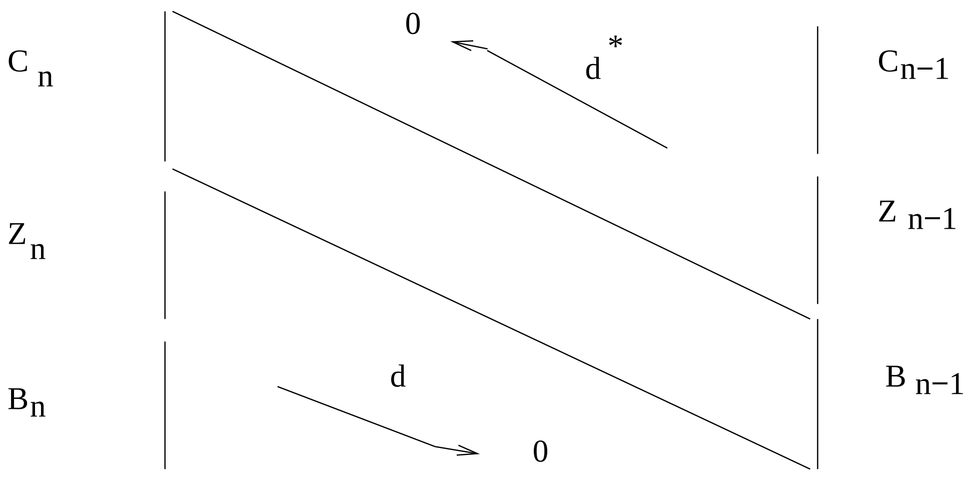
Universal Coefficient Theorem:

$$dim(H_n(C)) = dim(H^n(C))$$

Proof: $dim(H_n) = dim(Z_n) - dim(B_n)$ and $dim(H^n) = dim(Z^n) - dim(B^n)$.
So the theorem follows from

$$dimZ_n + dimB^n = dimC_n$$

$$= dimC^n = dimZ^n + dimB_n$$



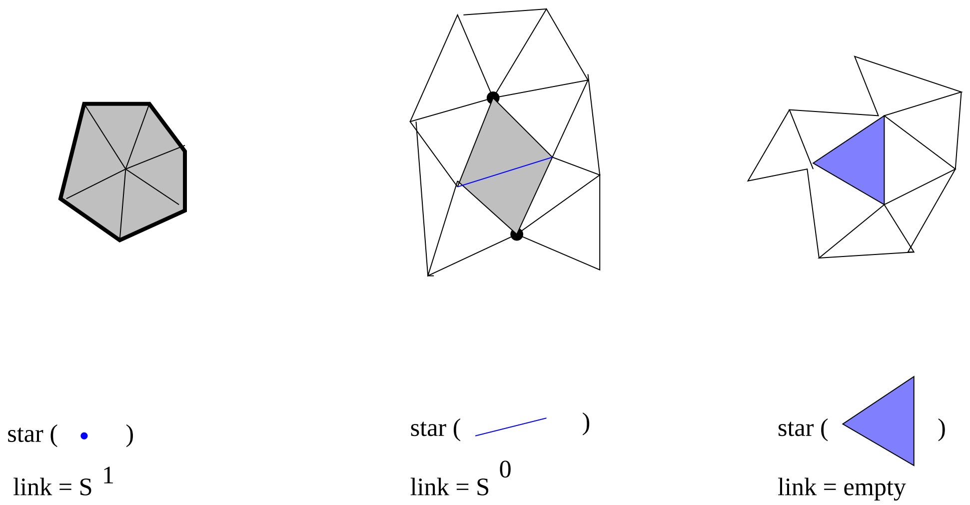
3. Manifolds and duality

3.1 Combinatorial d-manifolds

A *manifold* of dimension d is a topological space M for which every point lies in an open neighbourhood homeomorphic to the open d-dimensional unit disk D^d. M is *triangulated* by a simplicial complex K if M is a realization of K. K is also called a *triangulation* of M. For a p simplex $\sigma \in K$ define

$$star(\sigma) := \{\tau \in K | \, \sigma \leq \tau\}$$

$$link(\sigma) := \{\tau \in \overline{star(\sigma)} | \, \tau \cap \sigma = \emptyset\}$$



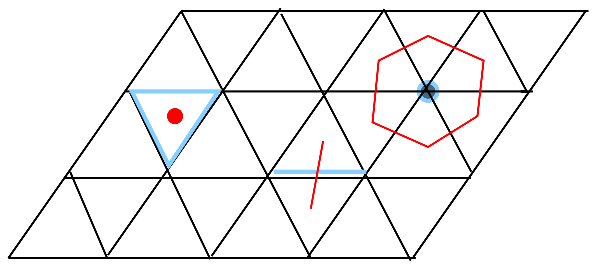
A *combinatorial d-manifold* is a d-dimensional manifold with a triangulation such that
(**) the link of every i-simplex triangulates a sphere of dimension $d - i - 1$.

