Lie Groups

Warning: work in progress!
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PART A: Theory of matrix groups

A1. Matrices and the exponential map.

Let $\mathbb{F}$ denote $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.1. *The operator norm* $||.||$ *for* $A \in M_n\mathbb{F}$ *is given by*

$$||A|| = \max_{|v|=1, v\in\mathbb{F}} \{|Av|\}.$$

Proposition 1.2. *For a matrix* $A = (a_{ij})$,

$$\begin{aligned} ||\lambda A|| &= |\lambda|\,||A|| \\ ||AB|| &\leq ||A||\,||B|| \\ ||A+B|| &\leq ||A|| + ||B|| \\ ||A|| &\leq \Sigma_{i,j}|a_{ij}|. \end{aligned}$$

Proposition 1.3. *If* $AB - BA = 0$ *then* $\exp(A+B) = \exp A \exp B$. *Hence,* $\exp A$ *is invertible with inverse* $(\exp A)^{-1} = \exp(-A)$.

Proposition 1.4. $\exp : M_n\mathbb{F} \to GL_n\mathbb{F}$ *maps a neighbourhood of 0 homeomorphically to a neighbourhood of* I. *More precisely,*

$$\begin{aligned} \exp(\log(A)) &= A \quad \text{for} \quad ||A - I|| < 1; \\ \log(\exp B) &= B \quad \text{for} \quad ||\exp(B) - I|| < 1. \end{aligned}$$

Proposition 1.5. $a(t) = \exp(tX)$ *is the unique differentiable solution for the two systems of equations*

$$\begin{aligned} &(i) \quad a'(t) = Xa(t) \quad \text{and} \quad a(0) = I; \\ &(ii) \quad a'(0) = X,\ a(0) = I, \quad \text{and} \quad a(s+t) = a(s)a(t). \end{aligned}$$

Campbell-Baker-Hausdorff formula 1.6. *Let* $X, Y \in M_n\mathbb{F}$.
Then for small t *there exists a function* $Z(t)$ *such that* $\exp tX \exp tY = \exp Z(t)$ *and*

$$Z(t) = \Sigma_{m=1}^{\infty} t^m Z_m(X,Y),$$

with $Z_m(X,Y)$ *an* $(m-1)$*-order Lie bracket of* X *and* Y.

A2. Matrix groups.

Definition 2.1. *A matrix group is a subgroup G of $GL_n\mathbb{F}$.*

Proposition 2.2. *Every element $A \in O_n$ is the product of hyper-plane reflections.*

A3. Lie algebras and tangent spaces.

Definition 3.1. *A Lie algebra over $\mathbb{F}$ is a vector space $\mathfrak{g}$ over $\mathbb{F}$ with an $\mathbb{F}$-bilinear map, called the Lie bracket,*

$$[\, , \,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g},$$

which is skew-symmetric and satisfies the Jacobi identity, i.e. for all $X, Y, Z \in \mathfrak{g}$

$$\begin{aligned}
&[X, Y] = -[Y, X], \\
&[[XY], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.
\end{aligned}$$

An $\mathbb{F}$-linear map $\Phi : \mathfrak{g} \to \mathfrak{h}$ is an $\mathbb{F}$-Lie algebra homomorphism if $\Phi[X, Y] = [\Phi(X), \Phi(Y)]$ for all $X, Y \in \mathfrak{g}$.

$\mathfrak{h} \subset \mathfrak{g}$ is an $\mathbb{F}$-Lie subalgebra if it is an $\mathbb{F}$-subspace which is closed under the Lie bracket: $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$.

$\mathfrak{n} \subset \mathfrak{g}$ is an $\mathbb{F}$-Lie ideal if it is an $\mathbb{F}$-subspace with $[\mathfrak{g}, \mathfrak{n}] \subset \mathfrak{n}$.

Definition 3.2. *Let G be a matrix group in $M_n\mathbb{F}$. The tangent space of G at $g \in G$ is*

$$T_gG = \{\gamma'(0) |\, \gamma \text{ is a differential curve with } \gamma(0) = g\}.$$

Proposition 3.3. *T_gG is a real vector subspace of $M_n(\mathbb{F})$.*

Definition 3.4. *Define $L(G) := T_IG$ and put $\dim G = \dim L(G)$*

Theorem 3.5.

1. *If $G \subset GL_n\mathbb{F}$ is a matrix group then $L(G)$ is an $\mathbb{R}$-Lie subalgebra of $M_n\mathbb{F}$.*
2. *If $G \subset GL_n\mathbb{C}$ is a matrix subgroup and $L(G)$ is a $\mathbb{C}$-subspace of $M_n\mathbb{C}$ then $L(G)$ is a $\mathbb{C}$-Lie subalgebra of $M_n\mathbb{C}$.*

A4. Manifolds and topology.

