Compactifying string topology

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UC Berkeley

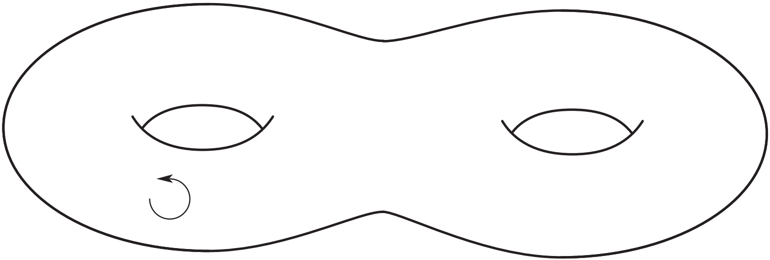
Algebraic Topology: Applications and New Developments
Stanford University, July 24, 2012

What is the algebraic topology of a manifold?

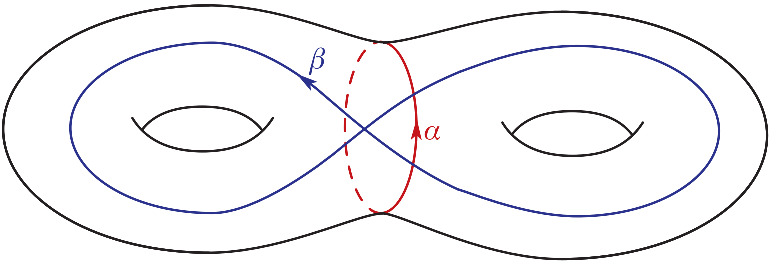
What can we say about the algebraic structure of the homology–or chains–of the free loop space of a manifold?

The Goldman Bracket

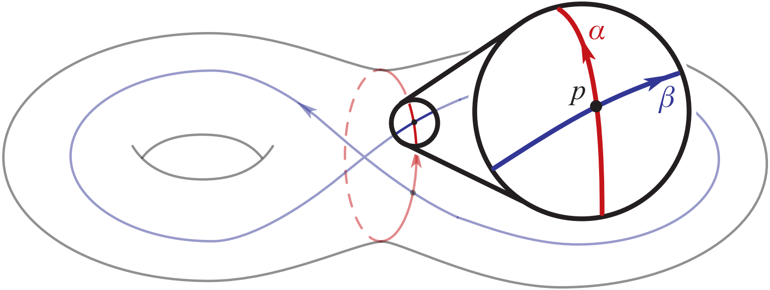
Fix an oriented surface Σ.



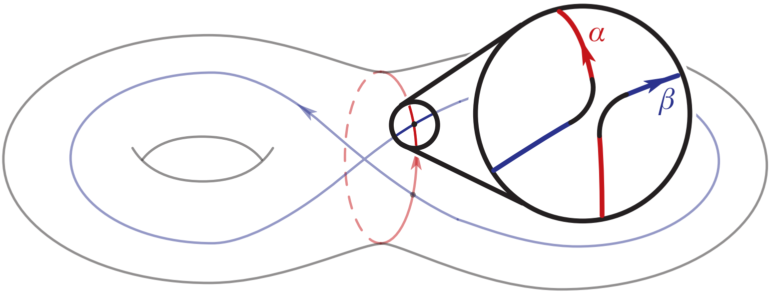
Consider two free homotopy classes α and β of closed curves on Σ.



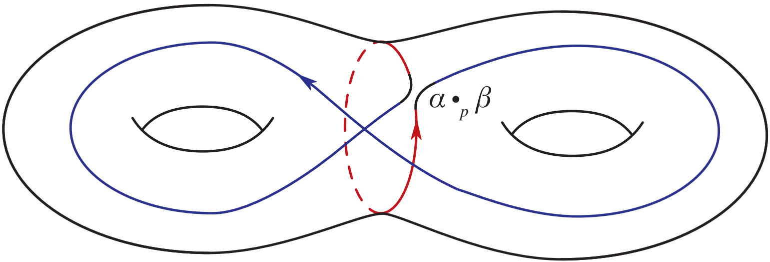
Consider representative curves that intersect one another only in transverse double points p.



Cut α and β at p and reconnect the strands in the other way that respects their orientation.



Let $\alpha \cdot_p \beta$ be the closed curve obtained by cutting and reconnecting.



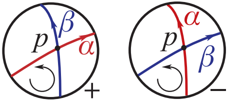
Each intersection point p of α and β gives a free homotopy class of closed curves $\alpha \cdot_p \beta$.

