Artin's map in stable homology

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Abstract: Using a recent theorem of Galatius [G] we identify the map on stable homology induced by Artin's injection of the braid group β_n into the automorphism group of the free group $\mathrm{Aut}F_n$.

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1. Definitions and results.

Let β_n be the braid group on n strings and $\mathrm{Aut}F_n$ the automorphism group of the free group F_n on n generators $x_1, \dots, x_n$. Artin [A] identified β_n as a subgroup of $\mathrm{Aut}F_n$ as follows. Let $\sigma_i \in \beta_n$ denote a standard generator, the braid which crosses the i-th over the $(i+1)$-st string. Artin's map

$$\phi : \beta_n \longrightarrow \mathrm{Aut}F_n$$

is defined by taking σ_i to the automorphism

$$\phi(\sigma_i) : x_j \mapsto \begin{cases} x_j & \text{if } j \neq i, i+1 \\ x_{i+1} & \text{if } j = i \\ x_{i+1}^{-1} x_i x_{i+1} & \text{if } j = i+1. \end{cases}$$

ϕ extends to a map from $\beta_\infty := \lim_{n\to\infty} \beta_n$ to $\mathrm{Aut}F_\infty := \lim_{n\to\infty} \mathrm{Aut}F_n$. We will describe this map of stable group on homology.

Theorem 1. *$\phi_* : H_*(\beta_\infty; k) \longrightarrow H_*(AutF_\infty; k)$ is trivial when $k = \mathbb{Q}$ or $k = \mathbb{Z}/p\mathbb{Z}$ for any odd prime p. It induces an injection on*

$$H_*(\beta_\infty; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}\,[\, x_i \mid \, deg\ (x_i) = 2^i - 1].$$

Theorem 1 is a corollary of the stronger, homotopy theoretic Theorem 3 below. Another way to state our results in more algebraic terms is in comparision to another homomorphism defined as follows. An element of the symmetric group Σ_n acts naturally by permutations of the generators on F_n. This defines an embedding $\Sigma_n \subset \mathrm{Aut}F_n$. Precomposition with the natural surjection from the braid group to the symmetric group defines the homomorphism

$$\pi : \beta_n \longrightarrow \Sigma_n \subset \mathrm{Aut}F_n.$$

As ϕ, π also commutes with limits and extends to a map of stable groups. Though these maps are very different, they induce the same map on homology.

Theorem 2. $\phi_* = \pi_* : H_*(\beta_\infty;\mathbb{Z}) \longrightarrow H_*(AutF_\infty;\mathbb{Z})$.

Remark 1: Unlike for (co)homology with trivial coefficients, as considered in this note, for (co)homology with twisted coefficients ϕ_* can be non-trivial even rationally, cf. for example the recent work of Kawazumi [K].

Remark 2: There are many other homomorphisms from β_n to $\mathrm{Aut}F_n$. Those that factor through the mapping class group are likely to be trivial in homology. Indeed, in [ST] we show that many algebraically and geometrically defined homomorphisms from the braid group to the mapping class group are homologically trivial, and hence so will the composition to $\mathrm{Aut}F_n$.

2. Translation into homotopy.

Juxtaposition of braids and disjoint union of sets respectively induce natural monoidal structures on the disjoint union of the classifying spaces $\coprod_{n\geq 0} B\beta_n$ and $\coprod_{n\geq 0} B\Sigma_n$. Their group completions can be identified respectively as

$$\mathbb{Z} \times B\beta_\infty^+ \simeq \Omega^2 S^2$$
$$\mathbb{Z} \times B\Sigma_\infty^+ \simeq \Omega^\infty S^\infty.$$

Here "+" denotes Quillen's plus construction with respect to the maximal perfect subgroup of the fundamental group; $\Omega^n S^n$ is the space of based maps from the n-sphere to itself and $\Omega^\infty S^\infty := \lim_{n\to\infty} \Omega^n S^n$. Recently, Galatius [G] was able to prove an analogue of the these results for the automorphism groups of free group:

$$\mathbb{Z} \times B\mathrm{Aut}F_\infty^+ \simeq \Omega^\infty S^\infty.$$

Furthermore, the inclusion $\Sigma_n \to \mathrm{Aut}F_n$ induces up to homotopy the identity map of $\Omega^\infty S^\infty$, cf. also [H]. It is also well-known, cf. [CLM], [S], that the surjection $\beta_n \to \Sigma_n$ induces on group completions up to homotopy the inclusion map $\Omega^2 S^2 \to \Omega^\infty S^\infty$. As the plus construction does not change the homology of the space Theorem 2 is therefore equivalent to the following.

