Smooth maps to the plane and Pontryagin classes

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(Jt work with Rui Reis)

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Pontryagin classes

TOP$(n) = \{$homeom. $\mathbb{R}^n \to \mathbb{R}^n\}$
O $= \bigcup$ O(n) , TOP $= \bigcup$ TOP(n)

FACT: BO $\simeq_{\mathbb{Q}}$ BTOP (Thom-Novikov, Kirby-Siebenmann), so $p_m \in H^{4m}(B\text{TOP};\mathbb{Q})$ def'd

FACT: $B\text{O}(n) \hookrightarrow B\text{TOP}(n)$ not $\simeq_{\mathbb{Q}}$ for odd $n \gg 0$ (not expected $\simeq_{\mathbb{Q}}$ for even $n \gg 0$)

DEFINE Euler $e \in H^{2m}(BS\text{TOP}(2m);\mathbb{Z})$ via Thom isom and square of Thom class.

Theorem A. $e^2 = p_m \in H^{4m}(BS\text{TOP}(2m);\mathbb{Q})$.

REMARK: Hard. Analog for $BS\text{O}(2m)$ with $\mathbb{Z}$ coeff known and easy.

REMARK. Implies $0 = p_m \in H^{4m}(B\text{TOP}(n);\mathbb{Q})$ when $n < 2m$.

We need $B\mathrm{TOP}(n)$

Smoothing thy (esp Morlet) gives ho cartesian

$$\begin{array}{ccc}
\mathrm{Diff}_\partial(D^n) & \xrightarrow{\nabla} & \Omega^n \mathrm{GL}_n(\mathbb{R}) \\
\downarrow & & \downarrow \\
\mathrm{Homeo}_\partial(D^n) & \xrightarrow{\nabla} & \Omega^n \mathrm{TOP}(n)
\end{array}$$

where horiz maps given by taking "derivatives". Since $\mathrm{Homeo}_\partial(D^n) \simeq \star$ by Alexander, get

$$\mathrm{Diff}_\partial(D^n) \simeq \Omega^{n+1}(\mathrm{TOP}(n)/\mathrm{O}(n))$$

CONVERSELY can use $\mathrm{Diff}_\partial(D^n)$ to learn about $B\mathrm{TOP}(n)$. Concordance thy: explore $\mathrm{Diff}_\partial(D^n)$ using ho fib sequences

$$\Omega\mathrm{Diff}_\partial(D^n) \longrightarrow \mathrm{Diff}_\partial(D^n \times D^1) \longrightarrow \mathscr{R}(n,1)$$

where $\mathscr{R}(n,1)$ space of smooth *regular*

$$f \colon D^n \times D^1 \to D^1$$

such that $f|\partial = \mathrm{proj}$. Regular means everywhere nonsing. Note O(1)-symmetry.

Diffeos and regular maps. Modify above: ho fibration sequences

$$\Omega^2 \mathsf{Diff}_\partial(D^n) \longrightarrow \mathsf{Diff}_\partial(D^n \times D^2) \longrightarrow \mathscr{R}(n,2)$$

where $\mathscr{R}(n,2)$ space of smooth regular

$$f\colon D^n \times D^2 \to D^2$$

such that $f|\partial = \mathrm{proj}$.
There is map

$$\begin{array}{rcl} \nabla\colon \mathscr{R}(n,2) & \longrightarrow & \Omega^{n+2}\left(\dfrac{\mathsf{GL}_{n+2}(\mathbb{R})}{\mathsf{GL}_n(\mathbb{R})}\right) \\ f & \mapsto & df \end{array}$$

Note O(2)-symmetry. Restrict to $S^1 = S\mathrm{O}(2)$. When n even, target rat homotopy equivalent to $K(\mathbb{Z}, n-3)$: therefore

$$[\nabla] \in H^{n-3}_{S^1}(\mathscr{R}(n,2);\mathbb{Q}) \ .$$

Theorem B. $[\nabla] = 0 \in H^{n-3}_{S^1}(\mathscr{R}(n,2);\mathbb{Q})$ for even $n \geq 4$.

