New directions for trace methods

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Background: K-theory is hard to compute

- Question: how to compute algebraic K-theory?
  Direct computation possible only in limited examples.
- One kind of answer: map to it something we can understand more easily.
- We do this using a massive generalization of taking the trace of a matrix.

  $\Longrightarrow$ Trace methods

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Trace methods

Dennis trace map:

$$BGL_n(R) \to B^{\mathrm{cyc}} GL_n(R) \to B^{\mathrm{cyc}} M_n(R) \to B^{\mathrm{cyc}} R \simeq HH(R),$$

- B^{cyc} is the cyclic bar construction
- $B^{\mathrm{cyc}} M_n(R) \to B^{\mathrm{cyc}} R$ is the trace map realizing Morita equivalence.
- Induces map $K(R) \to HH(R)$.

Even better: work over S rather than $\mathbb{Z}$ (Waldhausen)

Leads to topological Hochschild homology (THH) and the Bokstedt trace

$$K(R) \to THH(R).$$

This is better, but still not that close
e.g., $HH(\mathbb{Z})$ is trivial, and $THH(\mathbb{Z})$ is infinite but far from $K(Z)$.

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Trace methods and cyclic theories

- Lift of the Dennis trace map:

$$K(R) \to HC^{-}(R) \to HH(R),$$

  where $HC^{-}(R)$ is negative cyclic homology.
- Goodwillie shows this an infinitesimal rational equivalence.
- Lift of the Bokstedt trace map (Bokstedt-Hsiang-Madsen):

$$K(R) \to TC(R) \to THH(R),$$

  where $TC(R)$ is topological cyclic homology.
- Dundas and McCarthy show this is an infinitesimal p-adic equivalence.
- “often” a p-adic equivalence.

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Using TC to compute K-theory

Although K-theory is hard to compute, TC and especially THH are much easier to compute.

THH can be computed in all sorts of ways.
For example,

- natural filtration spectral sequences
- algebraic Hochschild homology computations

TC derives from the equivariant structure on THH

studied using equivariant stable homotopy theory.

In some important cases $K(R) \simeq_p TC(R)[0, \infty)$
$\implies$ can use TC to compute K

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Trace methods are wickedly successful

Through a lot of ingenuity and hard work, we now know how to calculate all sorts of things:

1. $K(Z_p^\wedge)$ (Bokstedt-Madsen)
2. Many cases of Quillen-Lichtenbaum (Hesselholt-Madsen)
3. $K(S) = A(*)$ [at some primes] (Rognes),
4. $A(\Sigma X)$ (Bokstedt-Carlsson-Cohen-Goodwillie-Hsiang-Madsen)
5. $K(ku)$ [after $V(1)$-localization] (Ausoni, Ausoni-Rognes)

What's next?

We may have come to the end of this line of attack.

- For one thing, we know the answers for many of the ring spectra we have names for.
- For another, it's become increasingly difficult to push these techniques.

There are new ideas. For example,

- homological approach of Bruner-Rognes
- Segal conjecture approach by Lunoe-Nielsen and Rognes building off Carlsson's work
- work stemming from Gerhardt's thesis

We're going to look at other directions.

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Goal of this talk: new directions

Some questions we might pursue:

- What kind of a thing is THH or TC?
  What structural properties do these theories have?
  (e.g., in analogy with K-theory)
- How can we interpret them conceptually?
  (e.g., from a motivic standpoint)
- Where do these theories (particularly TC) fit into the emerging story of field theories?
- Can we find new approaches to computing remaining named examples (i.e., $K(MU)$)?

I'll talk about work in these directions, providing some answers and many more questions and conjectures.

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Properties of THH / TC

Question first studied by Dundas and McCarthy:

Which of the structural theorems of K-theory are possessed by THH and TC?

We have complete answers to these questions (joint work with Mandell):

Essentially, THH and TC have all the same properties as K-theory:

- Additivity
- Localization
- Approximation

But the story of localization is a funny one.