Theorem 3. *On group completions ϕ induces up to homotopy the natural inclusion map $\Omega^2 S^2 \to \Omega^\infty S^\infty$.*

Proof of Theorem 1. Rationally the homology of Σ_n and $\Omega^\infty S^\infty$ is trivial, and hence ϕ_* is trivial on rational homology. Recall, F. Cohen in [CLM] describes the homology of the braid group with $\mathbb{Z}/p\mathbb{Z}$ coefficients for every prime p in terms of a one-dimensional generator $x_1 \in H_1(\beta_\infty; \mathbb{Z}/p\mathbb{Z})$ and powers of homology operation applied to x_1. Maps of double loop spaces commute with these homology operations. Hence, by Theorem 3, ϕ_* commutes with them so that its image is determined by its value on x_1. But

$$H_1(\mathrm{Aut}_\infty; \mathbb{Z}) = H_1(\Sigma_\infty; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z},$$

and hence it follows that ϕ_* is zero in all positive dimensions for all odd p. The class $x_1 \in H_1(\beta_\infty; \mathbb{Z}) = \pi_1 \Omega^2 S^2 = \mathbb{Z}$ corresponds to the Hopf map $S^3 \to S^2$ which maps under the inclusion map $\Omega^2 S^2 \to \Omega^\infty S^\infty$ to the non-zero element in the first homology. It is also well-known that the homology operations act freely on the homology of $\Omega^\infty S^\infty$, so that for $p = 2$ the map ϕ_* is an injection.

3. Proof of Theorem 3.

From the definition of $\phi(\sigma_i)$ it is clear that ϕ acts on the abelianisation of F_n by permutation of the generators. Hence, we have the following commutative diagram of groups:

$$\begin{array}{ccc} \beta_n & \xrightarrow{\phi} & \mathrm{Aut}F_n \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle L}\big\downarrow \\ \Sigma_n & \longrightarrow & \mathrm{GL}_n\mathbb{Z}. \end{array}$$

For a based topological space X, let $\mathcal{HE}(X)$ denote the topological monoid of its based homotopy equivalences. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Aut}F_n & \xleftarrow{\pi_0} & \mathcal{HE}(\bigvee_n S^1) \\ {\scriptstyle L}\big\downarrow & & \big\downarrow \\ \mathrm{GL}_n\mathbb{Z} & \xleftarrow{\pi_0} & \lim_k \mathcal{HE}(\bigvee_n S^k). \end{array}$$

The horizontal arrows π_0 are defined by taking connected components, and the top one is well-known to be a homotopy equivalence. Furthermore, Σ_n acts naturally on $\bigvee_n S^k$ by permutation of the summands in the wedge product. Hence the map $\Sigma_n \to \mathrm{GL}_n(\mathbb{Z})$ lifts to $\mathcal{HE}(\bigvee_n S^k)$. Thus on classifying spaces we yield the commutative diagram:

$$\begin{array}{ccc} B\beta_n & \xrightarrow{\phi} & B\mathrm{Aut}F_n \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow \\ B\Sigma_n & \longrightarrow & B\lim_k \mathcal{HE}(\bigvee_n S^k) \end{array}$$

The union over all $n \geq 0$ for each of the four spaces in the above diagram is a monoid, and all maps commute with the monoidal product. After group completion, we thus have:

$$\begin{array}{ccc} \mathbb{Z} \times B\beta_\infty^+ & \xrightarrow{\phi} & \mathbb{Z} \times B\mathrm{Aut}F_\infty^+ \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow \\ \mathbb{Z} \times B\Sigma_\infty^+ & \longrightarrow & \mathbb{Z} \times B\lim_k \mathcal{HE}(\bigvee_\infty S^k)^+. \end{array}$$

The space in the bottom right corner is Waldhausen's K-theory of a point, $A(*)$, and the bottom horizontal map is split by his trace map $tr : A(*) \to \Omega^\infty S^\infty$, cf. [W]. By Galatius' result [G] quoted above,

$$\mathbb{Z} \times B\mathrm{Aut}F_\infty^+ \longrightarrow A(*) \xrightarrow{tr} \Omega^\infty S^\infty$$

is a homotopy equivalence. Our final commutative diagram implies Theorem 3 (and Theorem 1) immediately:

$$\begin{array}{ccc} \mathbb{Z} \times B\beta_\infty^+ & \xrightarrow{\phi} & \mathbb{Z} \times \mathrm{Aut}F_\infty^+ \\ {\scriptstyle\pi}\downarrow & & \simeq\downarrow \\ \mathbb{Z} \times B\Sigma_\infty^+ & \xrightarrow{\simeq} & \Omega^\infty S^\infty. \end{array}$$

Remark 3: The injection ϕ defines a braided monoidal structure on the monoid $\coprod_{n\geq 0} B\mathrm{Aut}F_n$ in the sense of [F], cf. also [SW]. Hence it induces a map of double loop spaces on group completions. Any double loop map from $\Omega^2 S^2$ is determined by its image on S^0. However, apriori, it is not clear whether the induced double loop space structure on the group completion of $\coprod_{n\geq 0} B\mathrm{Aut}F_n$ is homotopic to the usual one on $\Omega^\infty S^\infty$. This does therefore not lead to an alternative proof of Theorem 3. On the other hand, Theorem 3 implies that the two double loop space structures on $\Omega^\infty S^\infty$ are indeed homotopic.

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