Definition 4.1. *A Hausdorff space M is a manifold of dimension n if for all $x \in M$ there exists an open neighbourhood U of x and an open subset $V \subset \mathbb{R}^n$ with a homeomorphism, called chart, $\phi_U : V \to U$.*

Definition 4.2. *(Informal) A smooth manifold is a manifold M with a choice of an open covering $\mathcal{U}$ and a collection of charts $\{\phi_U\}_{U\in\mathcal{U}}$ such that the composition*

$$(\phi_{U'})^{-1} \circ \phi_U : \phi_U^{-1}(U \cap U') \to \mathbb{R}^n$$

is infinitely often differentiable.

A map of smooth manifolds is differentiable if the associated maps of (restricted) coordinate charts are.

Theorem 4.3. *For a linear group $G \subset M_n\mathbb{F}$, $\exp$ maps $L(G)$ to G.*

Idea. The tangent space at any $g \in G$ is $T_gG = T_IGg = L(G)g$. $a(t) = \exp(tX)$ is a curve that starts in G at $a(0) = I$ and has differential $a'(t) = Xa(t)$. Thus, infinitesimally it stays in G. One proves that this is indeed enough to ensure that $a(t) \in G$. The proof shows furthermore that if $a(t)$ is any curve in $M_n\mathbb{F}$ starting at $a(0) = I$ with $a'(t) \in L(G)a(t)$ then $a(t) = \exp(X(t))$ with $X(t) =\in L(G)$ for small enough t. [See Rossman, p. 46ff.]

Corollary 4.4. *For a linear group G there exist a small neighbourhood U of $I \in G$ such that $\exp$ is onto the path component U_I of I in U.*

Corollary 4.5. $L(G) = \{X \in M_n\mathbb{F} \mid \exp tX \in G$ *for all* $t \in \mathbb{R}\}$.

Definition 4.6. *The group topology of a linear group G is the topology generated by the collection of sets*

$$B_\epsilon(g) := \{g \exp X \mid ||X|| < \epsilon, X \in L(G)\}$$

where $\epsilon > 0, g \in G$.

Note that this may not be the same as the subspace topology of G in $M_n\mathbb{F}$. For example $\mathrm{GL}_n\mathbb{Q}$ is discrete in its group topology but not as a subset of $M_n\mathbb{R}$. If G is a closed subset of $M_n\mathbb{F}$ then the two topologies agree. [See Rossmann, p. 87.]

From now on linear groups will always be considered with their group topology.

Proposition 4.7. *Matrix groups are smooth manifolds of dimension* $\dim G := \dim L(G)$.

Proof. The exponential map $\exp X$ restricted to a small enough open disk U in $L(G)$ defines a chart around $e \in G$. A chart around any $g \in G$ is given by $g \exp X$. The differentiability condition is satisfied as exp is smooth. □

Proposition 4.8. *The following are equivalent for a linear group G.*

1. *G is path-connected.*
2. *G is connected.*
3. *G is generated by any neighbourhood of the identity.*
4. *G is generated by* $\exp L(G)$.

The identity component G_I is a normal subgroup of G and the quotient group is discrete (but may not be linear).

A5. Lie correspondence.

Theorem 5.1. *There is a one-to-one correspondence between connected linear groups and linear Lie algebras which takes a group G to its Lie algebra $L(G)$ and a Lie algebra $\mathfrak{g}$ to the groups $\Gamma(\mathfrak{g})$ generated by the image of $\mathfrak{g}$ under* exp.

A differentiable map of smooth manifolds $f : M \to N$ induces a map of tangent spaces $Df_m : T_m M \to T_{f(m)} N$. If $\alpha(t)$ is a curve with $\alpha(0) = m$ and $\alpha'(0) = X$ then $Df_m X := (f \circ \alpha)'(0)$.

Recall from Problem Sheet 2, for a differentiable homomorphism $\phi : G \to H$ of linear groups, we have

$$\phi(\exp X) = \exp(\frac{d}{dt}\phi(\exp tX)|_0) = \exp(D\phi_I X).$$

We will use the notation $L(\phi) := D\phi_I$.

Theorem 5.2. *$L(\phi) : L(G) \to L(H)$ is a Lie algebra homomorphism.*

Lemma 5.3. *The following are equivalent for a homomorphism of linear groups $\phi : G \to H$*

(1) *ϕ is locally bijective (i.e. for all $g \in G$ there exists an open neighbourhood of U of g such that $\phi|_U$ is a homeomorphism onto a neighbourhood of $\phi(g)$).*
(2) *$L(\phi) : L(G) \to L(H)$ is a bijection.*

For such ϕ, $\ker \phi$ is discrete.

