Algebraic Topology
Michaelmas 2013
Question Sheet 5

Cell complexes, cellular homology, equivalence of homology theories, cochain complex and cohomology groups.

1. Express the following spaces as cell complexes and prove that, in each case, you have used the minimum number of cells: (i) $S^1 \vee S^1$; (ii) S^n; (iii) $\mathbb{R}P^n/RP^m$ for $n > m$.

2. The Euler characteristic $\chi(X)$ of a finite cell complex X is $\Sigma_{n\geq 0}(-1)^n c_n$ where c_n is the number of n-cells. Let Y be another finite cell complex. Describe a cell structure on $X \times Y$ and hence prove that $\chi(X \times Y) = \chi(X)\chi(Y)$. Deduce that $\chi(S^1 \times X) = 0$ for all X. How would you prove that $\chi(X)$ is independent of the cell decomposition of X, and indeed only depends on the homotopy type of X?

3. In this question let $H_*^{\triangle}$ denote simplicial and H_* denote singular homology of $\triangle$-complexes. Let $T : H_*^{\triangle} \to H_*$ denote the map between them induced by taking each n-simplex to its characteristic map. (i) Define relative homology for $H_*^{\triangle}$ and derive the corresponding long exact homology sequence. (ii) For a finite $\triangle$-complex X, let X^k denote the union of all l-simplices for $l \leq k$. Compute the relative homology of (X^k, X^{k-1}) and deduce that T induces an isomorphism in homology for this pair. (iii) Use induction on k and the Five Lemma to prove that T induces an isomorphism on homology for X.

4. Let $\phi \in C^k(X)$ and $\psi \in C^l(X)$ be cochains of a space X in dimension k and l. Prove that $\partial^*(\phi \cup \psi) = \partial^*\phi \cup \psi + (-1)^k \phi \cup \partial^*\psi$.

5. Give an explicit singular cocycle in $C^1(S^1)$ that represents a generator for $H^1(S^1)$.