# Cohomology theories

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## 1. Introduction

The origins of cohomology theory are found in topology and algebra at the beginning of the last century but since then it has become a tool of nearly every branch of mathematics. It's a way of life! Naturally, this article can only give a glimpse at the rich subject. We take here the point of view of algebraic topology and discuss only the cohomology of spaces.

Cohomology reflects the global properties of a manifold, or more generally of a topological space. It has two crucial properties: it only depends on the homotopy type of the space and is determined by local data. The latter property makes it in general computable.

To illustrate the interplay between the local and global structure, consider the Euler characteristic of a compact manifold; as will be explained below, cohomology is a refinement of the Euler characteristic. For simplicity, assume that the manifold $M$ is a surface and that we have chosen a way of dividing the surface into triangles. The Euler characteristic is then defined to be

$$\chi(M) = F - E + V$$

where $F$ denotes the number of faces, $E$ the number of edges, and $V$ the number of vertices in the triangulation. Remarkably, this number does not depend on the triangulation. Yet, this simple, easy to compute number can already distinguish the different types of closed, oriented surfaces: for the sphere we have $\chi = 2$, the torus $\chi = 0$, and in general for any surface $M_g$ of genus $g$

$$\chi(M_g) = 2 - 2g.$$

The Euler characteristic also tells us something about the geometry and analysis of the manifold. For example, the total curvature of a surface is equal to its Euler characteristic. This is the *Gauss-Bonnet Theorem* and an analogous result holds in higher dimensions. Another striking result is the *Poincaré-Hopf Theorem* which equates the Euler characteristic with the total index of a vector field and thus gives strong restrictions on what kind of vector fields can exist on a manifold. This interplay between global analysis and topology has been one of the most exciting and fruitful research areas and is most powerfully expressed in the celebrated *Atiyah-Singer Index Theorem* which determines the analytic index of the Dirac operator on a spin manifold in terms of cohomology classes.

## 2. Chain complexes and Homology

There are several different geometric definitions of the cohomology of a topological space. All share some basic algebraic structure which we will explain first.

A *chain complex* $(C_*, \partial_*)$

$$\ldots C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \ldots \xrightarrow{\partial_1} C_0 \tag{2.1}$$

is a collection of vector spaces (or $R$-modules more generally) $C_i, i \geq 0$, and linear maps ($R$-module maps) $\partial_i : C_i \to C_{i-1}$ with the property that for all $i$

$$\partial_i \circ \partial_{i+1} = 0. \tag{2.2}$$

The scalar fields one tends to consider are the rationals $\mathbb{Q}$, reals $\mathbb{R}$, complex numbers $\mathbb{C}$, or a primary field $\mathbb{Z}_p$, while the most important ring $R$ is the ring of integers $\mathbb{Z}$ though we will also consider localisations such as $\mathbb{Z}[\frac{1}{p}]$ which has the effect of suppressing any $p$-primary torsion information.

Of particular interest are the elements in $C_i$ that are mapped to zero by $\partial_i$, the $i$-dimensional *cycles*, and those that are in the image of $\partial_{i+1}$, the $i$-dimensional *boundaries*. Because of (2.2), every boundary is a cycle, and we may define the quotient vector space ($R$-module), the $i$-dimensional *homology*,

$$H_i(C_*; \partial_*) := \frac{\ker \partial_i}{\mathrm{im}\ \partial_{i+1}}. \tag{2.3}$$

$(C_*, \partial_*)$ is *exact* if all its cycles are boundaries. Homology thus measures to what extent the sequence (2.1) fails to be exact.

**2.1. Simplicial homology:** A triangulation of a surface gives rise to its *simplicial* chain complex: $C_2, C_1, C_0$ are the free abelian groups generated by the set of faces, edges and vertices respectively; $C_i = \{0\}$ for $i \geq 3$. The map $\partial_2$ assigns to a triangle the sum of its edges; $\partial_1$ maps an edge to the sum of its endpoints. If we are working with $\mathbb{Z}_2$ coefficients, this defines for us a chain complex as (2.2) is clearly satisfied; in general one needs to keep track of the orientations of the triangles and edges and take sums with appropriate signs, cf. (2.6) below. An easy calculation shows that for an oriented, closed surface $M_g$ of genus $g$ we have

(2.4)

$$H_0(M_g; \mathbb{Z}) = \mathbb{Z},\ H_1(M_g; \mathbb{Z}) = \mathbb{Z}^{2g},\ H_2(M_g; \mathbb{Z}) = \mathbb{Z},\ \text{and}\ H_i(M_g; \mathbb{Z}) = 0 \text{ for } i \geq 3.$$

Note that the Euler characteristic can be recovered as the alternating sum of the rank of the homology groups:

$$\chi(M) = \sum_{i=0}^{\dim M} (-1)^i\ \mathrm{rk}\ H_i(M; \mathbb{Z}). \tag{2.5}$$

Every smooth manifold $M$ has a triangulation, so that its simplicial homology can be defined just as above. More generally, simplicial homology can be defined for any simplicial space, i.e. a space that is built up out of points, edges, triangles, tetrahedra, etc. Formula (2.5) remains valid for any compact manifold or simplicial space.