Local bijections may not be global bijection as in the example $\exp : i\mathbb{R} \to S^1$.

Definition 5.4. *A locally bijective homomorphism $\pi : \tilde{G} \to G$ between connected linear groups is a covering.*

There are no coverings if every loop in G can be contracted to the constant loop, i.e. when G is simply connected.

Theorem 5.5. *Let G and H be connected linear groups, and $\Phi : L(G) \to L(H)$ be a map of Lie algebras. Then there exists a covering $\pi : \tilde{G} \to G$ and group homomorphism $\phi : \tilde{G} \to H$ such that*

$$L(\phi) = \Phi \circ L(\pi).$$

A6. The adjoint.

let G be any matrix group. For every element $a \in G$, conjugation defines a homomorphism $c_a(x) = axa^{-1}$, and hence induces a map of Lie algebras $\mathrm{Ad}\,(a) : L(G) \to L(G)$. Indeed $\mathrm{Ad}\,(a)$ is also given by conjugation $\mathrm{Ad}\,(a)(X) = aXa^{-1}$ as

$$a(\exp X)a^{-1} = \exp(aXa^{-1}).$$

Note that $\mathrm{Ad}\,(ab) = \mathrm{Ad}\,(a)\ \mathrm{Ad}\,(b)$ and $\mathrm{Ad}\,(a^{-1}) = \mathrm{Ad}\,(a)^{-1}$. Hence, Ad defines a group homomorphism taking $a \in G$ to the group of invertible linear transformations of $L(G)$ to itself, which we denote by $\mathrm{Aut}(L(G))$:

$$\mathrm{Ad}\,: G \longrightarrow \mathrm{Aut}(L(G)), \quad a \mapsto \mathrm{Ad}\,(a).$$

Choosing a basis for $L(G)$ over $\mathbb{F}$ we can identify $\mathrm{Aut}L(G)$ as the matrix group $\mathrm{GL}_n\mathbb{F}$ and its Lie algebra $L(\,\mathrm{Aut}L(G))$ with $M_n\mathbb{F}$; here $n = \dim L(G)$. We may consider the induced map on tangent spaces. The following proposition identifies it as

$$\mathrm{ad}\,: L(G) \longrightarrow L(\,\mathrm{Aut}(L(G))), \quad X \mapsto [X, \,],$$

that is $\mathrm{ad}\,(X)(Y) = [X, Y]$.

Proposition 6.1. *$Ad(\exp X) = \exp(adX)$, and hence $ad = L(\,Ad)$.*

Example 6.2.:

$$SU_2 = \{\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \mid a, b \in \mathbb{C},\, a\bar{a} + b\bar{b} = 1\}$$

is homeomorphic to $S^3 \subset \mathbb{C}^2$.

$$L(SU_2) = \{X = \begin{pmatrix} ix & -y + iz \\ y + iz & -ix \end{pmatrix} |x, y, z \in \mathbb{R}\} \simeq \mathbb{R}^3.$$

Note that the usual innerproduct for matrices on $L(SU_2)$ is given by $(X, Y) = \mathrm{tr}\,(X\bar{Y}^T) = -tr(XY)$ which is invariant under Ad. Hence Ad acts on the Lie algebra through isometrics, for a choice of an orthonormal basis on $L(SU_2)$,

$$\mathrm{Ad}\,: SU_2 \longrightarrow SO_3.$$

Claim: Ad is a covering map with kernel $\{I, -I\}$.

Topologically, this identifies SO_3 with $\mathbb{R}P^3$, the space of lines in $\mathbb{R}^4$ as SU_2 can be identified with the unit quarternions and hence with S^3 (see A.7).

A7. Classical groups and their Cartan subgroups.

Recall, $\mathbb{C} \simeq \mathbb{R}^2$ via $a + ib \mapsto (a, b)$. Similarly, one can identify the Hamiltonians $\mathbb{H} \simeq \mathbb{C}^2 \simeq \mathbb{R}^4$ via

$$a + ib + jc + kd \mapsto (a + ib, c + id) \mapsto (a, b, c, d).$$

Alternatively, $\mathbb{H}$ is the subring of $M_2\mathbb{C}$ of elements of the form

$$\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}, \quad z, w \in \mathbb{C}$$