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Theorem about spectral categories
(lifting triangulated categories)

Theorem

THH and TC satisfy Neeman's generalized version of Thomason-Trobaugh's localization theorem: exact sequence of triangulated categories $\mathcal{A} \to \mathcal{B} \to \mathcal{B}/\mathcal{A}$ yields

$$T(\mathcal{A}) \to T(\mathcal{B}) \to T(\mathcal{B}/\mathcal{A}) \to$$

a cofiber sequence of spectra.

$\Longrightarrow$ Mayer-Vietoris for schemes, projective bundle theorem.

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This is not the localization sequence in the Hesselholt-Madsen K-theory computations.

Hesselholt-Madsen localization sequence

Let R be a discrete valuation ring,
k = residue field
F = field of fractions

Thomason-Trobaugh cofiber sequence is

$$THH(R \text{ on } k) \to THH(R) \to THH(F) \to$$

$THH(R \text{ on } k)$ is not THH of a ring
but is THH of a spectral category
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$$THH(k) \to THH(R) \to THH(R \mid F) \to$$

$THH(R \mid F)$ is not THH of a ring
but is THH of a simplicial exact category.

Explanation

Two ways of encoding homotopy theory:

1. Mapping spaces (e.g., simplicial categories).
2. Weak equivalences (e.g., model categories).

We're used to these two agreeing: mapping complexes in simplicial model categories vs. the Dwyer-Kan simplicial localization.

We typically invert (localize) by changing both.
$\Longrightarrow$ Thomason-Trobaugh localization.

But instead can invert one set of equivalences in the mapping space and a different one in the weak equivalences
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Conceptual Explanation?

Questions

What does this mean, conceptually?

- What role does connectivity play here?
- Relation to log geometry of Rognes?
- Relation to Barwick's "virtual Waldhausen categories"?

Figure: Miriam explaining a subtle point in the argument.

New and improved models of THH and TC.

Classically, "do" THH and TC over S.

But easy to handle THH over arbitrary ground rings (EKMM).

This seems to lose the cyclotomic structure: Want

$$\phi^C(X^{\wedge C}) \simeq X.$$

Bokstedt originally solved using special coherence machinery that only works over S.

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Instead, we can use HHR norm!

Then $THH(R)$ is $N_e^{S^1} R$.
(Joint with Angeltveit, Gerhardt, Hill, and Lawson.)

This is closely related to perspective of Brun-Carlsson-Dundas.

Why useful?

1. Gives TC over other ground rings. Base-change spectral sequences, computations. (Mandell)
2. Progress towards TC of Thom spectra (extending work with Cohen and Schlichtkrull on THH of Thom spectra)

Theorem

Equivariant of homotopy type of THH of Thom spectrum is itself an equivariant Thom spectrum.

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Figure: Uncertainty about the use of ∞-categories

Interpreting THH and TC from a motivic perspective

Motivic means different things to different people.

For our purposes, it means co-representability of some motivic cohomology theory:

$$K_n(X) \cong \pi_n \operatorname{Rhom}(\mathbb{U}, \mathbb{M}(X)),$$

where $\mathbb{U}$ is the unit and $\mathbb{M}$ denotes the realization functor (i.e., the associated motive).

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Question

How can we interpret *THH* and *TC* in this fashion?

Motivic approaches, take 1: Category of cyclotomic spectra

THH yields an equivariant structure called a cyclotomic spectrum, generalizing properties of $\Sigma^\infty_+ \Lambda X$.

More basic question: how should we think about the homotopy theory of cyclotomic spectra?

One answer (joint with Mandell).

- There is a model category of pre-cyclotomic spectra.
- Cyclotomic spectra are a full triangulated subcategory.
- We have the following property in this category:

$$TC(R) = \mathrm{Rhom}(S, THH(R)).$$

- *THH* as a functor from ring spectra to cyclotomic spectra provides the realization functor $\mathbb{M}$.
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Motivic approaches, take 2: Non-commutative motives

Universal property of K-theory

K-theory is the initial functor from stable categories to spectra that "splits exact sequences". (Joint with Gepner and Tabuada.)