**2.2. Singular homology:** Let $X$ be any topological space, and let $\triangle^n$ be the oriented $n$-simplex $[v_0, \ldots, v_n]$ spanned by the standard basis vectors $v_i$ in $\mathbb{R}^{n+1}$.

The set of *singular* $n$-chains $S_n(X)$ is the free abelian group on the set of continuous maps $\sigma : \triangle^n \longrightarrow X$. The boundary of $\sigma$ is defined by the alternating sum of the restriction of $\sigma$ to the faces of the $\triangle^n$:

$$\partial_n(\sigma) := \sum_{i=0}^{n} (-1)^{-i} \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}. \tag{2.6}$$

One easily checks that the boundary of a boundary is zero, and hence $(S_*(X), \partial_*)$ defines a chain complex, and its homology is by definition the singular homology $H_*(X;\mathbb{Z})$ of $X$. For any simplicial space, the inclusion of the simplicial chains into the singular chains induces an isomorphism of homology groups. In particular, this implies that the simplicial homology of a manifold, and hence its Euler characteristic do not depend on its triangulation.

If in the definition of simplicial and singular homology we take free $R$-modules (where $R$ may also be a field) instead of free abelian groups, we then define the *homology* $H_*(X;R)$ *of* $X$ *with coefficients in* $R$. The *Universal Coefficient Theorem* describes the homology with arbitrary coefficients in terms of the homology with integer coefficients. In particular, if $R$ is a field of characteristic zero,

$$\dim\, H_n(X;R) = \mathrm{rk}\, H_n(X;\mathbb{Z}).$$

**2.3. Basic properties of singular homology:** While simplicial homology (and the more efficient *cellular homology* which we will not discuss) is easier to compute and easier to understand geometrically, singular homology lends itself more easily to theoretical treatment.

*(i) Homotopy Invariance: Any continuous map* $f : X \to Y$ *induces a map on homology* $f_* : H_*(X;R) \to H_*(Y;R)$ *which only depends on the homotopy class of* $f$.

In particular, a homotopy equivalence $f : X \to Y$ induces an isomorphism in homology. So, for example, the inclusion of the circle $S^1$ into the punctured plane $\mathbb{C}\backslash\{0\}$ is a homotopy equivalence, and thus

$$H_i(\mathbb{C}\backslash\{0\};R) \simeq H_i(S^1;R) = \begin{array}{ll} \mathbb{Z} & \text{for } i = 0,1 \\ 0 & \text{for } i \geq 2. \end{array}$$

For the point space we have $H_0(pt;R) = R$. Define *reduced homology* $\tilde{H}_*(X;R) :=$ ker $(H_*(X;R) \to H_*(pt;R))$.

*(ii) Dimension Axiom:* $\tilde{H}_i(pt;R) = 0$ *for all* $i$.

More generally, it follows immediately from the definition of simplicial homology that the homology of any $n$-dimensional manifold is zero in dimensions larger than $n$.

We mentioned in the introduction that homology depends only on local data, and this makes it in general computable. This is made precise by the *(iii) Mayer-Vietoris Theorem: Let $X = A \cup B$ be the union of two open subspaces. Then the following sequence is exact*

$$\begin{aligned} \ldots \longrightarrow & H_n(A \cap B; R) \longrightarrow H_n(A; R) \oplus H_n(B; R) \longrightarrow H_n(X; R) \xrightarrow{\partial} \\ & H_{n-1}(A \cap B; R) \longrightarrow \ldots \longrightarrow H_0(X; R) \to 0. \end{aligned}$$

On the level of chains, the first map is induced by the diagonal inclusion, while the second map takes the difference between the first and second summands. Finally, $\partial$ takes a cycle $c = a + b$ in the chains of $X$ that can be expressed as the sum of a chain $a$ in $A$ and $b$ in $B$ to $\partial c := \partial_n a = -\partial_n b$. For example, consider two cones, $A$ and $B$, on a space $X$ and identify them at the base $X$ to define the suspension $\Sigma X$ of $X$. Then $\Sigma X = A \cup B$ with $A, B \simeq pt$ and $A \cap B \simeq X$. The boundary map $\partial$ is then an isomorphism:

$$\tilde{H}_n(X; R) \simeq H_{n+1}(\Sigma X; R) \qquad \text{for all } n \geq 0. \tag{2.7}$$

From this one can easily compute the homolgoy of a sphere. First note that

$$\tilde{H}_0(X; \mathbb{Z}) = \mathbb{Z}^{k-1}$$

where $k$ is the number of connected components in $X$. Also, $S^n \simeq \Sigma S^{n-1} \simeq \cdots \simeq \Sigma^n S^0$. Thus, by (2.7),

$$H_n(S^n; \mathbb{Z}) \simeq \mathbb{Z} \quad \text{and} \quad \tilde{H}_*(S^n; \mathbb{Z}) = 0 \text{ for } * \neq n. \tag{2.8}$$

If $Y$ is a subspace of $X$, *relative homology* groups $H_*(X, Y; R)$ can be defined as the homology of the quotient complex $S_*(X)/S_*(Y)$. When $Y$ has a good neighbourhood in $X$ (i.e. is a neighbourhood deformation retract in $X$) then, by the *Excision Theorem*,

$$H_*(X, Y; R) \simeq \tilde{H}_*(X/Y; R)$$

where $X/Y$ denotes the quotient space of $X$ with $Y$ identified to a point. There is a long exact sequence

$$\begin{aligned} \ldots \longrightarrow & H_n(Y; R) \longrightarrow H_n(X; R) \longrightarrow H_n(X, Y; R) \xrightarrow{\partial} \\ & H_{n-1}(Y; R) \longrightarrow \ldots \longrightarrow H_0(X, Y; R) \longrightarrow 0. \end{aligned}$$