If $x = a + ib + jc + kd \in \mathbb{H}$, its conjugate is defined by $\bar{x} := a - ib - jc - kd$ and its norm by $N(x) := x\bar{x} = |x|^2$. The Hermitian product on $\mathbb{C}^n$ extends to a symplectic scalar product on $\mathbb{H}^n$:

$$< x, y >:= \Sigma_{i=1}^n x_i \bar{y}_i.$$

The group that preserves this product is the symplectic group

$$\begin{aligned} Sp_n =& \{A \in \mathrm{GL}_n\mathbb{H} | \, \bar{A}^T = A^{-1}\} \\ & \{A \in \mathrm{GL}_{2n}\mathbb{C} | \, A = \begin{pmatrix} X & Y \\ -\bar{Y} & \bar{X} \end{pmatrix} \in U_{2n}, X, Y \in M_n\mathbb{C}\} \end{aligned}$$

More generally, let $B : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}^n$ be a non-singular form over $\mathbb{K} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, i.e. $B(v, w) = 0$ for all w implies $v = 0$. We distinguish bilinear forms $(B(\alpha v, \beta w) = \alpha B(v, w)\beta)$ and sesquilinear forms$(B(\alpha v, \beta w) = \alpha B(v, w)\bar{\beta})$. A form is symmetric if $B(v, w) = B(w, v)$ and non-degenerate when $B(v, w) = 0$ for all w implies $v = 0$. Define

$$\mathrm{Aut}B := \{A \in \mathrm{GL}_n\mathbb{K} | \, B(Av, Aw) = B(v, w)\}.$$

Example: Non-degenerate symmetric bilinear forms over the reals are given by symmetric matrices. Indeed, one can always find a basis such that the matrix is diagonal with p entries 1 followed by $q = n - p$ entries -1. The associated automorphism group is denoted by $O(p, q)$. The group $O(3, 1)$ is the Lorentz group which plays a special role in physics.

Theorem 7.1. *For all $A \in G$ there exists a $P \in G$ such that if PAP^{-1} is of the following diagonal or block diagonal form:*

$$\begin{array}{cc} SU_n, & \textit{diag}\,(\lambda_1, \ldots, \lambda_n) \textit{ with } \lambda_j = \exp(2\pi i\theta_j) \textit{ and } \prod_j \lambda_j = 1; \\ SO_{2n+1}, & \textit{block-diag}\,(t_1, \ldots, t_n, 1) \textit{ with } t_i = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}; \\ Sp_n, & \textit{diag}\,(\lambda_1, \ldots, \lambda_n) \textit{ with } \lambda_j = \exp(2\pi i\theta_j); \\ SO_{2n}, & \textit{block-diag}\,(t_1, \ldots, t_n) \textit{ with } t_i \textit{as above.} \end{array}$$

Definition 7.2. *The matrices described in each case from a subgroup, called the Cartan subgroup.*

Corollary 7.3. *The exponential map is surjective for the classical groups SU_n, SO_n, and Sp_n.*

Proposition 7.4. *SO_n, SU_n, and Sp_n are compact.*

Definition 7.5. *$T \subset G$ is a torus if it is isomorphic to a product of circles $S^1 \times \cdots \times S^1$. It is a maximal torus if for any torus T' containing T we have $T' = T$.*

Proposition 7.6. *The Cartan subgroups are maximal abelian subgroups. Furthermore, any maximal torus in $G (= SU_n, SO_n, Sp_n)$ is conjugate in G to the Cartan subgroup.*

Let $N(T) := \{g \in G | \, gTg^{-1} \subset T\}$ be the normaliser of T in G.

Definition 7.7. *$W := N(T)/T$ is called the Weyl group.*

Example 7.8: The Weyl group of $G = \mathrm{SL}_n\mathbb{C}$ is $W = \Sigma_n$ which is of order $n!$.

$N(T)$ acts via conjugation on T and hence via Ad on $L(T)$. T is in the kernel of this action and hence the Weyl group W acts on $L(T)$. As T by Proposition 7.6 is maximal abelian, the kernel of the action is precisely T, and W acts effectively.

A8. Complexifications.

Let $\mathfrak{g}$ be a complex Lie algebra. A real Lie subalgebra $\mathfrak{g}_0$ is called a real form of $\mathfrak{g}$ if $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$. For example: $L(\mathrm{GL}_n\mathbb{C}) = M_n\mathbb{C}$ has real form $M_n\mathbb{R}$, and $L(\ \mathrm{SL}_n\mathbb{C}) = \{A \in M_n\mathbb{C} |\, trA = 0\}$ has real form $L(\ \mathrm{SL}_n\mathbb{R}) = \{A \in M_n\mathbb{R} |\, trA = 0\}$. Define

$$SO_n\mathbb{C} = \{A \in \mathrm{GL}_n\mathbb{C} |\, A^T A = I\};$$
$$Sp_n\mathbb{C} = \{A \in \mathrm{GL}_{2n}\mathbb{C} |\, A^T JA = J\}$$

where J is the block diagonal matrix $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$.