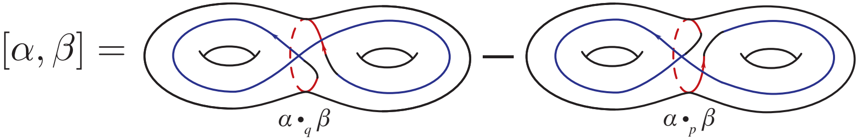
Let H be the $\mathbb{Q}$-vector space generated by the set of free homotopy classes of closed curves on Σ. (In general, H is infinite dimensional.)

Define

$$[\alpha, \beta] = \sum_{p \in \alpha \cap \beta} \pm \alpha \cdot_p \beta.$$

Signs depend on the orientation of Σ





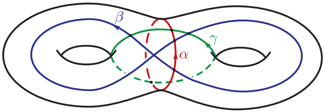
Definition (Goldman Bracket)

Extend $[\ ,\ ]$ linearly to obtain a map $[\ ,\ ] : H \otimes H \longrightarrow H$.

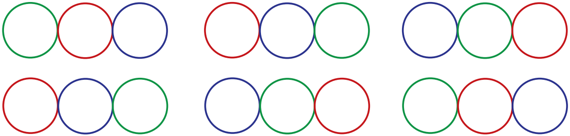
Theorem (Goldman)

The bracket is well defined and gives H the structure of a Lie algebra.

Idea of proof of Jacobi identity: terms cancel in pairs.

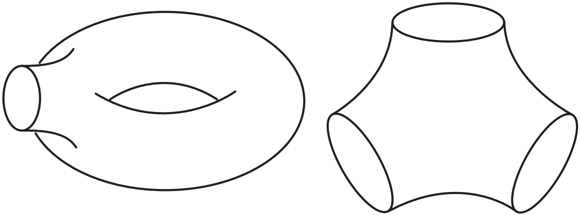


$[[\alpha, \beta], \gamma]$ $\quad$ $[[\beta, \gamma], \alpha]$ $\quad$ $[[\gamma, \alpha], \beta]$



Theorem (Gadgil)

A homotopy equivalence between compact, connected, oriented surfaces is homotopic to a homeomorphism if and only if it commutes with the Goldman bracket.



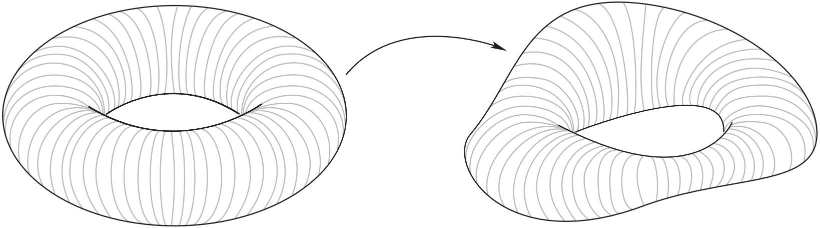
String bracket

Let M be a closed, oriented d-dimensional manifold.

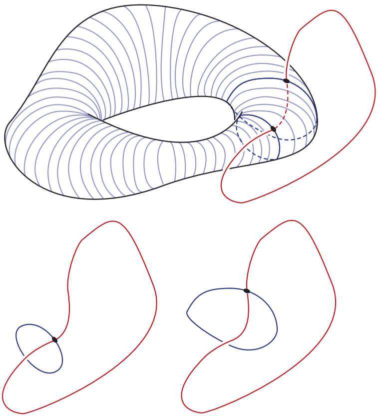
Let $d = 3$.

Let

- H_0 be the $\mathbb{Q}$-vector space generated by free homotopy classes of loops in M.
- H_1 be the $\mathbb{Q}$-vector space generated by homotopy classes of fibered tori in M.



Intersections



$$H_0 \otimes H_1 \longrightarrow H_0$$

String bracket

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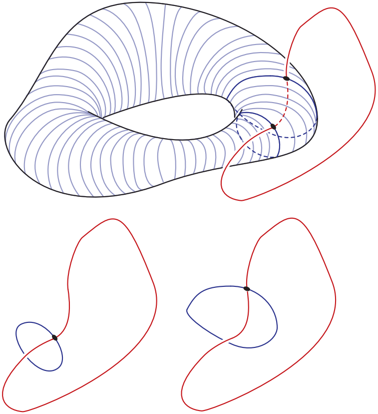

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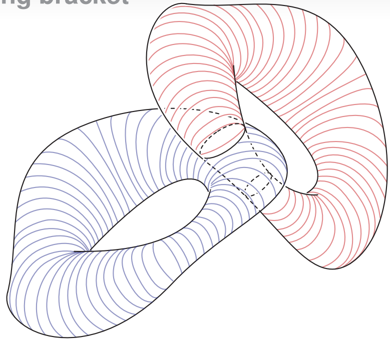