This and the Mayer-Vietoris sequence give two ways of breaking up the problem of computing the homology of a space into computing the homology of related spaces. An iteration of this process leads to the powerful tool of *spectral sequences.*

**2.4. Relation to homotopy groups:** Let $\pi_1(X, x_0)$ denote the fundamental group of $X$ relative to the base point $x_0$. These are the based homotopy classes of based maps from a circle to $X$.

(2. 9) *If $X$ is connected, then $H_1(X; \mathbb{Z})$ is the abelianization of $\pi_1(X, x_0)$.*

Indeed, every map from a (triangulated) sphere to $X$ defines a cycle and hence gives rise to a homology class. This gives us the Hurewicz map $h : \pi_*(X; x_0) \to H_*(X; \mathbb{Z})$.

In general there is no good despription of its image. However, if $X$ is $k$-connected with $k \geq 1$ then $h$ induces an isomorphism in dimension $k+1$ and an epimorphism in dimension $k+2$.

Though (2.9) indicates that homology cannot distinguish between all homotopy types, the fundamental group is in a sense the only obstruction to this. A simple form of the *Whitehead Theorem* states:

(2.10) *If a map $f : X \to Y$ between two simplicial complexes with trivial fundamental groups induces an isomorphism on all homology groups, then it is a homotopy equivalence.*

Warning: This does *not* imply that two simply connected spaces with isomorphic homology groups are homotopic! The existence of the map $f$ inducing this isomorphism is crucial and counter examples can easily be constructed.

## 3. Dual chain complexes and cohomology

The process of dualizing itself cannot be expected to yield any new information. Nevertheless, the cohomology of a space, which is obtained by dualizing its simplicail chain complex, carries important additional structure: it possesses a product, and moreover, when the coefficients are a primary field, it is an algebra over the rich Steenrod algebra. As with homology we start with the algebraic set up.

Every chain complex $(C_*, \partial_*)$ gives rise to a dual chain complex $(C^*, \partial^*)$ where $C^i = \hom_R(C_i, R)$ is the dual $R$-module of $C_i$; because of (2.2), the composition of two dual boundary morphisms $\partial^{i+1} : C^i \to C^{i+1}$ is trivial. Hence we may define the $i$-th dimensional *cohomology group* as

$$H^i(C^*, \partial^*) := \frac{\ker \partial^{i+1}}{\operatorname{im} \partial^i}. \tag{3.1}$$

Evaluation $(\sigma, \phi) \mapsto \phi(\sigma)$ descends to a *dual pairing*

$$H_n(C_*, \partial_*) \otimes_R H^n(C^*, \partial^*) \longrightarrow R,$$

and when $R$ is a field, this identifies the cohomology groups as the duals of the homology groups. More generally, the *Universal Coefficient Theorem* relates the two. A simple version states: let $(C_*, \partial_*)$ be a chain complex of free abelian groups (such as the simplicial or singular chain complexes) with finitely generated homology groups. Then

$$H^i(C^*, \partial^*) \simeq H_i^{free}(C_*, \partial_*) \oplus H_{i-1}^{tor}(C_*, \partial), \tag{3.2}$$

where $H_*^{tor}$ denotes the torsion subgroup of $H_*$ and $H_*^{free}$ denotes the quotient group $H_*/H_*^{tor}$.

**3.1. Singular cohomology:** The dual $S^*(X)$ of the singular chain complex of a space $X$ carries a natural pairing, the cup product, $\cup : S^p(X) \otimes S^q(X) \to S^{p+q}(X)$ defined by

$$(\phi_1 \cup \phi_2)(\sigma) := \phi_1(\sigma|_{[v_0,\ldots,v_p]})\, \phi_2(\sigma|_{[v_p,\ldots,v_{p+q}]}).$$

This descends to a multiplication on cohomology groups and makes $H^*(X;R) := \bigoplus_{n\geq 0} H^n(X;R)$ into an associative, graded commutative ring: $u \cup v = (-1)^{\deg u \deg v} v \cup u$.

The *Künneth Theorem* gives some geometric intuition for the cup product. A simple version states: for spaces $X$ and $Y$ with $H^*(Y;R)$ a finitely generated free $R$-module, the cup product defines an isomorphism of graded rings

$$H^*(X;R) \otimes_R H^*(Y;R) \longrightarrow H^*(X \times Y;R).$$

For example, for a sphere, all products are trivial for dimension reasons. Hence,

$$H^*(S^n;\mathbb{Z}) = \bigwedge^*(x) \tag{3.3}$$

is an exterior algebra on one generator $x$ of degree $n$. On the other hand, the cohomology of the $n$-dimensional torus $T^n$ is an exterior algebra on $n$ degree one generators,

$$H^*(T^n;\mathbb{Z}) = \bigwedge^*(x_1,\ldots,x_n). \tag{3.4}$$

The dual pairing can be generalised to the *slant* or *cap product*

$$\cap : H_n(X;R) \otimes_R H^i(X;R) \longrightarrow H_{n-i}(X;R)$$

defined on the chain level by the formula $(\sigma,\phi) \mapsto \phi(\sigma|_{[v_0,\ldots,v_i]})\,\sigma|_{[v_i,\ldots,v_n]}$.