Caveat: Not every triangulation of a manifold satisfies condition (**).

Exercise: If K satisfies condition (**) then so does $Sd(K)$.

3.2. Dual bock complex

Let K be a triangulation of a combinatorial d-dimensional manifold. Label the vertices of $Sd(K)$ by the dimension of the simplex in K that they belong to. For a p-simplex σ define its *dual block* $\hat{\sigma}$ as the union of simplices in $Sd(K)$ that contain the barycenter of σ as their lowest vertex.



simplices (in blue) and

their dual blocks (in red)

The *boundary* $\partial(\hat{\sigma})$ of a block is the union $\bigcup_{\sigma\subset\tau}\hat{\tau}$ of blocks $\hat{\tau}$; the block is homeomorphic to a $d-p-1$ sphere by the condition (**) on links in the triangulation K.

The *dual block chain complex* D is the chain complex in which the dual blocks of p-simplices form a basis for D_{d-p} and the boundary map is given by ∂.

Exercise: $\partial \circ \partial = 0$.

Block Lemma: The map that sends a block $\mu \in D_p$ to the sum $\Sigma_{\sigma \subset \mu} \sigma \in C_p(Sd(K))$ induces an isomorphism in homology

$$H_p(D) = H_p(Sd(K)) = H_p(K).$$

Proof: The proof is the same as that in section 2.4. Here X_p is the subcomplex of $Sd(K)$ which contains all simplices in a block of dimension p or less. Note that as before $H_q(X_p, X_{p-1}) = D_p$ if $p = q$ and 0 otherwise. $\diamondsuit$

3.3. Poincaré Duality

Theorem: Let M be a combinatorial d-dimensional manifold. For all $0 \leq p \leq d$,

$$H_{d-p}(M) \simeq H^p(M)$$

Proof: Let M be triangulated by K and $\sigma \in K_p$. The linear map

$$\psi : D_{d-p} \longrightarrow C^p(K), \quad \widehat{\sigma} \mapsto \sigma^*$$

that sends the dual block $\widehat{\sigma}$ to the dual basis element σ^* is a bijection on basis elements and hence an isomorphism of vector spaces. To prove the theorem, it remains to show that ψ commutes with the boundary maps:

$$\psi(\partial(\widehat{\sigma})) = \psi(\Sigma_{\sigma \subset \tau} \, \widehat{\tau}) = \Sigma_{\sigma \subset \tau} \, \tau^*$$

$$d^*(\psi(\widehat{\sigma})) = d^*(\sigma^*) = \Sigma_{\sigma \subset \tau} \, \tau^*$$

$\diamondsuit$

Corollary: $H_{d-p}(M) \simeq H_p(M)$

Proof: This follows from Poincare Duality and the Universal Coefficient Theorem. $\diamondsuit$

Example 1: If d is odd then $\chi(M) = 0$.

Example 2: If M is connected then

$$H_d(M) \simeq H_0(M) \simeq \mathbf{F}_2$$

and the only non-zero d-dimensional cycle is the *fundamental class* $[M] := \Sigma_{\sigma \in K_d} \sigma$.

3.4. Intersection theory

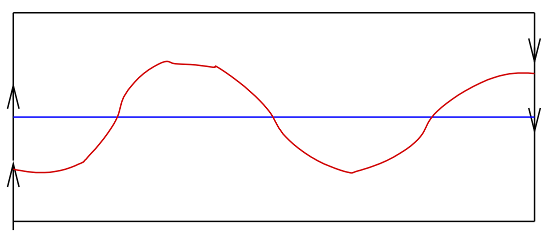
Let M be a d-dimensional combinatorial manifold with triangulation K.
Goal: to get a better geometric picture of P.D.

For $\sigma, \tau \in K_p$ the intersection $\sigma \cap \hat{\tau}$ is the barycenter of σ if $\tau = \sigma$ and is empty if $\tau \neq \sigma$. Define

$$< \sigma, \hat{\tau} > := 1 \text{ if } \sigma = \tau, \text{ and } = 0 \text{ if } \sigma \neq \tau$$

