Algebraic Topology
Miachaelmas 2013
Question Sheet 6

Künneth Formula, cup products, manifolds, Poincaré duality

1. Compute the homology and cohomology of $S^2 \times \mathbb{R}P^2$ and $S^1 \times F_g$ where F_g is an oriented surface of genus g.

2. Identify $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z}$ as a cyclic group for all $n, m \in \mathbb{N}$.

3. Consider the torus T as a $\triangle$-complex with two 2-simplices. Determine its simplicial homology and cohomology giving generators for each group. Compute the cup products in cohomology.

4. Using cup products, show that $X = S^2 \vee S^1 \vee S^1$ is not homotopic to T (even though they have the same homology).

5. Let M be a closed, connected, orientable manifold of dimension n.

   (i) Show that $H_{n-1}M$ is torsion free.

   (ii) Show that the rank of $H_{n-k}M$ is equal to that of H_kM.

   (iii) Show that the Euler characteristic of any odd dimensional manifold is zero.

6. For a map $f : M \to N$ between connected, orientable, closed manifolds of dimension n with fundamental classes $[M]$ and $[N]$, the degree of f is defined to be the integer $deg(f)$ such that $f_*([M]) = deg(f)[N]$. (Note the sign of the degree depends on the choice of fundamental classes.) Show that for any closed, connected, orientable manifold M of dimension n there is a degree k map $M \to S^n$ for any integer k. Show that every map $S^2 \to T$ has degree zero. [*Hint: Use cup products.*]