**3.2. Steenrod algebra:** The cup product on the chain level is homotopy commutative, but not commutative. Steenrod used this defect to define operations

$$Sq^i : H^n(X;\mathbb{Z}_2) \longrightarrow H^{n+i}(X;\mathbb{Z}_2)$$

for all $i \geq 0$ which refine the cup squaring operation: when $n = i$, then $Sq^n(x) = x \cup x$. These are natural group homomorphisms which commute with suspension. Furthermore, they satisfy the *Cartan* and *Adem Relations*

$$Sq^n(x \cup y) = \sum_{i+j=n} Sq^i(x) \cup Sq^j(y)$$

$$Sq^i Sq^j = \sum_{k=0}^{[i/2]} \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k \text{ for } i \leq 2j.$$

The mod-2 Steenrod algebra $\mathcal{A}$ is then the free $\mathbb{Z}_2$-algebra generated by the Steenrod squares $Sq^i$, $i \geq 0$, subject only to the Adem relations. With the help of Adem's relations Serre and Cartan found a $\mathbb{Z}_2$-basis for $\mathcal{A}$:

$$\{Sq^I := Sq^{i_1} \ldots Sq^{i_n} \quad | \quad i_j \geq 2i_{j+1} \text{ for all } j\ \}$$

The Steenrod algebra is also a Hopf algebra with a commutative comultipilication $\triangle : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ induced by

$$\triangle(Sq^n) := \sum_{i+j=n} Sq^i \otimes Sq^j.$$

The Cartan relation then implies that the mod-2 cohomology of a space is compatible with the comultiplcation, i.e. $H^*(X;\mathbb{Z}_2)$ is an algebra over the Hopf algebra $\mathcal{A}$.

There are odd primary analogues of the Steenrod algebra based on the reduced $p$-th power operations

$$P^i : H^n(X;\mathbb{Z}_p) \longrightarrow H^{n+2i(p-1)}(X;\mathbb{Z}_p)$$

with similar properties to $\mathcal{A}$.

One of the most striking applications of the Steenrod algebra can be found in the work of Adams on the vector fields on spheres problem: for each $n$ find the greatest number $k$, denoted $K(n)$, such that there is a $k$-field on the $(n-1)$-sphere $S^{n-1}$. Recall that a *$k$-field* is an ordered set of $k$ pointwise linear independent tangent vector fields. If we write $n$ in the form $n = 2^{4a+b}(2s+1)$ with $0 \leq b < 4$, Adams proved that $K(n) = 2^b + 8a - 1$. In particular, when $n$ is odd, $K(n) = 0$. We give an outline of the proof for this special case in the next section.

• The failure of associativity of the cup product at the chain level gives rise to secondary operations, the so called *Massey products*.

## 4. Cohomology of smooth manifolds

A smooth manifold $M$ of dimension $n$ can be triangulated by smooth simplices $\sigma : \triangle^n \to M$. If $M$ is compact, oriented, without boundary, the sum of these simplices define a homology cycle $[M]$, the *fundamental class* of $M$. The most remarkable property of the cohomology of manifolds is that they satisfy *Poincaré Duality*: taking cap product with $[M]$ defines an isomorphism:

$$D := [M]\cap : H^k(M;\mathbb{Z}) \xrightarrow{\simeq} H_{n-k}(M;\mathbb{Z}) \quad \text{for all } k. \tag{4.1}$$

In particular, for connected manifolds, $H^n(M;\mathbb{Z}) \simeq \mathbb{Z}$, and every map $f : M' \to M$ between oriented, compact closed manifolds of the same dimension has a *degree*: the integer deg $(f)$ such that $f^*$ in dimension $n$ is multiplication by deg $(f)$. For smooth maps, the degree is the number of points in the inverse image of a generic point $p \in M$ counted with signs:

$$\deg\,(f) = \sum_{p' \in f^{-1}(p)} \operatorname{sign}(p')$$

where sign $(p')$ is $+1$ or $-1$ depending on whether $f$ is orientation preserving or reversing in a neighbourhood of $p'$. For example, a complex polynomial of degree $d$ defines a map of the two dimensional sphere to itself of degree $d$: a generic point has $n$ points in its inverse image and the map it locally orientation preserving. On the other hand, a map of $S^{n-1}$ induced by a reflection of $\mathbb{R}^n$ reverses orientation and has degree $-1$. Thus, as degrees multiply on composing maps, the anti-podal map $x \mapsto -x$ has degree $(-1)^n$. As an application we prove:

*Every tangent vector field on an even dimensional sphere $S^{n-1}$ has a zero.*

*Proof*: Assume $v(x)$ is a vector field which is non-zero for all $x \in S^{n-1}$. Then $x$ is perpendicular to $v(x)$, and after rescaling, we may assume that $v(x)$ has length one. The function $F(x,t) = \cos(t)x + \sin(t)v(x)$ is then a well-defined homotopy from the identity map ($t = 0$) to the antipodal map ($t = \pi$). But this is impossible as homotopic maps induce the same map in (co)homology but we have already seen that the degree of the idendity map is 1 while the degree of the antipodal map is $(-1)^n = -1$ when $n$ is odd.