Easily $\Leftrightarrow$ to Thm A re e, p_m. (See next slide.)

Thm A is cor of Thm B
Category $\mathscr{J}$: objects f d real vector spaces V with $\langle\,,\rangle$, morphisms lin maps preserving $\langle\,,\rangle$. Cts functors on $\mathscr{J}$

$$E : V \mapsto B\mathrm{O}(V), \quad F : V \mapsto B\mathrm{TOP}(V)\ .$$

Statement C: *The incl $E \to F$ admits nat rat left homotopy inverse* (in shape of $F \to E'$ such that compos $E \to E'$ is $\simeq_{\mathbb{Q}}$).

Show: Thm $B \Rightarrow C \Rightarrow$ Thm A.

$C \Rightarrow$ Thm A: such $F \to E'$ must take e, p_m classes in E' to e, p_m classes in F.
Thm B $\Rightarrow C$: use orthog calculus (for cts functors from $\mathscr{J}$ to spaces). Since E rat polyn of deg 2, governed by $E(\mathbb{R}^\infty) = B\mathrm{O}$ and first/second rate of change at ∞, which are spectra with act of O(1), O(2) resp. Turns out enough to have rat splittings for incl $E(\mathbb{R}^\infty) \to F(\mathbb{R}^\infty)$ and maps betw first/second rates of change at ∞ induced by $E \to F$. First two splittings: obvious resp known, last: encoded in Thm B. Use smoothing thy descr of $\mathscr{R}(n,2)$ to decode.

Strategy for proving Thm B. Imitate Cerf's approach to $\mathscr{R}(n,1)$. This uses

$$\mathscr{R}(n,1) \subset \mathscr{W}(n,1)$$

where $\mathscr{W}(n,1)$ space of smooth

$$f\colon D^n \times D^1 \to \mathbb{R}^1$$

such that $f|\partial =$ proj and the crit pts of f are no worse than Morse or birth-death. He proves
– $\mathscr{W}(n,1)$ connected (easy)
– incl $\mathscr{R}(n,1)) \to \mathscr{W}(n,1)$ is 1-conn (hard)
and concludes that $\mathscr{R}(n,1)$ connected (has more general statement).
So we introduce $\mathscr{W}(n,2) \supset \mathscr{R}(n,2)$ and show

(i) inclusion $\mathscr{R}(n,2) \to \mathscr{W}(n,2)$ induces zero homomorphism in $\tilde{H}^{n-3}_{S^1}(-,\mathbb{Q})$

(ii) cocycle ∇ in formul of Thm B extends to cocycle on $\mathscr{W}(n,2)$ with enough S^1-invariance.

Proof of (i) uses h-principle machinery and mfd calculus. Proof of (ii) hands-on geometric work. Conclusion $[\nabla] = 0$ as stated in Thm B.

Details on $\mathscr{W}(n,2)$
DEF. $\mathscr{W}(n,2)$ space of smooth

$$f\colon D^n \times D^2 \to \mathbb{R}^2$$

such that $f|\partial =$ proj and the crit pts of f are no worse than fold, cusp, lips, beak-to-beak or swallowtail, and *the critical values of f are subject to some add conditions* (see below).

REMARK. The sing types listed above all have differential of rk 1 at crit pts. Classific of most common rk 1 sing germs $(\mathbb{R}^{n+2}, 0) \to (\mathbb{R}^2, 0)$ mostly indep of n. Fold and cusp are Whitney's sing, while lips, beak-to-beak and swallowtail are three ways in which two cusps can collide and annihilate each other.

REMARK. Conditions on critical values: only finitely many crit pts for each crit value; no more than one non-fold among these; if a non-fold among them, then all make different singular directions in target; if all are folds, then allow at most two of them to make a first order tangency in target.