and extend this bilinearly to a pairing

$$C_p(K) \times D_{d-p} \longrightarrow \mathbf{F}_2$$

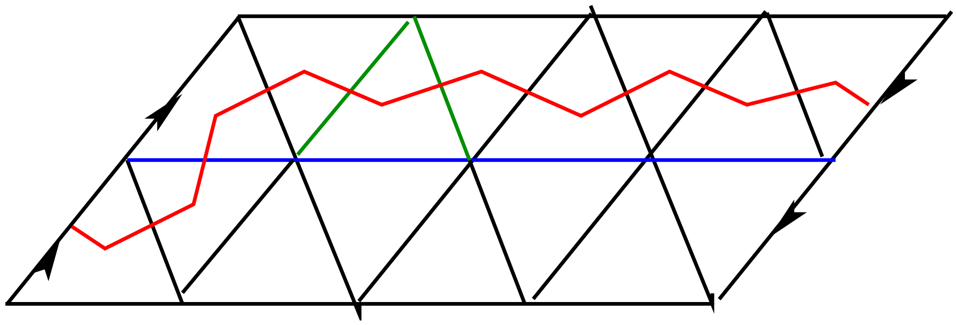


<c, d> = 1

Claim: The intersection number $< c, d >$ does not change if c or d is replaced by a homologous cycle.

Proof: If c is homologous to c_0 then there exists a $p+1$-chain e with $\partial(e) = c + c_0$.
Let τ be a summand of e.
Then for any $(d-p)$-simplex $\widehat{\sigma}$ (of d), $\tau \cap \widehat{\sigma}$ is either empty or, if σ is a face of τ, the 1-simplex from the barycenter of τ to the barycenter of σ.

As d is a cycle, the path of which $\widehat{\sigma}$ is a part needs to continue. The possibilities are that the path goes in and out of τ through faces both belonging to c, or both belonging to c_0, or one belonging to c and one belonging to c_0. In all cases the intersection number with c is the same as that with c_0. $\diamondsuit$



$|K| =$ Möbius band

$c =$ core of the Möbius band − blue

$c_0 =$ core: one blue simplex replaced by two green

$d =$ cycle in D_* homotopic to the core − red

Theorem: The bilinear map

$$H_p(M) \times H_{d-p}(M) \longrightarrow \mathbf{F}_2$$

$$([c],[d]) \mapsto < c, d >$$

defines a perfect pairing, i.e. the map

$$[d] \mapsto < \ , d >$$

defines an isomorphisms $H_{d-p}(M) \to (H_p(M))^*$.

Proof: By Poincaré Duality, the pairing

$$H_p(K) \times H_{d-p}(D) \longrightarrow \mathbf{F}_2$$

$$(\Sigma\, \sigma_i, \Sigma\, \widehat{\tau}_j) \mapsto \Sigma < \sigma_i, \widehat{\tau}_j >$$

is the same as the pairing

$$H_p(K) \times H^p(K) \longrightarrow \mathbf{F}_2$$

$$(\Sigma\, \sigma_i, \Sigma\, \tau_j^*) \mapsto \Sigma\, \tau_j^*(\sigma_i)$$

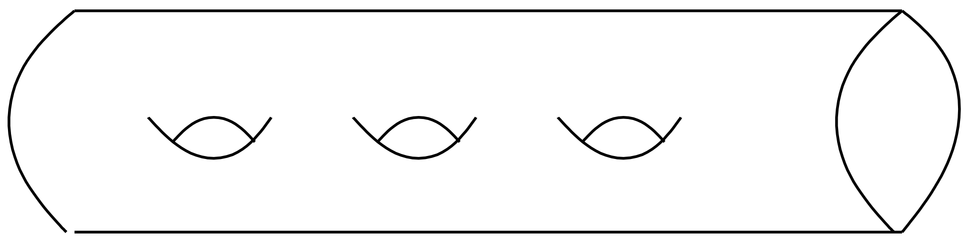
as $< \sigma, \widehat{\tau} >= \tau^*(\sigma)$ for all $\sigma, \tau \in K_p$.

The second pairing induces by the Universal Coefficient Theorem a perfect pairing $H^p(K) \simeq (H_p(K))^*$. $\diamondsuit$

3.5. Lefschetz Duality

A *manifold with boundary* of dimension d is a topological space M for which every point lies in an open neighbourhood homeomorphic to the open d-dimensional unit disk D^d or the half disk $D^d \cap (\mathbf{R}_{\geq 0} \times \mathbf{R}^{d-1})$. The points with an open neighbourhood of the second kind form the *boundary* ∂M of M.

A *combinatorial d-manifold with boundary* is a d-dimensional manifold with boundary and a triangulation such that
(**) the link of every i-simplex triangulates a sphere or a half-sphere of dimension $d - i - 1$.