• It is well-known that two self-maps of a sphere of any dimension are homotopic if and only if they have the same degree, i.e. $\pi_n(S^n) \simeq \mathbb{Z}$ for $n \geq 1$.

• When $M$ is not orientable, $[M]$ still defines a cycle in homology with $\mathbb{Z}_2$-coefficients, and $\cap[M]$ defines an isomorphism between the cohomology and homology with $\mathbb{Z}_2$ coefficients.

• As $[M]$ represents a homology class, so does every other closed (orientable) submanifold of $M$. It is however not the case that every homology class can be represented by a submanifold or linear combinations of such.

Cohomology is a contravariant functor. Poincaré duality however allows us to define, for any $f : M' \to M$ between oriented, compact, closed manifolds of arbitrary dimensions, a *transfer* or *Umkehr map*,

$$f^! := D^{-1} f_* D' : H^*(M';\mathbb{Z}) \longrightarrow H^{*-c}(M;\mathbb{Z})$$

which lowers the degree by $c = \dim M' - \dim M$. It satisfies the formula

$$f^!(f^*(x) \cup y) = x \cup f^!(y)$$

for all $x \in H^*(M;\mathbb{Z})$ and $y \in H^*(M';\mathbb{Z})$. When $f$ is a covering map then $f^!$ can be defined on the chain level by

$$f^!(x)(\sigma) := x\Big(\sum_{f(\tilde{\sigma})=\sigma} \tilde{\sigma}\Big)$$

where $x \in C^*(M')$ and $\sigma \in C_*(M)$.

**4.1. De Rham cohomology:** If $x_1, \ldots, x_n$ are the local coordinates of $\mathbb{R}^n$, define an algebra $\Omega^*$ to be the algebra generated by symbols $dx_1, \ldots, dx_n$ subject to the relations $dx_i dx_j = -dx_j dx_i$ for all $i, j$. We say $dx_{i_1} \ldots dx_{i_q}$ has degree $q$. The *differential forms* on $\mathbb{R}^n$ are the algebra

$$\Omega^*(\mathbb{R}^n) := \{C^\infty \text{ functions on } \mathbb{R}^n\} \otimes_{\mathbb{R}} \Omega^*.$$

The algebra $\Omega^*(\mathbb{R}^n) = \bigoplus_{q=0}^n \Omega^q(\mathbb{R}^n)$ is naturally graded by degree. There is a differential operator $d : \Omega^q(\mathbb{R}^n) \to \Omega^{q+1}(\mathbb{R}^n)$ defined by

$$(i) \text{ if } f \in \Omega^0(\mathbb{R}^n), \text{then } df = \sum (\partial f/\partial x_i)\, dx_i$$

$$(ii) \text{ if } \omega = \sum f_I dx_I, \text{ then } d\omega = \sum df_I dx_I.$$

$I$ stands here for a multi-index. For example in $\mathbb{R}^3$ the differential assigns to 0-forms (= functions) the gradient, to 1-forms the curl, and to 2-forms the divergence. An easy exercise shows that $d^2 = 0$ and the *q-th de Rahm cohomology* of $\mathbb{R}^n$ is the vector space

$$H^q_{deR}(\mathbb{R}^n) = \frac{\ker\, d : \Omega^q(\mathbb{R}^n) \longrightarrow \Omega^{q+1}(\mathbb{R}^n)}{\mathrm{im}\, d : \Omega^{q-1}(\mathbb{R}^n) \rightarrow \Omega^q(\mathbb{R}^n)}.$$

More generally, the de Rham complex $\Omega^*(M)$ and the cohomology $H^*_{deR}(M)$ for any smooth manifold $M$ can be defined.

Let $\sigma$ be a smooth, singular, real $(q+1)$-chain on $M$, and let $\omega \in \Omega^q(M)$. *Stokes Theorem* then says

$$\int_{\partial\sigma} \omega = \int_\sigma d\omega,$$

and therefore integration defines a pairing between the $q$-th singular homology and the $q$-th de Rahm cohomology of $M$. This pairing is exact and thus de Rahm cohomology is isomorphic to singular cohomology with real coefficients:

$$H^*_{deR}(M) \simeq (H_*(M;\mathbb{R}))^* \simeq H^*(M;\mathbb{R}).$$

Let $\Omega^*_c(M)$ denote the subcomplex of *compactly supported forms* and $H^*_c(M)$ its cohomology. Integration with respect to the first $i$ coordinates defines a map

$$\Omega^*_c(\mathbb{R}^n) \longrightarrow \Omega^{*-i}_c(\mathbb{R}^{n-i})$$

which induces an isomorphism in cohomology; note in particular $H^n_c(\mathbb{R}^n) = \mathbb{R}$. More generally, when $E \rightarrow M$ is an $i$-dimensional orientable, real vector bundle over a compact, orientable manifold $M$, integration over the fiber gives the *Thom Isomorphism*:

$$H^*_c(E) \simeq H^{*-i}_c(M) \simeq H^{*-i}_{deR}(M),$$

while for orientable fiber bundles $F \rightarrow M' \xrightarrow{f} M$ with compact, orientable fiber $F$, integration over the fiber provides another definition of the transfer map

$$f^! : H^*_{deR}(M') \longrightarrow H^{*-i}_{deR}(M).$$

**4.2. Hodge decomposition:** Let $M$ be a compact oriented Riemannian manifold of dimension $n$. The Hodge star operator, $*$, associates to every $q$-form a $(n-q)$-form. For $\mathbb{R}^n$ and any orthonormal basis $\{e_1, \ldots, e_n\}$, it is defined by setting

$$*(e_1 \wedge \cdots \wedge e_q) := \pm e_{p+1} \wedge \cdots \wedge e_n$$

where one takes $+$ if the orientation defined by $\{e_1, \ldots, e_n\}$ is the same as the usual one, and $-$ otherwise. Using local coordinate charts this definition can be extended to any $M$. Clearly, $*$ depends on the chosen metric and orientation of $M$. If $M$ is compact, we may define an inner product on the $q$-forms by

$$(\omega, \omega') := \int_M \omega \wedge *\,\omega'.$$

With respect to this inner product $*$ is an isometry. Define the *co-differential* via

$$\delta := (-1)^{np+n+1} * d* : \Omega^q(M) \longrightarrow \Omega^{q-1}(M),$$

and the *Laplace-Beltrami* operator via

$$\triangle := \delta d + d\delta.$$

The co-differential satisfies $\delta^2 = 0$ and is the adjoint of the differential. Indeed, for $q$-forms $\omega$ and $(q+1)$-forms $\omega'$:

$$(d\omega, \omega') = (\omega, \delta\omega'). \tag{4.2}$$

It follows easily that $\triangle$ is self-adjoint, and furthermore,

$$\triangle\omega = 0 \quad \text{if and only if} \quad d\omega = 0 \text{ and } \delta\omega = 0. \tag{4.3}$$

A form $\omega$ satisfying $\triangle\omega = 0$ is called *harmonic*. Let $\mathcal{H}^q$ denote the subspace of all harmonic $q$-forms. It is not hard to prove the *Hodge Decomposition Theorem* :

$$\Omega^q = \mathcal{H}^q \oplus \text{ im } d \oplus \text{ im } \delta.$$

Furthermore, by adjointness (4.2), a form $\omega$ is closed only if it is orthogonal to im $\delta$. On calculating the de Rham cohomology we can also ignore the summand im $d$ and find that:

*Each de Rham cohomology class on a compact oriented Riemannian manifold $M$ contains a unique harmonic representative, i.e. $H^q_{deR}(M) \simeq \mathcal{H}^q$.*

Warning: This isomorphism is an isomorphism of vector spaces and in general does not extend to an isomorphism of algebras.

## 5. Examples

We list the cohomology of some important examples.

*Projective spaces:* Let $\mathbb{R}P^n$ be real projective space of dimension $n$. Then

$$H^*(\mathbb{R}P^n; \mathbb{Z}_2) = \mathbb{Z}_2[x]/(x^{n+1})$$

is a stunted polynomial ring on one generator $x$ of degree one.
Similarly, let $\mathbb{C}P^n$ and $\mathbb{H}P^n$ denote complex and quaternionic projective space of real dimensions $2n$ and $4n$ respectively. Then

$$H^*(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}[y]/(y^{n+1}) \quad \text{and} \quad H^*(\mathbb{H}P^n; \mathbb{Z}) = \mathbb{Z}[z]/(z^{n+1})$$

are stunted polynomial rings with deg $(y) = 2$ and deg $(z) = 4$.

*Lie groups:* Let $G$ be a compact, connected Lie group of rank $l$, i.e. the dimension of the maximal torus of $G$ is $l$. Then

$$H^*(G, \mathbb{Q}) \simeq \bigwedge_{\mathbb{Q}}^{*}[a_{2d_1-1}, a_{2d_2-1}, \dots, a_{2d_l-1}],$$

where $|a_i| = i$ and $d_1, \ldots, d_l$ are the fundamental degrees of $G$ which are known for all $G$. Often this structure lifts to the integral cohomology. In particular we have:

$$H^*_{free}(SO(2k+1);\mathbb{Z})) \simeq \bigwedge_{\mathbb{Z}}^{*}[a_3, a_7, \ldots, a_{4k-1}]$$
$$H^*_{free}(SO(2k);\mathbb{Z})) \simeq \bigwedge_{\mathbb{Z}}^{*}[a_1, a_7, \ldots, a_{4k-5}, a_{2k-1}]$$
$$H^*(U(k);\mathbb{Z}) \simeq \bigwedge_{\mathbb{Z}}^{*}[a_1, a_3, \ldots, a_{2k-1}]$$

*Classifying spaces:* For any group $G$ there exists a classifying space $BG$, well-defined up to homotopy. Classifying spaces are of central interest to geometers and topologists for the set of isomorphism classes of principal $G$-bundles over a space $X$ is in one-to-one correspondence with the set of homotopy classes of maps from $X$ to $BG$. In particular, every cohomology class $c \in H^*(BG; R)$ defines a *characteristic class* of principle $G$-bundles $E$ over $X$: if $E$ corresponds to the map $f_E : X \to BG$ then $c(E) := f_E^*(c)$.