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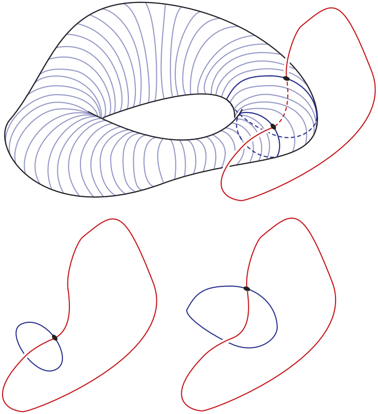

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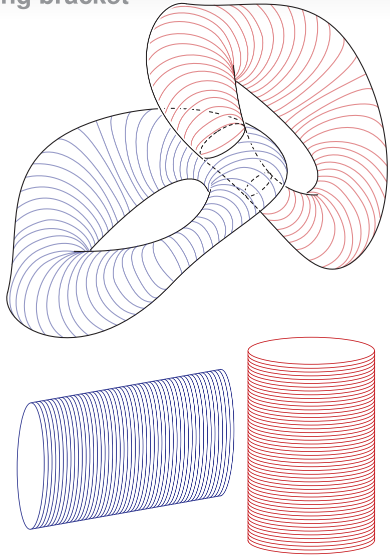

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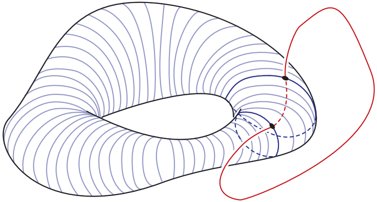

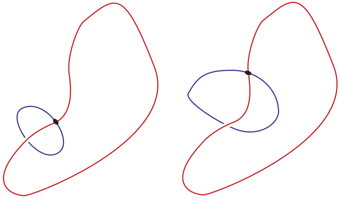

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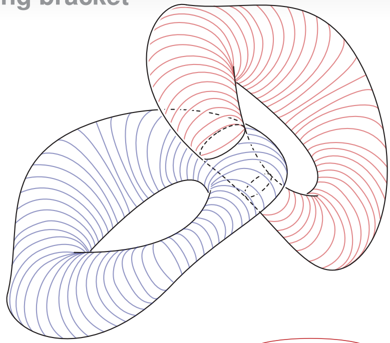

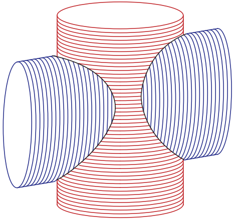

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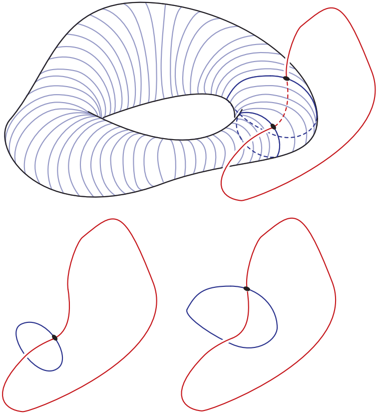

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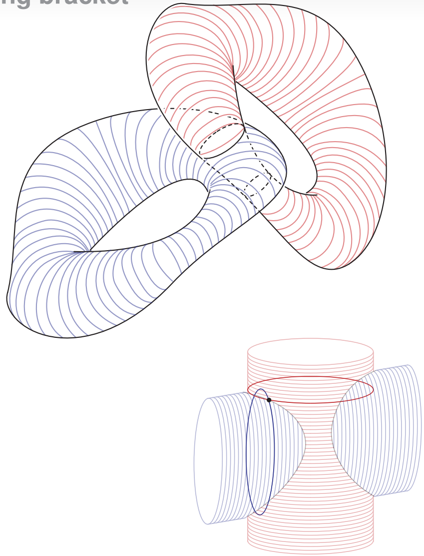

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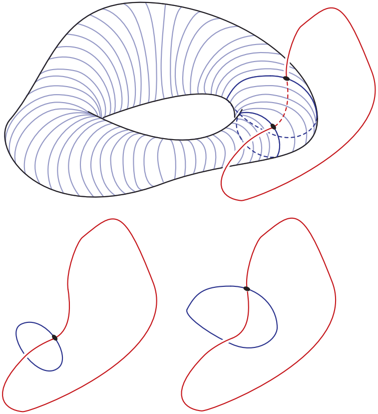