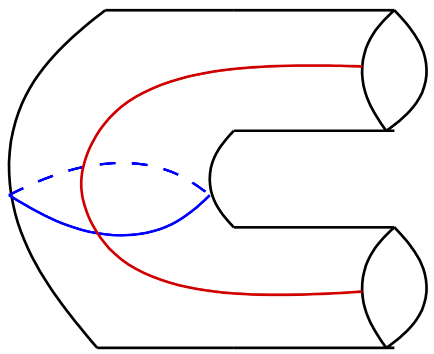


Theorem For a d-dimensional combinatorial manifold M with boundary, there are isomorphisms

$$H_{d-p}(M, \partial M) \simeq H^p(M) \text{ and } H_{d-p}(M) \simeq H^p(M, \partial M)$$

Equivalently, intersection defines a perfect pairing

$$H_p(M) \times H_{d-p}(M, \partial M) \longrightarrow \mathbf{F}_2$$



intersection of a cycle

with a relative cycle

in an annulus

Example: Annulus A

$H_0(A) = H_2(A, \partial A) = \mathbf{F}_2$

$H_1(A) = H_1(A, \partial A) = \mathbf{F}_2$

$H_2(A) = H_0(A, \partial A) = 0$

3.6. Alexander Duality

Let $X \subset S^d$ be the realization of a simplicial complex. Consider its ϵ-neighbourhood

$$N(X) := \{y \in B_\epsilon(x) \mid x \in X\}.$$

Its closure $\overline{N(X)}$ is a compact d-manifold with boundary. Its complement $Y := S^d \setminus N(X)$ is also a compact d-manifold with (the same) boundary.

Note: A simplicial complex version of $\overline{N(X)}$ and its complement can be constructed as follows. Let S^d be triangulated by K such that X is the realization of a subcomplex of K. Consider $Sd^2(K)$ and let $S(X)$ be its subcomplex representing X. Then $\overline{N(X)}$ can be replaced by the simplicial d-manifold (with boundary)

$$N := \bigsqcup_{u \in S(X)} \overline{star(u)}$$

Theorem: For all p,

$$\tilde{H}_p(X) = \tilde{H}^{d-p-1}(Y)$$

Notation: $\tilde{H}_p(X) := H_p(X, *)$.

Example 1: $X = \{0\} \subset S^d$
$N(X) = B^d \subset S^d$
$Y = S^d \setminus B^d$
$\tilde{H}_*(X) = 0 = \tilde{H}^*(Y)$

Example 2: $X = S^1 \subset S^3$
$\overline{N(X)}$ is a solid torus in S^3
$Y = S^3 \setminus N(X)$ is another solid torus in S^3

$$\tilde{H}_0(X) = 0 = \tilde{H}^2(Y)$$

$$\tilde{H}_1(X) = \mathbf{F}_2 = \tilde{H}^1(Y)$$

$$\tilde{H}_2(X) = 0 = \tilde{H}^0(Y)$$

Note: Alexander Duality implies that the homology of all knot complements are the same and can therefore not be used to distinguish knots.

Proof: Let $p < d-1$. Then

$$
\begin{array}{rcll}
\tilde{H}^{d-p-1}(Y) & = & H^{d-p-1}(Y) & \\
 & = & H_{p+1}(Y, \partial Y) & \text{by L.D.} \\
 & = & H_{p+1}(S^d, \overline{N(X)}) & \\
 & = & \tilde{H}_p(\overline{N(X)}) & \text{by l.e.s.} \\
 & = & \tilde{H}_p(X) &
\end{array}
$$

Let $p = d-1$. Then by a similar argument

$$
\begin{array}{rcl}
\tilde{H}^0(Y) \oplus \mathbf{F}_2 & = & H^0(Y) \\
 & = & H_d(Y, \partial Y) \\
 & = & H_d(S^d, \overline{N(X)}) \\
 & = & \tilde{H}_{d-1}(\overline{N}(X)) \oplus \mathbf{F}_2 \\
 & = & \tilde{H}_{d-1}(X) \oplus \mathbf{F}_2
\end{array}
$$

as the l.e.s. (in relative homology) gives

$$0 \to H_d(S^d) \to H_d(S^d, \overline{N(X)}) \to \tilde{H}_{d-1}(\overline{N(X)}) \to 0$$

We have used here that by L.D.

$$H_d(N(X)) = H^0(N(X), \partial N(X)) = 0$$

as every component of $\overline{N(X)}$ has non-empty boundary.

Finally, let $p \geq d$. Then $H^{d-p-1}(Y) = 0$ and so is

$$H_p(X) = H_p(\overline{N(X)}) = H^{d-p}(\overline{N(X)}, \partial\overline{N(X)}) = 0$$

by L.D. $\diamondsuit$

Example 3: X a torus embedded in S^3
By Alexander Duality,
$\tilde{H}_0(X) = 0 = \tilde{H}^2(Y)$
$\tilde{H}_1(X) = \mathbf{F}_2 \times \mathbf{F}_2 = \tilde{H}^1(Y)$
$\tilde{H}_2(X) = \mathbf{F}_2 = \tilde{H}^0(Y)$
The last equation implies that Y has two connected components. Indeed, $Y \simeq S^1 \cup S^1$, one circle corresponding to the interior of the torus and one to the exterior.

Exercise: Prove the Jordan Curve Theorem that says that every closed embedded curve in the plane divides the plane in an outside and an inside.