$BG$ can be constructed as the space of $G$-orbits of a contractible space $EG$ on which $G$ acts freely. Thus for example

$$B\mathbb{Z} = \mathbb{R}/\mathbb{Z} \simeq S^1$$
$$B\mathbb{Z}_2 = (\lim_{n\to\infty} S^n)/\mathbb{Z}_2 \simeq \mathbb{R}P^\infty$$
$$BS^1 = (\lim_{n\to\infty} S^{2n+1})/S^1 \simeq \mathbb{C}P^\infty$$

and more generally, infinite Grassmannian manifolds are classifying spaces for linear groups. When $G$ is a compact connected Lie group,

$$H^*(BG;\mathbb{Q}) \simeq \mathbb{Q}[x_{2d_1}, \ldots, x_{2d_l}]$$

with $d_i$ as above and $|x_i| = i$. In particular,

$$H^*(BSO(2k+1);\mathbb{Z}[1/2]) \simeq \mathbb{Z}[1/2][p_1, p_2, \ldots, p_k]$$
$$H^*(BSO(2k);\mathbb{Z}[1/2]) \simeq \mathbb{Z}[1/2][p_1, p_2, \ldots, p_{k-1}, e_k]$$
$$H^*(BU(k);\mathbb{Z}) \simeq \mathbb{Z}[c_1, c_2, \ldots, c_k]$$

where the Pontryagin, Euler, and Chern classes have degree $|p_i| = 4i, |e_k| = 2k$, and $|c_i| = 2i$ respectively.

*Moduli spaces:* Let $\mathcal{M}_g^n$ be the space of Riemann surfaces of genus $g$ with $n$ ordered, marked points. There are naturally defined classes $\kappa_i$ and $e_1, \ldots, e_n$ of degree $2i$ and 2 respectively. By Harer-Ivanov stability and the recent proof of the Mumford conjecture [Madsen-Weiss, preprint 2004], there is an isomorphism up to degree $* < \frac{3g}{2}$ of the rational cohomology of $\mathcal{M}_g^n$ with

$$\mathbb{Q}[\kappa_1, \kappa_2, \ldots] \otimes \mathbb{Q}[e_1, \ldots, e_n].$$

The rational cohomology vanishes in degrees $* > 4g-5$ if $n = 0$, and $* > 4g-4+n$ if $n > 0$. Though the stable part of the cohomology is now well understood, the structure of the unstable part, as proposed by Farber [Aspects of Mathematics, E33, Viehweg 1999], remains conjectural.

## 6. Generalised cohomology theories

The three basic properties 2.3 (i), (ii), (iii), appropriately dualized, hold of course also for cohomology. Furthermore, they (essentially) determine (co)homology uniquely as a functor from the category of simplicial spaces and continuous functions to the category of abelian groups. If we drop the dimension axiom (ii), we are left with homotopy invariance (i), and the Mayer-Vietoris sequence (iii). Abelian group valued functors satisfying (i) and (iii) are so called *generalised (co)homology theories.* $K$-theory and cobordism theory are two well-known examples but there are many more.

**6.1. $K$-theory:** The geometric objects representing elements in complex $K$-theory $K^0(X)$ are isomorphism classes of complex vector bundles $E$ over $X$. Vector bundles $E, E'$ can be added to form a new bundle $E \oplus E'$ over $X$, and $K^0(X)$ is just the group completion of the arising monoid. Thus for example for the point space we have $K^0(pt) = \mathbb{Z}$. Tensor product of vector bundles $E \otimes E'$ induces a multiplication on $K$-theory making $K^*(X)$ into a graded commutative ring.

In many ways $K$-theory is easier than cohomology. In particular, the groups are 2-periodic: all even degree groups are isomorphic to the reduced $K$-theory group $\tilde{K}^0(X) :=$ coker $(K^0(pt) = \mathbb{Z} \to K^0(X))$, and all odd degree groups are isomorphic to $K^{-1}(X) := \tilde{K}^0(\Sigma X)$.

The theory of characteristic classes gives a close relation between the two cohomology theories. The Chern character map, a rational polynomial in the Chern classes,

$$ch : K^0(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow H^{even}(X;\mathbb{Q}) := \oplus_{k \geq 0} H^{2k}(X;\mathbb{Q}),$$

is an isomorphism of rings. Thus the $K$-theory and cohomology of a space carry the same rational information. But they may have different torsion parts. This became an issue in string theory when $D$-brane charges which had formerly been thought of as differential forms (and hence cohomology classes) were later reinterpreted more naturally as $K$-theory classes by Witten [J. Higher Energy Phys. 1998, no.12].

• There are real and quarternionic $K$-theory groups which are 8-periodic.

**6.2. Cobordism theory:** The geometric objects representing an element in the oriented cobordism group $\Omega^n_{SO}(X)$ are pairs $(M, f)$ where $M$ is a smooth, orientable $n$-dimensional manifold and $f : M \to X$ is a continuous map. Two pairs $(M, f)$ and $(M', f')$ represent the same cobordism class if there exists a pair $(W, F)$ where $W$ is a $n+1$-dimensional, smooth, oriented manifold with boundary $\partial W = M \cup -M'$ such that $F$ restricts to $f$ and $f'$ on the boundary $\partial W$. Disjoint union and Cartesian product of manifolds define an addition and multiplication so that $\Omega^*_{SO}(X)$ is a graded, commutative ring.