(related characterization by Barwick)

- The category $\mathrm{Cat}^{\mathrm{ex}}_{\mathrm{stab}}$ of small stable categories has internal homs: exact functors $\mathcal{C} \to \mathcal{D}$ are *right-compact* $\mathcal{C}^{\mathrm{op}} \wedge \mathcal{D}$-modules.
- Moral construction: Applying K-theory to $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$ to get Mot, the category of non-commutative motives.
- Colimit preserving functors Mot $\to$ Sp are filtered-colimit preserving functors $\mathrm{Cat}^{\mathrm{ex}}_{\mathrm{stab}} \to$ Sp that split exact sequences.
- This is designed to give co-representability:

$$K(R) = \mathrm{Rhom}_{\mathrm{Mot}}(S, R).$$

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What is the motivic interpretation of THH and TC?

Co-representability result implies that the cyclotomic trace corresponds to

$$1 \in \pi_0(THH(S)) \cong \pi_0(S) \cong \mathbb{Z}$$

One answer: produce Mot_{TC} and Mot_{THH} by "applying THH or TC to $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$".

Related to Morava's Tannakian viewpoint.

Question

What is the multiplication on $TC(S)$ and $K(S)$?

Another answer given by Bloch's p-typical curves.

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p-typical curves and endomorphism K-theory

For a ring R, have truncated polynomial rings $R[x]/x^n$.

Following Betley-Schlichtkrull, we can index $\{K(R[x]/x^n)\}$ on the category $\mathbb{I}$ with objects natural numbers and maps $F_s, R_s \colon rs \to s$ (usual cyclotomic relations).

Projection maps and transfer maps (associated to $x \mapsto x^n$) assemble to form a diagram over $\mathbb{I}$.

Can form two p-typical curve limits:

$$C(R) = \lim_{\mathcal{I}} \Omega \tilde{K}(R[x]/x^n).$$

(using all maps, and only using projections)

After completion, these are respectively $TR(R)$ and $TC(R)$. (Betley-Schlichtkrull, Hesselholt, Lindenstrauss-McCarthy).

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Motivic view of all this

Related to K-theory of R-modules equipped with an endomorphism. (Joint with Gepner and Tabuada.)

Theorem

$$K\,\mathrm{End}(\mathcal{C}) = \mathrm{Rhom}_{\mathrm{Mot}}(S[t], \mathcal{C})$$

Also gives a characterization of the rational Witt vectors. (Dwyer)

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Figure: Sometimes arguments fall apart. . .

Various approaches to thinking about iterated K-theory:

For a commutative ring spectrum R, $K(R)$ is a commutative ring. $\implies$ iterate by forming $K(K(R))$.

But also can do "categorical" iterations.

1. Take K-theory of the stable category Mot (e.g., take the dualizable or compact objects). (Kontsevich)
2. Take K-theory of a suitable Waldhausen structure on the category of stable categories itself. (Toen-Vezzosi)

Conjecture

These are the same.

Iterated K-theory and the K-theory of n-categories

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Algebraic K-theory of higher n-categories

(Joint with Ayala)

First idea: recursive definition. $\implies$ Define K-theory of categories enriched in Waldhausen categories, then categories enriched in categories enriched in Waldhausen categories, etc.

Pre-theorem: Can also do all at once.

Recovers examples above when we take Mot or $\mathrm{Cat}^{\mathrm{ex}}_{\mathrm{stab}}$!

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What relationship does this bear to $K(K(S))$?

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A trace for the K-theory of higher n-categories

Perspective: should be a higher trace, related to the "iterated trace"

$$K(K(\cdots K(\mathcal{C})) \to THH(THH(\cdots THH(\mathcal{C})))$$

This arises as the "color by an object" map (generalizing the Dundas-McCarthy inclusion of objects model of the trace) into composition cohomology (Ayala-Rozenblyum).

Can take limits over finite-sheeted covers of framed n-manifolds to form something like TC.

Recovers the "covering homology" of Brun-Carlsson-Dundas, structures of Carlsson-Douglas-Dundas in the commutative case.

Once again, equivariant structure involves the norm!

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Th-th-th-that's all, folks