4. Persistent Homology

Motivation: data analysis.

4.1. Definitions

Let $(K(\bullet), \phi_\bullet)$ be a sequence of finite simplicial complexes $K(i), i = 0, \dots N$ and simplicial maps ϕ_i

$$K(0) \xrightarrow{\phi_0} \cdots \xrightarrow{\phi_{i-1}} K(i) \xrightarrow{\phi_i} \cdots \xrightarrow{\phi_{N-1}} K(N).$$

For each degree $p \geq 0$, this gives rise to a sequence of maps in homology

$$H_pK(0) \xrightarrow{\phi_0} \cdots \xrightarrow{\phi_{i-1}} H_pK(i) \xrightarrow{\phi_i} \cdots \xrightarrow{\phi_{N-1}} H_pK(N).$$

The p-th *persistent homology groups* are the images

$$H_p^{i,j} := \text{Im } (\phi_{j-1} \circ \cdots \circ \phi_i : H_pK(i) \to H_pK(j))$$

A non-zero homology class $\alpha \in H_pK(i)$ is said to be *born* in $H_pK(i)$ if it is not in the image of ϕ_{i-1} and it is said to *die* in $H_pK(j)$ if

$$\phi_{j-2} \circ \cdots \circ \phi_i(\alpha) \neq 0 \text{ and } \phi_{j-1} \circ \cdots \circ \phi_i(\alpha) = 0$$

For such a class α, define

$$\textit{persistence } (\alpha) := j - i$$

If α does not die for any j, define

$$\textit{persistence } (\alpha) := \infty$$

Graphically we represent the persistence of the class α in the two cases by the half open intervals $[i, j)$ and $[i, \infty)$, its *barcode* .

4.2. Examples

Example 1: Consider the sphere S^2 embedded in $\mathbf{R}^3$ and height function $h : S^2 \to \mathbf{R}$ as in the figure below. Triangulate S^2 by K such that each $h^{-1}([0, a_i])$ is triangulated by a sub-complex $K(i) \subset K$. Consider the system $(K(\bullet), \phi_\bullet)$ where each ϕ_i is the inclusion map. Note the following homotopies