• Similarly, unoriented, complex, or spin cobordism groups can be defined.

**6.3. Elliptic cohomology:** Quillen proved that complex cobordism theory is universal for all *complex oriented* cohomology theories, that is those cohomology theories that allow a theory of Chern classes. In a complex oriented theory the first Chern class of the tensor product of two line bundles can be expressed in terms of the first Chern class of each of them via a two variable power series: $c_1(E \otimes E') = F(c_1(E), c_1(E'))$. $F$ defines a formal group law and Quillen's theorem [Bull. Amer. Math. Soc. vol. 75 (1969)] asserts that the one arising from complex cobordism theory is the universal one.

Vice versa, given a formal group law one may try to construct a complex oriented cohomology theory from it. In particular, an elliptic curve gives rise to a formal group law and an elliptic cohomology theory. Hopkins et al. have described and studied an inverse limit of these elliptic theories, which they call the theory of topological modular forms, *tmf*, as the theory is closely related to modular forms. In particular there is a natural map from the groups $tmf_{2n}(pt)$ to the group of modular forms of weight $n$ over $\mathbb{Z}$. After inverting a certain element (related to the discriminant), the theory becomes periodic with period $24^2 = 576$.

Witten [Springer Lecture Notes in Mathematics, vol. 1326 (1998)] showed that the purely theoretically constructed elliptic cohomology theories should play an important role in string theory: the index of the Dirac operator on the free loop space of a spin manifold should be interpreted as an element of it. But unlike for ordinary cohomology, $K$-theory, and cobordism theory we do not (yet) know a good geometric object representing elements in this theory without which its use for geometry and analysis remains limited. Segal speculated some 20 years ago that conformal field theories should define such geometric objects. However, though progress in this direction has been made, the search for a good geometric interpretation of elliptic cohomology (and *tmf*) remains an active and important research area.

**6.4. Infinite loop spaces:** Brown's representability theorem implies that for each (reduced) generalised cohomology theory $h^*$ we can find a sequence of spaces $E_*$ such that $h^n(X)$ is the set of homotopy classes $[X, E_n]$ from the space $X$ to $E_n$ for all $n$. Recall that the Mayer-Vietoris sequence implies that $h^n(X) \simeq h^{n+1}(\Sigma X)$. The suspension functor $\Sigma$ is adjoint to the based loop space functor $\Omega$ which takes a space $X$ to the space of based maps from the circle to $X$. Hence,

$$h^n(X) = [X, E_n] = [\Sigma X, E_{n+1}] = [X, \Omega E_{n+1}],$$

and it follows that every generalised cohomology theory is represented by an *infinite loop space*

$$E_0 \simeq \Omega E_1 \simeq \cdots \simeq \Omega^n E_n \simeq \ldots.$$

Vice versa, any such infinite loop space gives rise to a generalised cohomology theory.

One may think of infinite loop spaces as the abelian groups up to homotopy in the strongest sense. Indeed, ordinary cohomology with integer coefficients is represented by

$$\mathbb{Z} \simeq \Omega S^1 \simeq \Omega^2 \mathbb{C}P^\infty \simeq \cdots \simeq \Omega^n K(n, \mathbb{Z}) \ldots,$$

where by definition the Eilenberg-Maclane space $K(n,\mathbb{Z})$ has trivial homotopy groups for all dimensions not equal $n$ and $\pi_n K(n,\mathbb{Z}) = \mathbb{Z}$. Complex $K$-theory is represented by

$$\mathbb{Z} \times BU \simeq \Omega(U) \simeq \Omega^2(BU) \simeq \Omega^3(U) \simeq \ldots.$$

The first equivalence is Bott's celebrated *Periodicity Theorem.* Finally, oriented cobordism theory is represented by

$$\Omega^\infty MSO := \lim_{n\to\infty} \Omega^n \mathrm{Th}(\gamma_n)$$

where $\gamma_n \to BSO_n$ is the universal $n$-dimensional vector space over the Grassmannian manifold of oriented $n$-planes in $\mathbb{R}^\infty$, and $\mathrm{Th}(\gamma_n)$ denotes its Thom space.

A good source of infinite loop spaces are symmetric monoidal categories. Indeed every infinite loop space can be constructed from such a category: the symmetric monoidal structure gives corresponds the homotopy abelian group structure. For example, the category of finite dimensional, complex vector spaces and their isomorphisms gives rise to $\mathbb{Z} \times BU$. To give another example, in quantum field theory one considers the $(d+1)$-dimensional cobordism category with objects the compact, oriented $d$-dimensional manifolds and their $(d+1)$-dimensional cobordisms as morphisms. Disjoint union of manifolds makes this category into a symmetric monoidal category. The associated infinite loop space and hence generalised cohomology theory has recently been identified as a $(d+1)$-dimensional slice of oriented cobordism theory [Galatius-Madsen-Tillmann-Weiss, preprint 2005].

**See also**

**Characteristic classes. K-theory. Spectral sequences. Moduli spaces. Intersection theory. Index theorems. Equivariant cohomology and the Cartan model.**

**Further reading**

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