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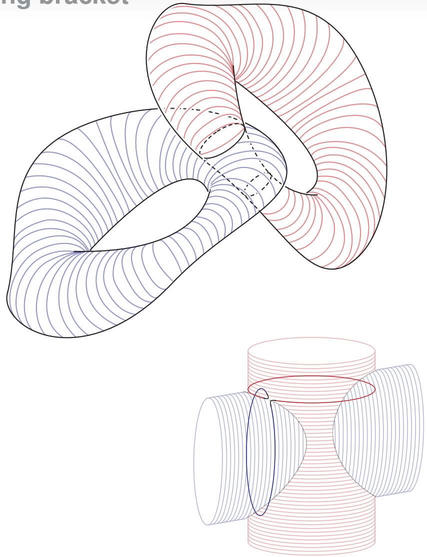

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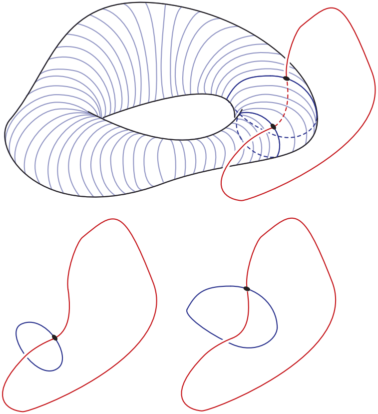

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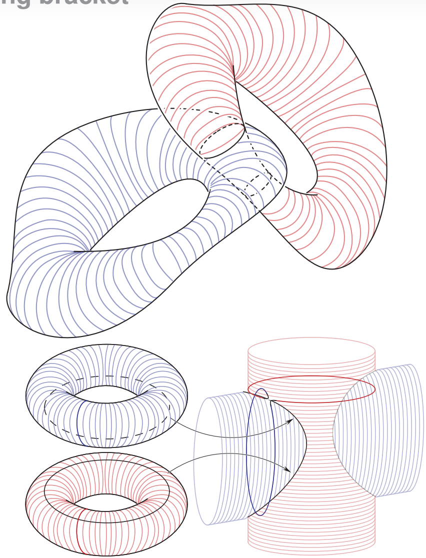

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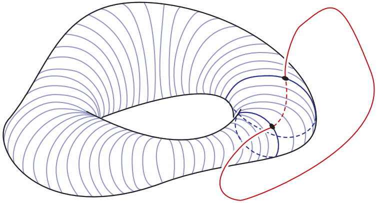

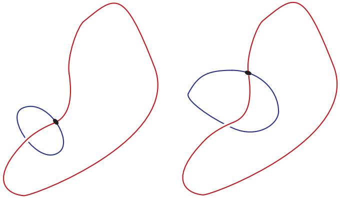

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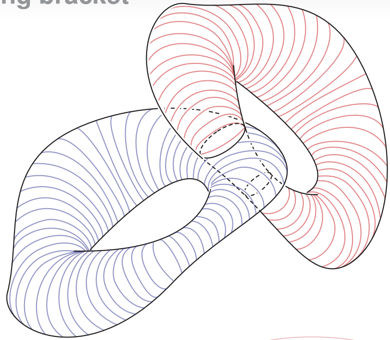

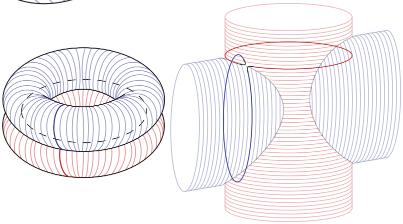

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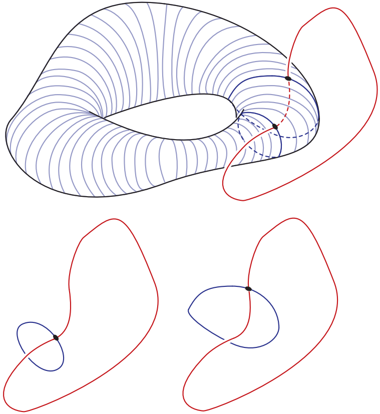

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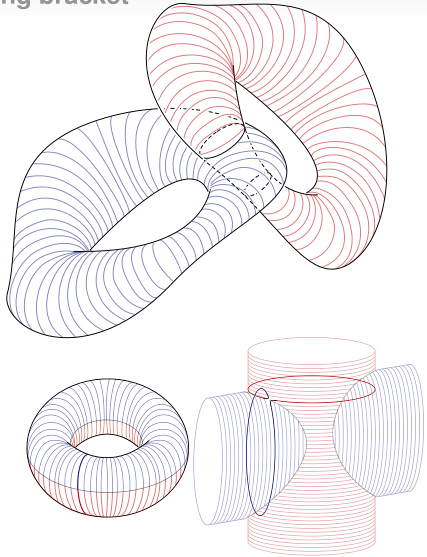