Proposition 8.1.

(1) *$L(SU_n)$ is a real form of $L(\ SL_n\mathbb{C})$;*
(2) *$L(SO_n)$ is a real form of $L(SO_n\mathbb{C})$;*
(3) *$L(U_n)$ is a real form of $L(GL_n\mathbb{C})$.*

Note that both $L(U_n)$ and $L(\mathrm{GL}_n\mathbb{R})$ are real forms of $L(\mathrm{GL}_n\mathbb{C})$ but they are not isomorphic (except for $n = 1$).

Given any real Lie algebra $\mathfrak{g}$ there is an associated complex Lie algebra, its complexification $\mathfrak{g}_\mathbb{C} := \mathfrak{g} \oplus i\mathfrak{g}$ with bracket

$$[X_1 + iX_2, Y_1 + iY_2] = ([X_1, Y_1] - [X_2, Y_2]) + i([X_2, Y_1] + [X_1, Y_2]).$$

In the proposition above, we identified the Lie algebra $L(G)$ of certain compact matrix groups G as the real form of $L(G_\mathbb{C})$ for certain groups $G_\mathbb{C}$. $G_\mathbb{C}$ is called the complexification of G.

The next result describes the complexification $H = T_\mathbb{C}$ of the Cartan subgroup of G (which will also be called the Cartan subgroup of $G_\mathbb{C}$) and its Lie algebra $L(H)$. The names for the type is conventional; the subscript gives the dimension of the Cartan subgroup.

Proposition 8.2.

Type	$G_\mathbb{C}$	H	$L(H)$
A_{n-1}	$SL_n\mathbb{C}$	$(\epsilon_1, \ldots, \epsilon_n)$, $\prod_j \epsilon_j = 1$,	$(\lambda_1, \ldots, \lambda_n)$, $\Sigma_j\lambda_j = 0$,
B_n	$SO_{2n+1}\mathbb{C}$	$(\epsilon_1, \ldots, \epsilon_n, \epsilon^{-1}, \ldots, \epsilon_n^{-1}, 1)$,	$(\lambda_1, \ldots, \lambda_n, -\lambda_1, \ldots, -\lambda_n, 0)$
C_n	$Sp_n\mathbb{C}$	$(\epsilon_1, \ldots, \epsilon_n, \epsilon^{-1}, \ldots, \epsilon_n^{-1})$,	$(\lambda_1, \ldots, \lambda_n, -\lambda_1, \ldots, -\lambda_n)$
D_n	$SO_{2n}\mathbb{C}$	$(\epsilon_1, \ldots, \epsilon_n, \epsilon^{-1}, \ldots, \epsilon_n^{-1})$,	$(\lambda_1, \ldots, \lambda_n, -\lambda_1, \ldots, -\lambda_n)$

Any matrix which commutes with a diagonal matrix with distinct diagonal entries is itself diagonal. Hence, the Cartan subgroups of the groups G and $G_\mathbb{C}$ listed above are maximal abelian subgroups of G and $G_\mathbb{C}$. Equally, $L(H)$ is a maximal abelian subalgebra of $L(G_\mathbb{C})$.

A9. Roots and weights.

Let G be compact and connected ($G = SU_n, SO_n, Sp_n$), T be the Cartan subgroup (or more generally a maximal torus), $G_{\mathbb{C}}$ be its complexification and $H = T_{\mathbb{C}}$ its Cartan subgroup. We want to analyse the map of Lie algebras

$$\mathrm{ad} : L(H) \to L(\ \mathrm{Aut}(L(G_{\mathbb{C}})))$$

given by $X \mapsto [X,]$. In particular, we want to describe a decomposition of $L(G_{\mathbb{C}})$ into the eigenspaces of $\mathrm{ad}\,(X)$ for all X simultaneously:

$$L(G_{\mathbb{C}}) = L(H) \oplus \Sigma_{\alpha \in \Phi} \mathbb{C} E_\alpha. \tag{9.1}$$

The α in the decomposition (9.1) are linear combinations of the λ_j as in the description of $L(H)$ in Proposition 8.2, now thought of as coordinates, i.e. as $\mathbb{C}$-linear functionals

$$\lambda_j : L(H) \longrightarrow \mathbb{C}.$$

Definition 9.2. *The non-trivial joint eigenvalues $\alpha \in \Phi$ of $ad(L(H))$ are the roots of G. The collection Φ is the root system. Note that the roots are elements of the dual space $L(H)^*$, and if E_α is the corresponding eigenvector then*

$$ad\,(X)E_\alpha = [X, E_\alpha] = \alpha(X)E_\alpha.$$

Example Let $G_{\mathbb{C}} =\ \mathrm{SL}_n\mathbb{C}$, and let E_{jk} denote the matrix with one non-zero entry $e_{jk} = 1$. Let $X =\ \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n) \in L(H)$. Then

$$\begin{aligned} \mathrm{ad}\,(X)E_{jk} &= XE_{jk} - E_{jk}X \\ &= \lambda_j E_{jk} - \lambda_k E_{jk} \\ &= (\lambda_j - \lambda_k)E_{jk} \end{aligned}$$