$K(0) = \emptyset$

$K(1) \simeq D^2 \simeq *$

$K(2) \simeq D^2 \sqcup D^2 \simeq * \sqcup *$

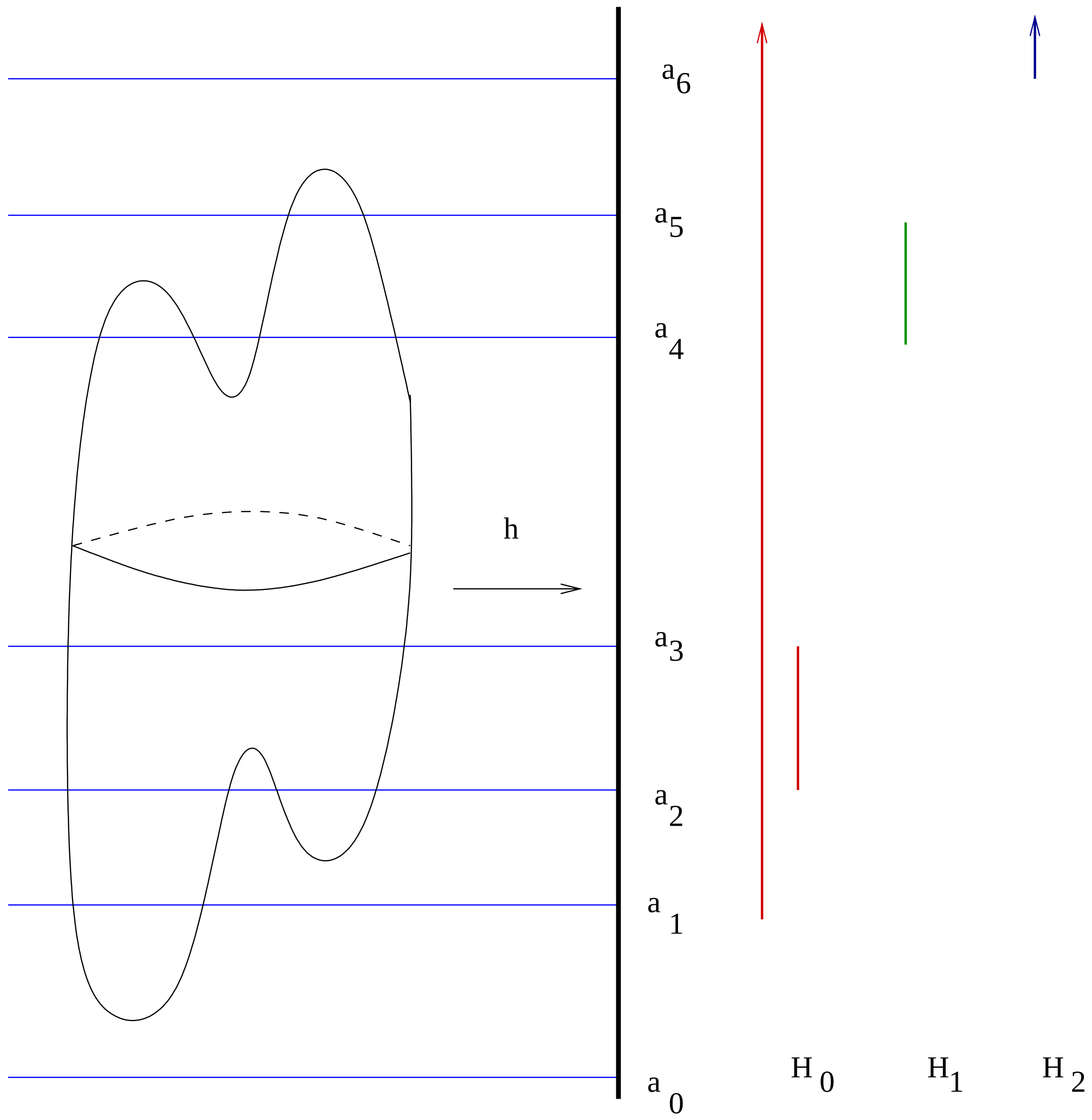
$K(3) \simeq D^2 \simeq *$

$K(4) \simeq S^2 \setminus (D^2 \sqcup D^2) \simeq S^1$

$K(5) \simeq D^2$

$K(6) = S^2$

We track the birth and death of classes for each $p = 0, 1, 2$ in barcodes as in the figure below.



h
a_6
a_5
a_4
a_3
a_2
a_1
a_0
H_0
H_1
H_2

Example 2: Let K be the 2-skeleton of a standard 3-simplex and filter it by skeletons:

$$K^0 = K_0 \subset K^1 \subset K^2 = K$$

$K(0)$ has 4 points: v_0, v_1, v_2, v_3

$K(1)$ is a graph with 4 vertices and 6 edges which is homotopic to the one-point union of 3 circles. To see this, contract the 4 edges containing v_0 to zero length. Then the 3 circles are formed by the edges $e_1 = [v_1, v_2], e_2 = [v_2, v_3]$ and $e_3 = [v_1, v_3]$.

$K(2)$ is homotopic to the sphere.

We describe the persistent homology for each degree $p = 0, 1, 2$.

$H_0K(0) = (\mathbf{F}_2)^4 =< v_0, v_1, v_2, v_3 >$ is generated by the four vertices. Change the basis to

$$\alpha_0 = v_0,\ \alpha_1 = v_1 - v_0,\ \alpha_2 = v_2 - v_0,\ \alpha_3 = v_3 - v_0$$

which have persistence $\infty, 1, 1, 1$ respectively, with associated barcodes $[0, \infty), [0, 1), [0, 1), [0, 1)$.

$H_1K(1) = (\mathbf{F}_2)^3 =< e_1, e_2, e_3 >$ and all three die in $H_1K(2) = 0$. So e_1, e_2, e_3 have persistence $1, 1, 1$ with associated barcodes $[1, 2), [1, 2)$, $[1, 2)$.

$H_2K(2) = \mathbf{F}_2$ and the non-zero class has persistence ∞ and associated barcode $[2, \infty)$.

4.3. Module Structure

Consider the total p-th persistent homology group of a sequence $(K(\bullet), \phi_\bullet)$

$$PH_p = \bigoplus_{i=0}^{N} H_p(K(i))$$

PH_p has a natural, graded module structure over the ring of polynomials $\mathbf{F}_2[t]$ where t acts on a class $\alpha \in H_pK(i)$ by $t.\alpha := \phi_i(\alpha)$.

Theorem: (Carlsson, Zomorodian 2005)

$$PH_p \simeq \bigoplus_{i=0}^{N} \bigoplus_{j>i} U(i,j)^{\beta_{ij}}$$

where $i \in \mathbf{N}$ and $j \in \mathbf{N} \cup \{\infty\}$ and

$$U(i,j) = (F_2[t]/(t^j))(t^i)$$

In other words, $U(i,j) \cap H_pK(l) = \mathbf{F}_2^{\beta_{i,j}}$ if $l \in [i,j)$ and zero otherwise.

Sketch: $\mathbf{F}_2[t]$ is a principal ideal domain and $\{PH_p\}_{\geq p}$ is a finitely generated module over it as we only consider finite sequences of finite simplicial complexes. Furthermore, the module is graded and the theorem follows by the general structure theory for such modules. $\diamondsuit$

The isomorphism class of PH_p determines and is determined uniquely by the multiplicities β_{ij}. This is the rank of the subspace of elements in H_p that are born at $H_pK(i)$ and die at $H_pK(j)$.