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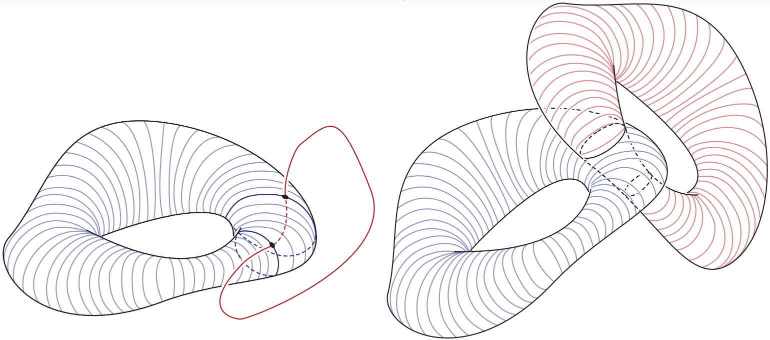
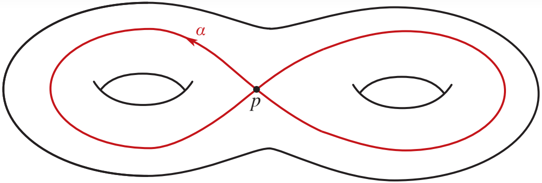


The string bracket for d-dimensional manifolds M is defined analogously.

Theorem (Chas-Sullivan)

Let M be a closed, oriented d-dimensional manifold, let $LM = Maps(S^1, M)$ be its free loop space and let $H_*^{S^1}(LM)$ be the S^1-equivariant homology of LM. Then the string bracket gives $H_*^{S^1}(LM)$ the structure of a graded Lie algebra. When $d = 2$ and $* = 0$, then the string bracket coincides with the Goldman bracket.

Turaev's cobracket



Let H' be the quotient of H by the subspace generated by nullhomotopic loops.

Theorem (Turaev)

The string bracket induces a well-defined bracket $[\ ,\ ] : H' \otimes H' \to H'$*, the cobracket* $\Delta : H' \to H' \otimes H'$ *is well defined and* $(H', [\ ,\ ], \Delta)$ *is a Lie bialgebra.*

Again, the cobracket Δ generalizes to higher dimensions.

Theorem (Chas-Sullivan)

Let $M \subset LM$ *be the subspace of constant loops. Then* $(H_*^{S^1}(LM, M), [\ ,\ ], \Delta)$ *is a graded involutive Lie bialgebra.*

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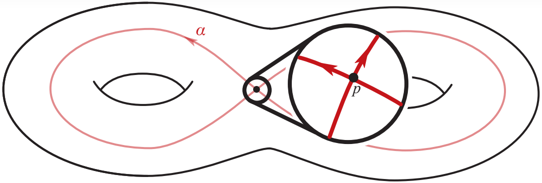

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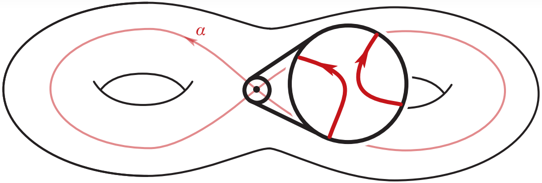

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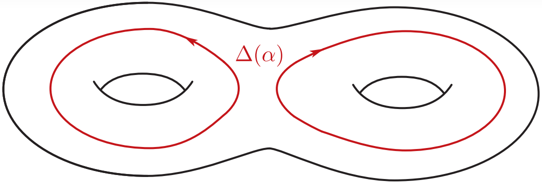

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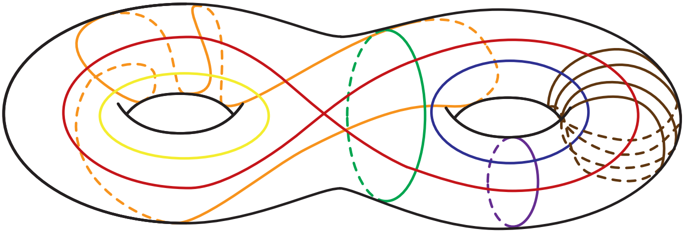
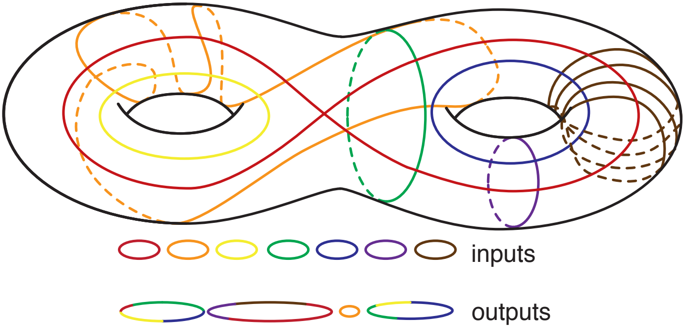