Hence, $\Phi = \{\pm(\lambda_j - \lambda_k)\}_{j<k}$ is a set of non-trivial joint eigenvalues of ad. A simple dimension count gives that this must be a complete set.

Proposition 9.3. *For the classical groups, the roots systems are as follows ($j < k$):*

$$\begin{array}{cc} SL_n(\mathbb{C}) & \pm(\lambda_j - \lambda_k) \\ SO_{2n+1}(\mathbb{C}) & \pm(\lambda_j \pm \lambda_k), \pm\lambda k \\ Sp_n(\mathbb{C}) & \pm(\lambda_j \pm \lambda_k), \pm 2\lambda_k \\ SO_{2n}(\mathbb{C}) & \pm(\lambda_j \pm \lambda_k), \prod(\pm 1) = 1 \end{array}$$

Defintion 9.4. *The set of roots with positive signs $\Phi^+ \subset \Phi$ are called the positive roots. A root $\alpha \in \Phi^+$ is called simple if it is not the sum of two positive roots.*

$N(T)$ acts on $L(T^*)$ and $L(H)^*$ via $s.\alpha(X) := \alpha(s^{-1}Xs)$.

Proposition 9.5. *The Weyl group W permutes the roots (or equivalently the eigenvectors).*

For the Cartan subalgebras H (as listed in Proposition 8.2), $L(H)^*$ is spanned by $\lambda_1, \ldots, \lambda_n$. For type B_n, C_n, and D_n, they are linear indenpendent, while for type A_{n-1} they satisfy the relation $\Sigma_i \lambda_i = 0$.

Definition 9.6. $\{l_1\lambda_1 + \ldots l_n\lambda_n | \, l_i \in \mathbb{Z}\}$ *is the set of weights.*

For example, all roots are weights. Any functional $L(H) \to \mathbb{C}$ is a map of Lie algebras as $L(H)$ is abelian. The weights are those functionals that correspond to a group homomorphism, i.e. those of the form $L(\phi)$ for some homomorphism $\phi : H \to \mathbb{C}^*$.

The real subspace $L := \{l_1\lambda_1 + \ldots l_n\lambda_n | \, l_i \in \mathbb{R}\}$ of $L(H)^*$ has an innerproduct: $(\lambda, \lambda') = \Sigma_i \, l_i l_i'$.

Definition 9.7.

(1) $\lambda \in L$ *is dominant if* $(\lambda, \alpha) \geq 0$ *for all* $\alpha \in \Phi^+$.
(2) $\lambda \in L$ *is dominant if* $(\lambda, \alpha) > 0$ *for all* $\alpha \in \Phi^+$.
(3) $\lambda \in L$ *is regular if* $(\lambda, \alpha) \neq 0$ *for all* $\alpha \in \Phi$.
(4) $\lambda \in L$ *is higher than* μ *if* $\lambda = \mu + \Sigma_\alpha a_\alpha \alpha$, *for some* $a_\alpha \geq 0, \alpha \in \Phi^+$.

For example, A_2 has highest root $\lambda_1 - \lambda_3$, $B_2 = C_2$ has highest root $\lambda_1 + \lambda_2$.

Remark: A Lie algebra of dimension > 1 is called simple if it does not contain any nontrivial ideals. In particular its centre is trivial. It truns out that the Lie algebras of the above classical groups are all simple, with the exception of D_1 and D_2. There are only a handful additional isomorphism classes of simple, complex Lie algebras $(G_2, F_4, E_6, E_7, E_8)$.

PART B: Representation theory

B1. First notions.

Definition 1.1 Let G be a (linear) group. Then a *representation* V of G is a finite dimensional vector space V and a (differentiable) homomorphism

$$\rho : G \longrightarrow \operatorname{Aut}(V) = \mathrm{GL}(V).$$

A map of G-representations $(V_1, \rho_1) \to (V_2, \rho_2)$ is a linear map $T : V_1 \to V_2$ such that

$$T(\rho_1(g)v) = \rho_2(g)T(v)$$

for all $g \in G$ and $v \in V$.

Example: Every linear group $G \subset \mathrm{GL}(\mathbb{C})$ has a natural linear action on $V = \mathbb{C}^n$.