Corollary: To each PH_p we can associate a well-defined barcode: the union of βij copies of half open intervals $[i, j)$ for each $i = 0, \ldots, N$

4.4. Stability

In Example 4.2.1, the persistent homology one calculates clearly depends on

- the choice of intervals,
i.e. the choice of $\{a_0, \ldots, a_6\}$, and

- the choice of height function h.

Example: Consider the standard embedding of the sphere S^2 in $\mathbf{R^3}$. Whatever intervals we choose, there will be exactly two half infinite intervals describing the one dimensional persistent homology in degree 0 and 2 respectively. This differs from the barcode computed in Example 4.2.1 only by some short intervals in the barcodes, which should be interpreted as ‘noise’.

In the example we see that persistent homology is *stable*. To express this stability more formally we need a topology on the set of persistence homologies.

For a fixed k, to the persistent homology data $\{PH_k\}$ associate a *persistence diagram* D in $\mathbf{R^2} \cup \mathbf{R} \times \{\infty\}$: a point (i, j) represents the multi-set of independent classes born in degree i and dying in degree j. In addition (to simplify definitions below) we add the points on the diagonal with infinite multiplicity.

The *bottleneck-distance* between two diagrams D and D' is defined by

$$d_B(D, D') := \inf_{\mu: D \to D'} \sup_{x \in D} ||x - \mu(x)||_\infty.$$

where μ is a bijection.

Assumtions:

• X is triangulable, i.e. $X = |K|$ for some finite simplicial complex K, and

• $h : X \to \mathbf{R}$ is such that the homology groups $H_k(h^{-1}(-\infty, a])$ are finite dimensional for all $a \in \mathbf{R}$, and change only at finitely many points in between $a_0, \ldots, a_N$ (such as Morse functions on a manifold).

Consider the persistent homology associated to these finite 'cut' (similar to Example 4.2.1), and let $D(h)$ be the associated persistence diagram.

Theorem:
(Cohen-Steiner, Edelsbrunner, Harer 2007)
For height functions $f, g : X \to \mathbf{R}$,

$$d_B(D(f), D(g)) \leq ||f - g||_\infty := \sup_{x \in X} |f(x) - g(x)|.$$

5. Persistent homology in applications

Data sets are finite sets often taken from some background metric space such as $\mathbf{R}^3$. These sets are discrete and have no interesting topology as such. To capture the topology of the underlying object, our first task is to associate a simplicial complex to the data set that reflects its shape.

5.1. Čech complexes

Let X be a topological space and $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a covering, or simply a collection of subsets of X. We define the *nerve* $N\mathcal{U}$ of $\mathcal{U}$ to be the abstract simplicial complex with one p-simplex for each set $\{\alpha_0, \ldots, \alpha_p\}$ with

$$U_{\alpha_0} \cap \cdots \cap U_{\alpha_p} \neq \emptyset$$

The i-face is the set without U_{α_i}.

We will not prove the following theorems. Nevertheless, they should give some feeling for the topology of the nerve of a covering.

Theorem 1: Assume $\mathcal{U}$ is an open, numerable covering. Furthermore assume $\bigcap_{\alpha \in I} U_\alpha$ is empty or contractible for all $I \subset A$. Then

$$|N\mathcal{U}| \simeq X$$

$$\diamondsuit$$

Let X be a metric space and for $\epsilon > 0$ and $A \subset X$ a finite subset consider the open coverings $\mathcal{U}_\epsilon = \{B_\epsilon(x)\}_{x \in X}$ and $\mathcal{U}_{A,\epsilon} = \{B_\epsilon(x)\}_{x \in A}$ where $B_\epsilon(x)$ denotes the open ϵ-ball around x. Denote the corresponding nerve complexes by $\bar{C}(X, \epsilon)$ and $\bar{C}(A, \epsilon)$.

Note that $\bar{C}(X, \epsilon) \to \bar{C}(X, \mu)$ and $\bar{C}(A, \epsilon) \to \bar{C}(A, \mu)$ for all $\epsilon < \mu$.

Theorem 2: Let M be a compact Riemannian manifold. Then there exists an $e > 0$ such that for all $0 < \epsilon < e$

$$|\bar{C}(X, \epsilon)| \simeq M$$

and for all $0 < \epsilon < e$ there exists a finite set $A \subset M$ such that

$$|\bar{C}(A, \epsilon)| \simeq M$$

The complexes $\bar{C}(X, \epsilon)$ and $\bar{C}(A, \epsilon)$ are referred to as the *C̄ech complexes*.

