Mock exam/ Practice questions

1. What is a *Lie algebra*? What is a *linear Lie algebra*? Define the *exponential map* for linear Lie algebras.
   State (a version of) the *Campbell-Baker-Hausdorff* formula, and deduce that the image of any linear Lie algebra is a group.
   Let SO_3 denote the set of 3×3 orthogonal groups with determinant 1. Find a linear Lie algebra $\mathfrak{g}$ such that SO_3 is the image of $\mathfrak{g}$ under the exponential map. Prove your assertion.

2. Let G be a linear group and LG its Lie algebra. Prove that the following are equivalent:
   i. G is path connected by differentiable paths.
   ii. G is connected.
   iii. G is generated by a neighbourhood of the identity.
   iv. G is generated by $\exp LG$.
   [*You may assume that the image of the exponential map exp* $: LG \to G$ *contains a neighbourhood of the identity.*]
   Show that exp is a homomorphism of groups if and only if G is abelian. Deduce that a connected abelian Lie group is the product of a torus and a vector space,$G \simeq T^r \times \mathbb{R}^s$.

3. Describe the Lie algebra of the unitary group U_n and the general complex linear group $GL_n\mathbb{C}$. What is the relation between them? Also describe the Cartan subgroup H of $GL_n\mathbb{C}$ and the Cartan Lie algebra $L(H)$.
   What is a *weight* of H? Find (with proof) all weights of H. What is a *root* of $(L(Gl_n\mathbb{C}), L(H))$? What is the set Φ of roots in this case? Give proof.
   Define the *Weyl group* W of $(GL_n\mathbb{C}, H)$.
   Show that W acts on the roots Φ. Hence show that W is the symmetric group on n letters.
   State carefully the *Weyl character formula* for U_n.

4. Define the terms *(complex) representation*, *irreducible representation*, and *character* of a linear group G. Show that every character of G is a class function.
   State and prove *Schur's Lemma*.
   Outline a proof for: Two representations of a compact linear group G are equivalent if and only if they have the same character.
   Find all the irreducible characters of the cyclic group of order n.