Cutting and reconnecting at intersection points yields generalized operations

$$H^{\otimes k} \longrightarrow H^{\otimes \ell}.$$

String topology operations



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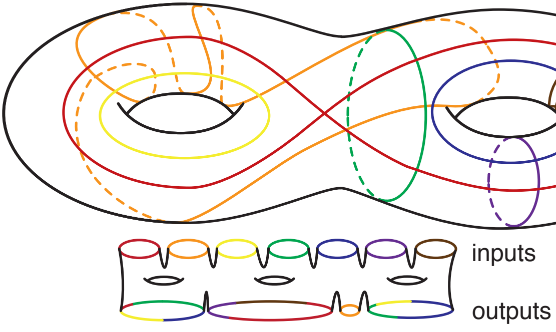

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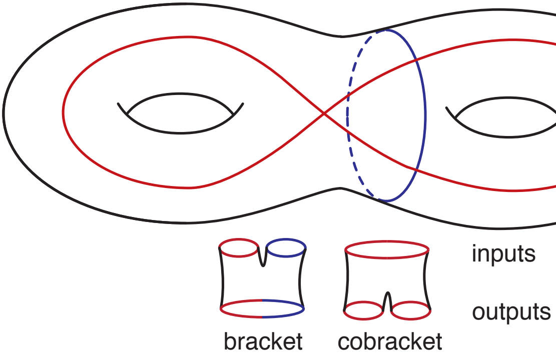

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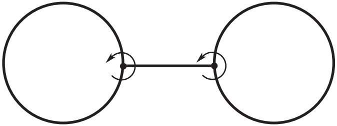
String diagrams organize more complicated intersections giving rise to k-to-ℓ operations.

Definition

A fatgraph is a graph together with a cyclic order of half-edges at each vertex.



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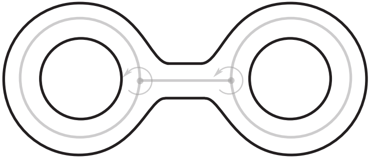

A fatgraph determines an orientable *ribbon* surface with boundary that contains the fatgraph as a deformation retract.

String Diagrams

Definition

A *string diagram* of type (g, k, ℓ) is a sequence of marked metric fatgraphs $\Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_N$, constructed inductively:

- Γ_0 is k disjoint "input" circles (each of length 1)
- Γ_{n+1} is constructed from Γ_n by adjoining a collection of metric trees (each satisfying a metric condition) along their leaves

such that Γ_N has genus g and $k + \ell$ boundary components, k of which correspond to Γ_0, the remaining ℓ are called "outputs," together with "spacing parameters" $s \in (0, 1]^{N-1}$.

Definition

A string diagram is *simple* if $N = 1$ and $\Gamma_1 - \text{edges}(\Gamma_0)$ is a forest.

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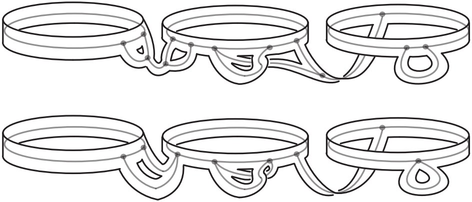

Proposition

Let $\mathcal{S}$ be the space of string diagrams, $\mathcal{SS}$ the space of simple string diagrams, and $\mathcal{C}$ be the space of chord diagrams. Then $\mathcal{S}$ is a finite cell complex, $\mathcal{SS}$ is a union of open cells, and $\mathcal{C}$ is a subcomplex.

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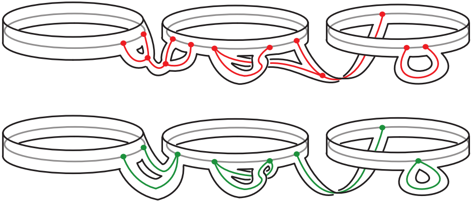

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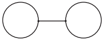
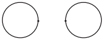
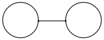
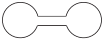
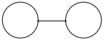
String diagrams and string topology operations

Maps( , *M*)

Maps( , *M*) $\xleftarrow{\rho_{in}}$ *Maps*( , *M*) $\xrightarrow{\rho_{out}}$ *Maps*( , *M*)

String diagrams and string topology operations

$(LM)^k \xleftarrow{\rho_{in}} Maps($  $, M) \xrightarrow{\rho_{out}} (LM)^\ell$

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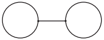
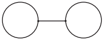
$(LM)^k \xleftarrow{\rho_{in}} Maps($  $, M) \xrightarrow{\rho_{out}} (LM)^\ell$

$(LM)^k \xleftarrow{\rho_{in}} Maps($  $, M) \xrightarrow{\rho_{out}} (LM)^\ell$

Let $\chi = 2g - 2 + k + \ell$.