5.2. Applications to data sets

Let A be a finite set of points in some background metric space (such as $\mathbf{R}^3$). To capture the topology of the underlying object we associate a simplicial complex such as the Čech complex $\bar{C}(A, \epsilon)$. However, it is difficult to choose the right ϵ. One can decide on an appropriate 'scale' (microscopic, normal, or telescopic) but it would be difficult to decide at the start what ϵ (with or without glasses) might catch the most relevant information. We are led to consider a sequence

$$0 < \epsilon_1 \cdots < \epsilon_N$$

and the persistent cohomology of

$$\bar{C}(A, \epsilon_1) \longrightarrow \ldots \longrightarrow \bar{C}(A, \epsilon_N)$$

Example 1: Let A be a collection of points on the surface of a donut-shaped balloon floating in 3-space. The surface is 2-dimensional torus. In the light of Theorem 2, we expect

$$H_p \bar{C}(A, \epsilon) = H_p(S^1 \times S^1) \qquad (*)$$

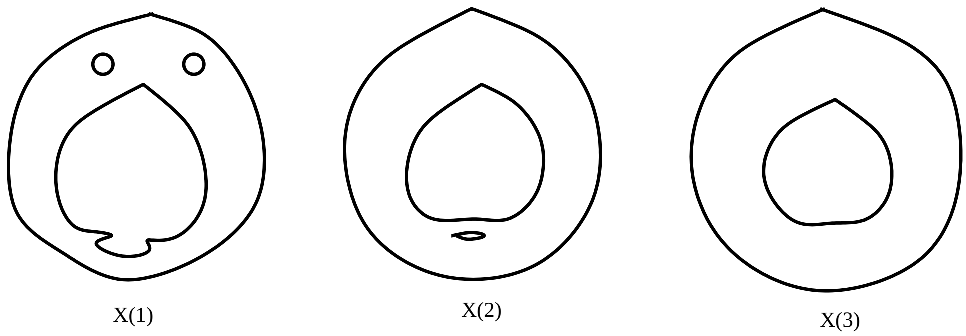
which equals $\mathbf{F}_2, (\mathbf{F}_2)^2, \mathbf{F}_2$ for $p = 0, 1, 2$ and zero otherwise. Clearly, this depends on the right choice of ϵ. But more crucially, the data set A must be good enough:

it needs to be dense enough so that for some choice of ϵ the union $X = \bigcup_{x \in A} B_\epsilon(x)$ of the ϵ balls contains the surface of the balloon and at the same is homotopic to it, i.e. the holes of the balloon are not filled in X.

Under these conditions, (*) will indeed hold. [In order to apply Theorem 2, we need to intersect the open balls in $\mathbf{R}^3$ with the torus to get open balls on the torus which is a Riemannian manifold with the induced metric from $\mathbf{R}^3$.]

If we only knew that the data was collected from a balloon (i.e. an oriented, compact surface without boundary) but did not know what shape it was, the fact that the first homology has rank 2 would now allow us to conclude (!) that the balloon had the shape of a donut.

In this example, we assumed a near perfect data set. This is quite unrealistic. In the next example, we will see how persistent homology can deal with noise and flaws in the data set.



Example 2: Consider a data set A taken from an annulus in the plane $\mathbf{R}^2$. Let us assume that for $\epsilon_1, \epsilon_2, \epsilon_3$, the unions $X(1), X(2), X(3)$ of the ϵ_i balls are as pictured above. Then the associated 1st persistent homology group is

$$PH_1 = H_1X(1) \oplus H_1X(2) \oplus H_1X(3)$$

$$= (\mathbf{F}_2)^3 \oplus (\mathbf{F}_2)^2 \oplus \mathbf{F}_2$$

with associated barcode

$$[1, \infty), [1, 2), [1, 2), [2, 3)$$

Recall that the annulus is homotopic to S^1 and has first homology $\mathbf{F}_2$. We see that the short bars in the barcode correspond to errors in the data: the first two, $[1, 2)$ and $[1, 2)$, are due to the uneven distribution of the data collected, while the third, $[2, 3)$, is due to data the location of which was recorded incorrectly.

References and further reading

G. Carlsson, *Topology and data*, Bulletin A.M.S. **46** (2009), 255-308.

D. Cohen-Steiner, H. Edelsbrunner, J.L. Harer, *Stability of persistence diagrams*,Discrete Comput. Geom. **37** (2007), 103-120.

H. Edelsbrunner, J.L. Harer, *Persistent homology: A survey*, Contemporary Mathematics **452** A.M.S. (2008), 257-282.

H. Edelsbrunner, J.L. Harer, Computational Topology, A.M.S., 2010

S. Weinberger, *What is ... Persistent Homology?*, Notices A.M.S. **58** (2011), 36-39.

A.J. Zomorodian, G. Carlsson, *Computing persistent homology*, Discrete Comput. Geom. **33**(2005), 249-274.