Example: The adjoint representation is a representation of G on its Lie algebra $V = L(G)$.

Given two representations V and W one can define two new representations, their *direct sum* $V \oplus W$ with $g(v,w) := (gv, gw)$, and their *tensor product* $V \otimes W$ with $g(v \otimes w) := gv \otimes gw$.

Example: Let V be a real representation of G and consider $\mathbb{C}$ as a trivial representation of G. $V \otimes \mathbb{C}$ is the *complexification* of V. As a real representation it is isomorphic to the direct sum $V \oplus V$.

We will mainly be interested in representations of compact linear groups.

Theorem 1.2. *Let G be a compact linear group, $\mathcal{C}(G)$ the vector space of its real continuous functions. Then there exists a unique function*

$$\int_G : \mathcal{C}(G) \longrightarrow \mathbb{R}$$

which is (i) linear and monotone, (ii) normed (i.e. $\int_G 1 = 1$), and (iii) bi-invariant (i.e. $\int_G f(gx)dx = \int_G f(x)dx = \int_G f(xg)dx$ for $g \in G$).

We will take this theorem on trust.

Example: When G is finite and $f \in \mathcal{C}(G)$,

$$\int_G f = \frac{1}{|G|} \sum_{g \in G} f(g).$$

Proposition 1.3. *If V is a representation of a compact (linear) group G, then it has a positive definite Hermitian form which is invariant under G.*

Definition 1.4. In the situation of Proposition 1.3, with respect to an orthonormal basis, G acts through orthogonal or unitary matrices. We call such representations *orthogonal* or *unitary*.

Definition 1.5. A linear subspace W of a representation V is *G-stable* if $gw \in W$ for all $g \in G$ and $w \in W$. V is *irreducible* if it does not contain any proper G-stable subspaces.

Proposition 1.6. *If G is a compact (linear) group, then every G-representation V is a direct sum of irreducible G-spaces.*

Example: This is not true for non-compact groups: Let G be the group of real upper-triangular 2×2-matrices. It acts on $V = \mathbb{R}^2$. Then the subspace spanned by $(1,0)^t$ is stable but no other subspace is. So V cannot be the direct sum of irreducible representations.

Lemma 1.7. *(Schur's Lemma)*
Let G be a linear group, V and W be irreducible G-representations.

1. *If $T : V \to W$ is a G-map then $T = 0$ or T is an isomorphism.*
2. *If $T : V \to W$ is a G-isomorphism and V is complex then T is multiplication by some $\lambda \in \mathbb{C}$.*

Corollary 1.8. *If G is Abelian then every irreducible complex representation is one dimensional.*

Corollary 1.9. *Let G be compact (linear), (V, π) and (W, ρ) be two irreducible G-representations, and $T : V \to W$ be a linear map. Define*

$$T^0 := \int_G \rho(g) T \pi(g^{-1}).$$

1. *If π is not equivalent to ρ then $T^0 = 0$.*
2. *If $\pi = \rho$ and is complex then $T^0 = \lambda I$ where $\lambda = \; trT/\dim V$.*

Corollary 1.10. *(Schur's orthogonality relations for matrix coefficients)*
Let G be compact (linear), (V, π) and $(W\rho)$ be irreducible orthogonal or unitary representations.

1. *If π is not equivalent to ρ then $\int_G \pi_{ij}(g)\overline{\rho_{kl}(g)} = 0$.*
2. *If $\pi = \rho$ is complex then $\int_G \pi_{ij}(g)\overline{\rho_{kl}(g)} = \frac{1}{\dim V}\delta_{ik}\delta_{jl}$.*

B2. Characters.

Definition 2.1. Let G be a linear group and V be a real (or complex) G-representation. Its *character* is the map

$$\chi_V : G \longrightarrow \mathbb{R} \quad (\text{or } \mathbb{C}); \qquad g \mapsto \operatorname{tr}(ghg^{-1}).$$

Proposition 2.2. *Let G be a linear group and V and W be two representations.*

(1) χ_V *is continuous;*
(2) $\chi_V(ghg^{-1}) = \chi(h)$*;*
(3) $\chi_V(I) = \dim V$*;*
(4) $\chi_{V\oplus W} = \chi_V + \chi_W$*;*
(5) $\chi_{V\otimes W} = \chi_V \chi_W$*;*
(6) *if G is compact then* $\chi_V(g^{-1}) = \overline{\chi_V(g)}$.

Corollary 2.3. *Let G be a compact linear group and (V, π) and (W, ρ) be two irreducible representations. Then*

$$(\chi_V, \chi_W) := \int_G \chi_V(g)\overline{\chi_W(g)}$$

is 0 if π and ρ are not equivalent, and equal to 1 if they are euqivalent and complex.