Definition (Cohen-Godin)

Given $\Gamma \in \mathcal{SS}$, $(\rho_{in})_! : H_*(LM)^{\otimes k} \to H_{*-\chi d}(Maps(\Gamma, M))$ and

$$\mu_\Gamma = (\rho_{out})_* \circ (\rho_{in})_! : H_*(LM)^{\otimes k} \longrightarrow H_{*-\chi d}(LM)^{\otimes \ell}.$$

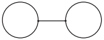
Theorem

Simple string diagrams satisfy a gluing condition and the construction respects gluing. ($H_0(\mathcal{SS})$ acts on $H_*(LM)$; the construction yields a "positive boundary" TQFT.)

Theorem (Chataur)

$H_*(\mathcal{SS})$ acts on $H_*(LM)$.

String diagrams and string topology operations

$(LM)^k \xleftarrow{\rho_{in}} Maps($  $, M) \xrightarrow{\rho_{out}} (LM)^{\ell}$

Let $\chi = 2g - 2 + k + \ell$.

Definition (P.-Rounds)

Let $C_*(\mathcal{C})$ be the cellular chains of $\mathcal{C}$ and let $C_*(LM)$ be the singular chains of LM.

$$\lambda : C_*(\mathcal{C}) \otimes C_*(LM)^{\otimes k} \longrightarrow C_{*-\chi d}(LM)^{\otimes \ell}$$

Theorem

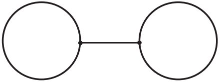
λ is a chain map.

For $\Gamma \in \mathcal{C} \cap \mathcal{SS}$, $\lambda(\Gamma, -)$ induces μ_Γ on homology.

Idea of chain-level construction

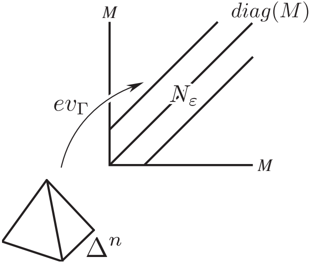
$$C_*(\mathcal{C}) \otimes C_*(LM^k) \longrightarrow C_{*-\chi d}(LM^\ell)$$

Let M be a compact, oriented, Riemannian manifold of dimension d, with injectivity radius ε.

Fix $\Gamma =$ 

Let $\sigma : \Delta^n \to LM \times LM$ be a singular simplex, $\sigma(t) : S^1 \sqcup S^1 \to M$.

Ingredients:



- Let N_ε be an ε-neighborhood of the diagonal
  $M \to M \times M$
- Representative of Thom class of diagonal
  $U \in C^d(N_\varepsilon, \partial N_\varepsilon)$
- Evaluation map $ev_\Gamma : \Delta^n \to M \times M$,
  evaluate $\sigma(t)$ at chord endpoints of Γ.
- Let $S_\varepsilon = ev_\Gamma^{-1}(N_\varepsilon)$.

- N_ε: ε-neighborhood of diagonal
- $U \in C^d(N_\varepsilon, \partial N_\varepsilon)$
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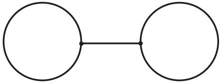

Step 1:

$$C_*(\Delta^n) \xrightarrow{j} C_*(\Delta^n, \Delta^n - S_\varepsilon) \xrightarrow{s} C_*(S_\varepsilon, \partial S_\varepsilon) \xrightarrow{\cap ev^*(U)} C_{*-d}(S_\varepsilon)$$

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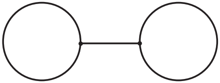

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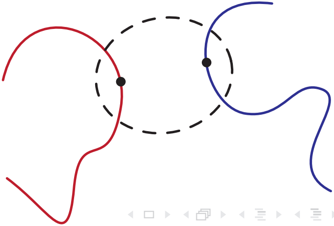
Step 2:

If $t \in S_\varepsilon$, then $\sigma(t) : S^1 \sqcup S^1 \longrightarrow M$ sends chord endpoints of Γ into an ε-ball in M.

$$S_\varepsilon \xrightarrow{\heartsuit} Maps(\Gamma, M)$$



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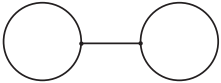

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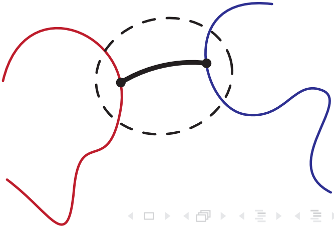
Step 2:

If $t \in S_\varepsilon$, then $\sigma(t) : S^1 \sqcup S^1 \to M$ may be extended to $\Gamma \to M$: map chord to geodesic segment

$$S_\varepsilon \xrightarrow{\heartsuit} Maps(\Gamma, M)$$



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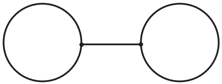

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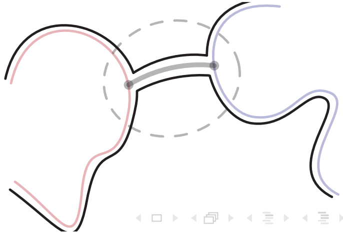

Step 3:

Restrict to outputs.

$$Maps(\Gamma, M) \xrightarrow{\rho_{out}} LM$$

Step 1:

$$C_*(\Delta^n) \xrightarrow{j} C_*(\Delta^n, \Delta^n - S_\varepsilon) \xrightarrow{s} C_*(S_\varepsilon, \partial S_\varepsilon) \xrightarrow{\cap ev^*(U)} C_{*-d}(S_\varepsilon)$$

Step 2:

$$S_\varepsilon \xrightarrow{\heartsuit} Maps(\Gamma, M) \rightsquigarrow C_*(S_\varepsilon) \xrightarrow{\heartsuit_*} C_*(Maps(\Gamma, M))$$

Step 3:

$$Maps(\Gamma, M) \xrightarrow{\rho_{out}} LM \rightsquigarrow C_*(Maps(\Gamma, M)) \xrightarrow{(\rho_{out})_*} C_*(LM)$$

Definition

Define $\lambda(\Gamma, \sigma) \in C_{*-d}(LM)$ as

$$((\rho_{out})_* \circ \heartsuit_* \circ \cap ev^*(U) \circ s \circ j)([\Delta^n])$$

where $[\Delta^n]$ is the fundamental chain of Δ^n. Extend linearly to

$$\lambda(\Gamma, -) : C_*(LM \times LM) \to C_{*-d}(LM).$$

The construction generalizes $C_*(\mathcal{C}) \otimes C_*(LM^k) \longrightarrow C_{*-\chi d}(LM^\ell)$.

The space of metric fatgraphs is a model for the moduli space of Riemann surfaces $\mathcal{M}$. Therefore $\mathcal{SS} \hookrightarrow \mathcal{M}$. This inclusion is not a homotopy equivalence in general.

Theorem (Godin, Kupers)

$H_*(\mathcal{M})$ acts on $H_*(LM)$.

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Theorem (Godin, Kupers)

$H_*(\mathcal{M})$ acts on $H_*(LM)$.

Theorem (Drummond-Cole-P.-Rounds, in progress)

$C_*(\mathcal{S})$ acts on $C_*(LM)$.

Moduli space and string topology operations

Conjecture (To-do list)

- *The space of string diagrams $\mathcal{S}$ is homeomorphic to a compactification of moduli space $\mathcal{M}$ with the homotopy type of $\mathcal{M}$.*
- *The action of $C_*(\mathcal{S})$ acts on $C_*(LM)$ induces known action of $H_*(\mathcal{M})$ on $H_*(LM)$.*
- *The chain map $C_*(\mathcal{S}) \otimes C_*(LM)^{\otimes k} \to C_{*-\chi d}(LM)^{\otimes \ell}$ factors through $C_*(\mathcal{S}) \otimes C_*(LM)^{\otimes k} \to C_*(\mathcal{S}/_\sim) \otimes C_*(LM)^{\otimes k}$ induced by $\mathcal{S} \to \mathcal{S}/_\sim$, quotient by an equivalence relation.*
- *The quotient space $\mathcal{S}/_\sim$ is homotopy equivalent to Bödigheimer's harmonic compactification $\overline{\mathcal{M}}$ of $\mathcal{M}$.*
- *Relations among chain-level operations–algebraic structure of $C_*(LM)$– which are not evident in homology-level construction are revealed by action of $C_*(\overline{\mathcal{M}})$.*
- *Formulate the full open-closed theory.*

Basu has used a (different) string topology construction to define a coalgebra structure which is *not* a homotopy invariant.

Question

To what extent is the algebraic structure of $C_*(LM)$ an invariant of the homotopy type of M?

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To what extent is the algebraic structure of $C_*(LM)$ an invariant of the homotopy type of M?

Tamanoi has shown that homology classes in the image of the stabilization map

$$H_*(\mathcal{M}_{g,k+\ell}) \longrightarrow H_*(\mathcal{M}_{g+1,k+\ell})$$

act trivially on $H_*(LM)$.

Question

Is there a manifold M and a homology class in the image of $H_*(\mathcal{M}) \longrightarrow H_*(\overline{\mathcal{M}})$ for which the associated string topology operation on is nontrivial?

Thank you!