Corollary 2.4. *Let G be a compact linear group. Two representations are equivalent if and only if they have the same character.*

Recall that $(f, g) := \int_G f\bar{g}$ defines an inner product on the space of continuous functions. Hence, Corollary 2.3 may be interpreted to say that the characters for irreducible representations are orthogonal, and in particular linearly independent. Furthermore, Schur's orthogonality relation for matrix coefficients may be rephrased as follows. If $\{\pi^\lambda\}_\lambda$ is a complete set of non-isomorphic irreducible representations, then the coordinate functions $\{\pi^\lambda_{i,j}\}_{\lambda,i,j}$ are orthogonal.

Note that $(\chi_V, \chi_V) = 1$ if and only if V is irreducible.

Theorem 2.5. *(Peter-Weyl)*
Let G be comact linear. Then $\{\pi^\lambda_{i,j}\}_{\lambda,i,j}$ is a complete set of orthogonal functions on G. I.e. every continuous function on G can be approximated by finite linear combinations of the coordinate functions of irreducible characters (in the L^2-norm).

B2.1 Characters of the torus.

Let $T = S^1 \times \ldots S^1$ be an n dimensional torus. Consider the commutative diagram

$$\begin{array}{ccc} L(T) \simeq 2\pi i\mathbb{R}^n & \xrightarrow{L(\omega)=\lambda} & 2\pi i\mathbb{R} \subset \mathbb{C} \\ {\scriptstyle\exp}\Big\downarrow & & {\scriptstyle\exp}\Big\downarrow \\ T \simeq 2\pi i\mathbb{R}^n/2\pi i\mathbb{Z}^n & \xrightarrow{\omega=e^\lambda} & U_1 = S^1 \subset \mathbb{C}^*. \end{array}$$

Irreducible characters of T are one dimensional unitary representations ω. Hence, $L(\omega) : L(T) \to \mathbb{C}$, $2\pi i(\theta_1, \ldots, \theta_n) \mapsto 2\pi i(l_1\theta_1 + \cdots + l_n\theta_n)$ factors through $i\mathbb{R} \subset \mathbb{C}$. Furthermore, $\exp \circ L(\omega)$ has to factor through T. Thus $(l_1\theta_1 + \cdots + l_n\theta_n)$ has to be in $\mathbb{Z}$ for all $(\theta_1, \ldots, \theta_n)$ in $\mathbb{Z}^n$. Thus $(l_1, \ldots, l_n)$ is in $\mathbb{Z}^n$. Vice versa, any $\lambda = (l_1, \ldots, l_n)$ in $\mathbb{Z}^n$ gives rise to a homomorphism

$$e^\lambda : T \longrightarrow S^1; \qquad (\epsilon_1, \ldots, \epsilon_n) \mapsto \epsilon_1^{l_1} \ldots \epsilon_n^{l_n}.$$

So the set of irreducible characters of the torus T is the set $\{e^\lambda\}_\lambda \simeq \mathbb{Z}^n$ and every character of T is of the form

$$\chi = \sum_\lambda m_\lambda e^\lambda$$

with $m_\lambda \in \mathbb{N}$.

B3. Weyl character formula.

Let G be compact linear, T be a maximal torus in G, and $W = N(T)/T$ be its Weyl group. $s \in W$ acts on the characters of T by $s.\phi(t) := \phi(sts^{-1})$.

Lemma 3.1. *If χ is a character of G then its restriction $\chi|_T = \phi$ to T is W-invariant.*

Lemma 3.2. *Every character χ of G is determined by its restriction $\chi|_T = \phi$ to the maximal torus T.*

Define
$$\triangle := \prod_{\alpha \in \Phi_+} (e^{\alpha/2} - e^{-\alpha/2}) \qquad \text{and} \qquad \rho := \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha$$

Theorem 3.3. *(Weyl Character Formula)*
As a function of T, the irreducible characters of G are precisely
$$\chi_\lambda = \frac{1}{\triangle} \sum_{s \in W} \textit{sign } (s) e^{s(\lambda+\rho)}$$
where λ is the dominant (or heighest) weight of the character.

Note that part of the claim is that χ_λ is a well-defined map $L(T) \to \mathbb{C}$ and that it factors through $\exp : L(T) \to T$. The trivial one-dimensional representation corresponds to $\lambda = 0$.

Theorem 3.4. *(Weyl Dimension Formula)*
The dimension of the representation with highest weight λ is
$$\frac{\prod_{\alpha \in \Phi_+} (\lambda + \rho, \alpha)}{\prod_{\alpha \in \Phi_+} (\rho, \alpha)}.$$